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Research Article

Formal Lagrangian Operad

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Given a symplectic manifold M, we may define an operad structure on the the spaces O^k of the Lagrangian submanifolds of $(\overline{M})^k \times M$ via symplectic reduction. If M is also a symplectic groupoid, then its multiplication space is an associative product in this operad. Following this idea, we provide a deformation theory for symplectic groupoids analog to the deformation theory of algebras. It turns out that the semiclassical part of Kontsevich's deformation of $C^\infty(\mathbb{R}^d)$ is a deformation of the trivial symplectic groupoid structure of $T^*\mathbb{R}^d$.

1. Introduction

Symplectic groupoids, in the extended symplectic category, may be thought as the analog of associative algebras in the category of vector spaces. For the latter, a deformation theory exists and is well known. In this paper, we will present a conceptual framework as well as an explicit deformation of the trivial symplectic groupoid over $\mathbb{R}^d$. In fact, rephrased appropriately, most constructions of the deformation theory of algebras can be extended to symplectic groupoids, at least for the trivial one over $\mathbb{R}^d$. Our guideline will be the Kontsevich deformation of the usual algebra of functions over $\mathbb{R}^d$, $(C^\infty(\mathbb{R}^d),\cdot)$. Namely, the usual pointwise product of functions $S_0^2(f,g) = fg$ generates a suboperad, the product suboperad, $\mathcal{O}_S^n = \{S_0^n\}$, of the endomorphism operad $\mathcal{O}$ of $C^\infty(\mathbb{R}^d)$, where S_0^n is the n-multilinear map defined by $S_0^n(f_1,\ldots,f_n) = f_1 f_2 \cdots f_n$. For each n one may choose the vector subspace $\mathcal{O}_{\mathrm{def}}^n \subset \mathcal{O}^n$ of n-multidifferential operators. The operad structure of $\mathcal{O}$ induces an operad structure on $\mathcal{O}_S + \mathcal{O}_{\mathrm{def}}$, which in turn generates an operad structure on $\mathcal{O}_{\mathrm{def}}$ which is, however, nonlinear. Then, γ is a deformation of the usual product S_0^2, that is, an element $\gamma \in \mathcal{O}_{\mathrm{def}}^2$ such that $S_0^2+\gamma$ is still an associative product, if γ is a product in the induced deformation operad $\mathcal{O}_{\mathrm{def}}$. We may also consider the formal version by replacing $\mathcal{O}_{\mathrm{def}}$ by the formal power series in ϵ, $\epsilon\mathcal{O}_{\mathrm{def}}[[\epsilon]]$. Kontsevich in [1] gives an explicit formal deformation of the product of functions over $\mathbb{R}^d$,

$$S_\epsilon = S_0^2 + \sum_{n=1}^{\infty} \epsilon^n \sum_{\Gamma \in G_{n,2}} W_\Gamma B_\Gamma, \tag{1.1}$$

where the W_Γ's are the Kontsevich weights and the B_Γ's are the Kontsevich bidifferential operators associated to the Kontsevich graphs of type $(n, 2)$ (see [2] for a brief introduction).

If we consider the trivial symplectic groupoid $\mathrm{T}^*\mathbb{R}^d$ over $\mathbb{R}^d$, we see that the multiplication space

$$\Delta_2^n := \left\{ (p_1, x), (p_2, x), (p_1 + p_2, x) : p_1, p_2 \in \mathbb{R}^{d*}, x \in \mathbb{R}^d \right\} \tag{1.2}$$

generates an operad $\mathcal{O}_\Delta^n = \{\Delta_n\}$, where

$$\Delta_n := \left\{ (p_1, x), \ldots, (p_n, x), (p_1 + \cdots + p_n, x) : p_i \in \mathbb{R}^{d*}, x \in \mathbb{R}^d \right\}. \tag{1.3}$$

Δ_2 is a product in this operad. The compositions are given by symplectic reduction as the Δ_n's are Lagrangian submanifolds of $\overline{(T^*\mathbb{R}^d)^n} \times T^*\mathbb{R}^d$. The main difference with the vector space case is that there is no "true" endomorphism operad where $\mathcal{O}_\Delta$ would naturally embed into. Thus, the question of finding a deformation operad for $\mathcal{O}_\Delta$ must be taken with more care. The first remark is that the Δ_n may be expressed in terms of generating functions

$$S_0^n(p_1, \ldots, p_n, x) = (p_1 + \cdots + p_n)x. \tag{1.4}$$

Namely, $\Delta_n = \text{graph}\, dS_0^n$. The idea is to look at the operad structure induced on the generating functions by symplectic reduction. In fact it is possible to find a vector space of special functions $\mathcal{O}_{\mathrm{def}}^n$ for each n such that $\mathcal{O}_\Delta + \mathcal{O}_{\mathrm{def}}$ remains an operad. The formal version of it gives a surprising result. Namely, we may find an explicit deformation of the trivial generating function S_0^2, it is given by the formula

$$S_\epsilon = S_0^2 + \sum_{n=1}^{\infty} \epsilon^n \sum_{\Gamma \in T_{n,2}} W_\Gamma \widehat{B}_\Gamma, \tag{1.5}$$

where the W_Γ are the Kontsevich weights and the $\widehat{B}_\Gamma$ are the symbols of the Kontsevich bidifferential operators and the sum is taken over all Kontsevich trees $T_{n,2}$. This formula may be seen as the semi-classical part of Kontsevich deformation quantization formula.

As a last comment, note that Kontsevich derives its star product formula from a more general result. In fact, he shows that $U = \sum_n \epsilon^n U_n$, where

$$U_n(\xi_1, \ldots, \xi_n) = \sum_{\Gamma \in G_n} W_\Gamma B_\Gamma(\xi_1, \ldots, \xi_n) \tag{1.6}$$

for $\xi_i \in \Gamma(\wedge^{d_i} TM)$, $i = 1, \ldots, d$ is an L_∞-morphism from the multivector fields to the multidifferential operators on $\mathbb{R}^d$. In our perspective, we may still write

$$\widetilde{U}_n(\xi_1, \ldots, \xi_n) = \sum_{\Gamma \in T_n} W_\Gamma \widehat{B}_\Gamma(\xi_1, \ldots, \xi_n) \tag{1.7}$$

summing over Kontsevich trees instead of Kontsevich graphs and replacing multidifferential operators by their symbols. Exactly, as in Kontsevich case,

$$S_\epsilon = S_0^2 + \sum_{n \geq 1} \epsilon^n \widetilde{U}_n(\alpha, \ldots, \alpha) \tag{1.8}$$

is an associative deformation of the generating function of the trivial symplectic groupoid $T^*\mathbb{R}^d$. However, it is still not completely clear how to define "semi-classical L_∞-morphisms".

Organization of the Paper

In Section 2, we describe the endomorphism operad $\mathcal{O}(M) = \mathrm{Hom}(M^{\otimes^n}, M)$ associated to any object M in a monoidal category. We explain what is an associative product S on M in a monoidal category and we define the product suboperad $\mathcal{O}_S(M)$ of $\mathcal{O}(M)$. If the category is further associative, we may choose a deformation operad for S, which is a choice, for each $n \in \mathbb{N}$ of a vector subspace $\mathcal{O}^n_{\mathrm{def}}$ such that $\mathcal{O}_S + \mathcal{O}_{\mathrm{def}}$ is still an operad. We describe the deformations of S in terms of products in $\mathcal{O}_{\mathrm{def}}$. As an example of this construction, we expose Kontsevich product deformation in this language. At last, we show that the extended symplectic category, although not being a true category, exhibits monoidal properties allowing us to carry the precedent construction up to a certain point. Then, we focus on the trivial symplectic groupoid over $\mathbb{R}^d$ case and define the product operad associated to its multiplications space. We give a deformation operad on a local form, the local deformation operad. In particular, we show that any local deformation of the trivial product gives rise to a local symplectic groupoid over $\mathbb{R}^d$. We conclude this section by defining equivalence between deformations of the trivial generating function and we show that two equivalent deformations induce the same local symplectic groupoid.

In Section 3, we describe the combinatorial tools needed to give a formal version of the local Lagrangian operad. As the problem consists mainly in taking Taylor's series of some implicit equations we need devices to keep track of all terms to all orders. The crucial point is that these implicit equations, describing the composition in the local Lagrangian operad, have a form extremely close to a special Runge-Kutta method: the partitioned implicit Euler method. We borrow then some techniques form numerical analysis of ODEs to make the expansion at all orders.

In the last section, we describe the formal Lagrangian operad, which is the perturbative version of the local one, in terms of composition of bipartite trees. We give in particular the product equation in the formal deformation operad in terms of these trees. At last, we restate the main theorem of [2] in this language. This tells us that the semi-classical part of Kontsevich star product on $\mathbb{R}^d$ is a product in the formal deformation operad of the cotangent Lagrangian operad in d dimensions.

Paper Genesis and Subsequent Works

This paper was inspired in large part by the unpublished note [3], in which the notion of lagrangian operads first appeared, and from the Ph.D. thesis [4]. It was originally conceived as a development of [2], providing a framework (the theory of operads), in which the results and computations of the latter article could be understood in a cleaner and more conceptual manner: each Taylor series expansion arising in [2] can be seen as a certain composition in the formal lagrangian operad over $T^*\mathbb{R}^n$.

The combinatorics of bicolored Runge-Kutta trees was borrowed from the numerical analysis of ODE (see [5]). We used it first in [2] to expand the structure equation (also called the "SGA equation") for symplectic groupoid generating functions in formal power series. Actually, this combinatorics happens to control the compositions in the formal lagrangian operad over $T^*\mathbb{R}^d$. It is very reminiscent of the one used, in the context of bicolored operads, to define versions of operad morphisms "up to homotopy" (see [6] and also [7]). However, in the case of the formal lagrangian operad over $T^*\mathbb{R}^d$, we are not dealing with weak structures or weak maps of any kind, at least in a direct way. The actual nature of the relationship between these two formally similar but contextually different combinatorics, if any, is unknown to the authors' best knowledge.

As far as geometric quantization of Poisson manifolds using symplectic groupoid techniques is concerned, recent works seem to indicate that the language of symmetric monoidal categories is better suited than the one of operads, namely, the microsymplectic category developed in [8] is a better fit than the notion of lagrangian operads for understanding functorial aspects of geometric quantization. At any rate, the endomorphism operad of $T^*\mathbb{R}^d$ in the microsymplectic category contains, as a suboperad, the local lagrangian operad constructed in the present paper (see [8]).

However, there is no formal version of the microsymplectic category to date, and the combinatorics presented here to deal with the compositions in the formal lagrangian operad over $T^*\mathbb{R}^d$ have no equivalent in terms of a "formal microsymplectic category"; this is, at the time of writing, still a work in progress.

2. Product in the Extended Symplectic Category

2.1. Basic Constructions and Kontsevich Deformation

In this section, we describe, in any monoidal category, a natural generalization of an associative algebra structure over a vector space. It is the notion of product in the endomorphism operad $\mathcal{O}(M)$ of an object M in the category. If the category is further additive, we explain what is a deformation of a product $S \in \mathcal{O}^2(M)$ and construct a non-linear operad, the deformation operad $\mathcal{O}_{\text{def}}(M, S)$ associated to S in which any product is equivalent to a deformation of S. We present the well-known Kontsevich deformation of the usual product of functions over $\mathbb{R}^d$ in this language. At last, we see that most parts of this construction, can be applied to the extended symplectic category, leading to the notion of Lagrangian operad.

Definition 2.1. An operad $\mathcal{O}$ consists of

(1) a collection of sets $\mathcal{O}^n$, $n \geq 0$,

(2) composition laws

$$\begin{aligned}&\mathcal{O}^n \times \mathcal{O}^{k_1} \times \cdots \times \mathcal{O}^{k_n} \longrightarrow \mathcal{O}^{k_1+\ldots+k_n}\\ &(F, G_1, \ldots, G_n) \longmapsto F(G_1, \ldots, G_n)\end{aligned} \tag{2.1}$$

satisfying the following associativity relations:

$$\begin{aligned}&F(G_1, \ldots, G_n)(H_{11}, \ldots, H_{1k_1}, \ldots, H_{n1}, \ldots, H_{nk_n})\\ &\quad = F(G_1(H_{11}, \ldots, H_{1k_1}), \ldots, G_n(H_{n1}, \ldots, H_{nk_n})),\end{aligned} \tag{2.2}$$

(3) a unit element $I \in \mathcal{O}^1$ such that $F(I, \ldots, I) = F$ forall $F \in \mathcal{O}^n$.

It usually also requires some equivariant action of the symmetric group. We do not require this here.

The structure we have just defined should then be called more correctly "nonsymmetric operad". However, we will simply keep using the term "operad" instead of "non symmetric operad" in the sequels.

Product in a Monoidal Category

We consider here a monoidal category $\mathcal{C}$. We denote by $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ the product bifunctor and by $\mathbf{e} \in \mathcal{C}$ the neutral object. Let us recall that we have the following canonical isomorphisms:

$$(A \otimes B) \otimes C \simeq A \otimes (B \otimes C), \qquad \mathbf{e} \otimes A \simeq A \otimes \mathbf{e} \simeq A \tag{2.3}$$

for all $A, B, C \in \mathrm{Obj}\mathcal{C}$.

Let $\mathcal{C}$ be a monoidal category and let an object $M \in \mathrm{Obj}\mathcal{C}$. We define the endomorphism operad of M in the following way:

(1) $\mathcal{O}^n(M) := \mathrm{Hom}(M^{\otimes n}, M)$, $\mathcal{O}^0(M) := \mathrm{Hom}(e, M)$,

(2) $F(G_1, \ldots, G_n) := F \circ (G_1 \otimes \cdots \otimes G_n)$,

(3) the unit is given by $\mathrm{id}_M \in \mathcal{O}^1(M)$.

The operad axioms follow directly from the bifunctoriality of $\otimes$, that is,

$$\begin{aligned}&(f \otimes g) \circ (\psi \otimes \phi) = (f \circ \psi) \otimes (g \circ \phi)\\ &\mathrm{id}_M \otimes \cdots \otimes \mathrm{id}_M = \mathrm{id}_{M\otimes\cdots\otimes M}.\end{aligned} \tag{2.4}$$

If M is an object of a monoidal category $\mathcal{C}$, we may define a product on M.

Definition 2.2. An associative product (In [9], Gerstenhaber and Voronov call it a multiplication.) on an operad $\mathcal{O}$ is an element $S \in \mathcal{O}^2$ such that $S(I,S) = S(S,I)$. An associative product on M is an associative product in the endomorphism operad $\mathcal{O}(M)$. In the sequel, we will constantly use the term product to mean in fact associative product.

Given a product $S \in \mathcal{O}^2$, the associativity of the operad implies that, for any $F \in \mathcal{O}^k$, $G \in \mathcal{O}^l$ and $H \in \mathcal{O}^m$ we have,

$$\begin{aligned} S(F, S(G,H)) &= S(I,S)(F,G,H) \\ &= S(S,I)(F,G,H) \\ &= S(S(F,G),H). \end{aligned} \tag{2.5}$$

This notion is the natural generalization of an associative product on a vector space. Namely, if M is a vector space, $\mathcal{O}^2(M)$ is the set of bilinear maps on M. As in this case $\mathcal{O}^0(M) = \mathrm{Hom}(\mathbb{C}, M) = M$, we have that $S : \mathcal{O}^0(M) \times \mathcal{O}^0(M) \to \mathcal{O}^0(M)$ is an associative product on M.

Product Deformation in a Monoidal Additive Category

Suppose we have a product $S \in \mathcal{O}^2(M)$, where M is an object of a monoidal category $\mathcal{C}$. If the category $\mathcal{C}$ is further additive, we may try to deform S, that is, to find an element $\gamma \in \mathcal{O}^2(M)$ such that $S + \gamma$ is still a product.

At this point, the standard way is to introduce the Hochschild complex of the linear operad $\mathcal{O}(M)$, to define the bilinear Gerstenhaber bracket and the Hochschild differential associated with the product S. A deformation of S would then be a solution of the Maurer-Cartan equation written in the Hochschild differential graded Lie algebra controlling the deformations of S.

We will however rephrase slightly this deformation theory in a way that will allow us to deal with categories whose hom-sets are still linear spaces but with a morphism composition that does not respect this linear structure, as it will be the case in the next sections.

The first step is to notice that a product $S \in \mathcal{O}^2(M)$ generates a suboperad $\mathcal{O}_S(M)$, which we call a *product operad*, in $\mathcal{O}(M)$ with only one point in each degree:

$$\begin{gathered} \mathcal{O}_S^0(M) := \emptyset, \quad \mathcal{O}_S^1(M) := \{I\}, \quad \mathcal{O}_S^2(M) := \{S\}, \\ \mathcal{O}_S^3(M) := \{S(S,I)\}, \quad \mathcal{O}_S^4(M) := \{S(S(S,I),I)\}, \dots, \text{etc.} \end{gathered} \tag{2.6}$$

To simplify the notation we will denote by S_0^n the unique element in $\mathcal{O}_S^n(M)$.

Remark 2.3. The product operad $\mathcal{O}_S(M)$ is a suboperad of $\mathcal{O}(M)$ but not a linear suboperad, namely, for each $n \in \mathbb{N}$, $\mathcal{O}_S^n(M)$ is not a linear subspace of $\mathcal{O}^n(M)$ (it contains only a single point).

Definition 2.4. Let M be an object of an additive monoidal category $\mathcal{C}$ and let $S \in \mathcal{O}^2(M)$ be a product. A *deformation operad*, $\mathcal{O}_{\text{def}}(M,S)$, for S is the data, for each $n \in \mathbb{N}$, of a linear subspace $\mathcal{O}^n_{\text{def}}(M,S) \subset \mathcal{O}^n(M)$ such that the difference

$$R\left(\gamma;\gamma^1,\dots,\gamma^n\right) := \left(S_0^n+\gamma\right)\left(S_0^{k_1}+\gamma^1,\dots,S_0^{k_n}+\gamma^n\right) - S_0^{k_1+\cdots+k_n} \tag{2.7}$$

is in $\mathcal{O}^{k_1+\cdots+k_n}_{\text{def}}(M,S)$ for all $\gamma \in \mathcal{O}^n_{\text{def}}(M,S)$, $\gamma^i \in \mathcal{O}^{k_i}_{\text{def}}(M,S)$, and $i=1,\dots,n$.

Remark 2.5. $\mathcal{O}_S + \mathcal{O}_{\text{def}}$ is a suboperad of $\mathcal{O}(M)$ but not a linear one: the spaces $\mathcal{O}^n_S + \mathcal{O}^n_{\text{def}}(M,S)$ are not linear subspaces but affine ones.

Proposition 2.6. *Let $\mathcal{O}_{\text{def}}(M,S)$ be a deformation operad for a product $S \in \mathcal{O}^2(M)$. Then the compositions*

$$\gamma\left(\gamma^1,\dots,\gamma^n\right) := R\left(\gamma;\gamma^1,\dots,\gamma^n\right), \tag{2.8}$$

defined by (2.7) *gives $\mathcal{O}_{\text{def}}(M,S)$ together with the unit* $0 \in \mathcal{O}^1_{\text{def}}(M,S)$ *the structure of an operad.*

Proof. The proof is direct using only (2.7) and the operad structure of the endomorphism operad $\mathcal{O}(M)$. □

Remark 2.7. Although each of its degrees is a linear subspace, $\mathcal{O}_{\text{def}}(M,S)$ is not a linear operad since its compositions, the Rs, are not multilinear.

Definition 2.8. We say that an element $\gamma \in \mathcal{O}^2_{\text{def}}(M,S)$ is a *deformation of the product S* w.r.t. the deformation operad $\mathcal{O}_{\text{def}}$ if $S+\gamma$ is still a product in $\mathcal{O}_S + \mathcal{O}_{\text{def}}$.

Remark 2.9. All what we have said still applies if we start with any linear operad instead of the endomorphism operad of an object in an additive monoidal category. This allows us to define a notion of product deformations in a specific class of deformations (which is given by the data of the deformation operad) in general linear operads.

Proposition 2.10. *Let $S \in \mathcal{O}^2(M)$ be a product. Take an element $\gamma \in \mathcal{O}^2_{\text{def}}(M,S)$. Then, γ is a deformation of the product S if and only if γ is a product in $\mathcal{O}_{\text{def}}(M,S)$. In particular,* $0 \in \mathcal{O}^2_{\text{def}}(M,S)$ *is always a product in the deformation operad of S.*

Proof. γ is a deformation of S if and only if

$$(S+\gamma)(S+\gamma,I) = (S+\gamma)(I,S+\gamma), \tag{2.9}$$

which is equivalent to

$$S_0^3 + R(\gamma;\gamma,0) = S_0^3 + R(\gamma;0,\gamma). \tag{2.10}$$

□

From now on, we will write 0_1 for the identity element of the deformation operad which is the zero of $\mathcal{O}^1_{\text{def}}$ and 0_2 for the trivial product of the deformation operad which is the 0 element in $\mathcal{O}^2_{\text{def}}(M,S)$.

Notice that neither $\mathcal{O}_S(M)$ nor $\mathcal{O}_S(M)+\mathcal{O}_{\text{def}}(M,S)$ is a linear operad in the sense that, although the compositions are multilinear, the spaces for each degree are not vector spaces but affine spaces. On the other hand, the spaces for each degrees of the deformation operad $\mathcal{O}_{\text{def}}(M,S)$ are vector spaces, but the induced operad compositions are not linear in general.

We may however introduce the Gerstenhaber bracket of the deformation operad

$$[,] : \mathcal{O}^k_{\text{def}}(M,S) \times \mathcal{O}^l_{\text{def}}(M,S) \longrightarrow \mathcal{O}^{k+l-1}_{\text{def}}(M,S) \tag{2.11}$$

defined by

$$[F,G] = F \circ G - (-1)^{(k-1)(l-1)} G \circ F, \tag{2.12}$$

where

$$F \circ G = \sum_{i=1}^{k} (-1)^{(i-l)(l-1)} R\left(F; 0_1, \ldots, 0_1, \underbrace{G}_{i\text{th}}, 0_1, \ldots, 0_1 \right). \tag{2.13}$$

This bracket is not bilinear. An important fact concerning this bracket is that,

$$\frac{1}{2}[\gamma,\gamma] = R(\gamma;\gamma,0_1) - R(\gamma;0_1,\gamma), \tag{2.14}$$

which means that γ is a product in the deformation operad if and only if

$$\frac{1}{2}[\gamma,\gamma] = 0. \tag{2.15}$$

Moreover, we may define an equivalent of the Hochschild differential

$$d : \mathcal{O}^n_{\text{def}}(M,S) \longrightarrow \mathcal{O}^{n+1}_{\text{def}}(M,S), \tag{2.16}$$

$$\begin{aligned} dF &:= [0_2,F] = R(0_2;F,0_1) + (-1)^{n-1} R(0_2;0_1,F) \\ &\quad -(-1)^{n-1} \sum_{i=1}^{n} (-1)^{i-1} R\left(F; 0_1, \ldots, 0_1, \underbrace{0_2}_{i\text{th}}, 0_1, \ldots, 0_1 \right). \end{aligned} \tag{2.17}$$

It turn out that d is still a coboundary operator.

Proposition 2.11. *d defined by (2.17) is a coboundary operator, that is, $d^2 = 0$. Moreover, $\gamma \in \mathcal{O}^2_{\mathrm{def}}(M,S)$ satisfies product equation $(1/2)[\gamma,\gamma] = 0$, in $\mathcal{O}_{\mathrm{def}}(M,S)$ if and only if*

$$d\gamma + \gamma\left(\gamma, S_0^1,\right) - \gamma\left(S_0^1, \gamma\right) = 0. \tag{2.18}$$

Proof. Using (2.7) we obtain d in terms of the endomorphism compositions

$$\begin{aligned} dF &= S_0^2\left(F, S_0^1\right) + (-1)^{n-1} S_0^2\left(S_0^1, F\right) \\ &- (-1)^{n-1}\sum_{i=1}^{n}(-1)^{i-1} F\left(S_0^1, \ldots, \underbrace{S_0^2}_{i\text{th}}, \ldots, S_0^1 \right). \end{aligned} \tag{2.19}$$

The result follows directly from the linearity of the compositions in the endomorphism operad. Using again (2.7) we get

$$\begin{aligned} \frac{1}{2}[\gamma,\gamma] &= R(\gamma;\gamma,0_1) - R(\gamma;0_1,\gamma) = S_0^2\left(\gamma, S_0^1\right) \\ &+ \gamma\left(S_0^2, S_0^1\right) + \gamma\left(\gamma, S_0^1\right) - S_0^2\left(S_0^1, \gamma\right) - \gamma\left(S_0^1, S_0^2\right) - \gamma\left(S_0^1, \gamma\right), \end{aligned} \tag{2.20}$$

which gives (2.18). □

A formal deformation S_ϵ of S is a formal power series

$$S_\epsilon = \epsilon S_1 + \epsilon^2 S_2 + \cdots \in \mathcal{O}^n_{\mathrm{form}}(M,S) := \epsilon \mathcal{O}^n_{\mathrm{def}}(M,S) \otimes k[[\epsilon]], \qquad n \in \mathbb{N}_*, \tag{2.21}$$

where ϵ is a formal parameter and $\mathcal{O}_{\mathrm{def}}(M,S)$ is a deformation operad for S, such that $S + S_\epsilon$ is a product in $\mathcal{O}_S(M) + \mathcal{O}_{\mathrm{form}}(M,S)$.

Equivalently, one may say that S_ϵ must satisfy

$$[S_\epsilon, S_\epsilon] = 0, \tag{2.22}$$

or, thanks to (2.18) that the S_i's satisfy at each order $n \in \mathbb{N}_*$ the following recursive equation:

$$dS_n + H_n(S_{n-1}, \ldots, S_1) = 0, \tag{2.23}$$

where

$$H_n(S_{n-1}, \ldots, S_1) = \sum_{n=i+j} S_i\left(S_j, S_0^1\right) - S_i\left(S_0^1, S_i\right). \tag{2.24}$$

The Kontsevich Product Deformation

Consider the category of real vector spaces. In this category we take the real vector space $M = C^\infty(\mathbb{R}^d)$ of smooth functions on $\mathbb{R}^d$. The endomorphism operad of $C^\infty(\mathbb{R}^d)$ is

$$\mathcal{O}^n(M) = \left\{ n\text{-multilinear maps from } C^\infty\left(\mathbb{R}^{\mathrm{d}}\right)^{\otimes n} \text{ to } C^\infty\left(\mathbb{R}^d\right) \right\}. \tag{2.25}$$

The usual product of functions induces a product in $\mathcal{O}(M)$, namely,

$$S_0^2(F,G)(f_1,\ldots,f_k,g_1,\ldots,g_l) = F(f_1,\ldots,f_k)G(g_1,\ldots,g_l), \tag{2.26}$$

for $F \in \mathcal{O}^k(M)$ and $G \in \mathcal{O}^l(M)$.

The induced product operad is

$$\mathcal{O}_S^n(M) = \{S_0^n\}, \tag{2.27}$$

where

$$S_0^n(f_1,\ldots,f_n) = f_1 f_2 \cdots f_n. \tag{2.28}$$

As deformation operad, we take

$$\mathcal{O}_{\mathrm{def}}^n(M,S) := \left\{ n\text{-multidifferential operators on } C^\infty\left(\mathbb{R}^d\right) \right\}. \tag{2.29}$$

The induced coboundary operator on $\mathcal{O}_{\mathrm{def}}(M,S)$ is the Hochschild coboundary operator,

$$\begin{aligned} dF(f_1,\ldots,f_n) &= F(f_1,\ldots,f_n)f_{n+1} + (-1)^{n-1} f_1 F(f_2,\ldots,f_{n+1}) \\ &\quad - (-1)^{n-1} \sum_{i=1}^{n} (-1)^{(i-1)} F(f_1,\ldots,f_{i-1}, f_i f_{i+1}, f_{i+2},\ldots,f_{n+1}). \end{aligned} \tag{2.30}$$

and the product equation

$$d\gamma + \gamma\left(\gamma, S_0^1,\right) - \gamma\left(S_0^1, \gamma\right) = 0, \tag{2.31}$$

is nothing but the usual Maurer-Cartan equation.

Kontsevich in [1] shows that there exists a formal deformation

$$S \in \mathcal{O}_S^2(M) + \epsilon \mathcal{O}_{\mathrm{def}}^2(M)[[\epsilon]] \tag{2.32}$$

of S_0^2. He provides the explicit formula for this deformation

$$S = S_0^2 + \sum_{n=1}^{\infty} \epsilon^n \sum_{\Gamma \in G_{n,2}} W_\Gamma B_\Gamma, \tag{2.33}$$

where the $G_{n,2}$ are the Kontsevich graphs of type $(n, 2)$, W_Γ is their associated weight, and B_Γ is their associated bidifferential operator (and [1] for more precisions).

2.2. Monoidal Structure of $\mathcal{SYM}$

Let us recall that the extended symplectic "category" $\mathcal{SYM}$ is given by

$$\begin{gathered} \mathrm{Obj} = \{\text{symplectic manifolds}\}, \\ \mathrm{Hom}(M, N) = \left\{ L \subset \overline{M} \times N : L \text{ is Lagrangian} \right\}, \end{gathered} \tag{2.34}$$

where $\overline{M}$ denotes the symplectic manifold M with opposite symplectic structure $-\omega$. The identity morphism of $\mathrm{Hom}(M, M)$ is the diagonal

$$\mathrm{id}_M := \Delta_M = \left\{ (m, m) \subset \overline{M} \times M \right\}. \tag{2.35}$$

The composition of two morphisms $L \in \mathrm{Hom}(M, N)$ and $\widetilde{L} \in \mathrm{Hom}(N, P)$ is given by the composition of canonical relations

$$\widetilde{L} \circ L := \pi_{M \times P}\left(\left(L \times \widetilde{L} \right) \cap (M \times \Delta_N \times P) \right) \subset \overline{M} \times P. \tag{2.36}$$

Everything works fine except the fact that the composition $\widetilde{L} \circ L$ may fail to be a Lagrangian submanifold of $\overline{M} \times P$. It is always the case when $L \times \widetilde{L}$ intersects $M \times \Delta_N \times P$ cleanly (see [10] for more precisions).

Let us pretend for a while that $\mathcal{SYM}$ is a true category or, better, that we have selected special symplectic manifolds and special arrows between them such that the composition is always well-defined.

We define the tensor product between two objects M and N of $\mathcal{SYM}$ as the Cartesian product

$$M \otimes N := M \times N, \tag{2.37}$$

and the tensor product between morphisms as

$$L_1 \otimes L_2 := \{(m, a, n, b) : (m, n) \in L_1, \quad (a, b) \in L_2\} \in \mathrm{Hom}(M \otimes A, N \otimes B), \tag{2.38}$$

for $L_1 \in \mathrm{Hom}(M, N)$ and $L_2 \in \mathrm{Hom}(A, B)$.

The neutral object is $\{*\}$, the one-point symplectic manifold. The following proposition tells us that $\mathcal{SYM}$ would be a monoidal category if it were a true category.

Proposition 2.12. *The following statements hold.*

(1) *Consider* $L_1 \in Hom(M, A)$, $L_2 \in Hom(N, B)$, $L_3 \in Hom(A, X)$ *and* $L_4 \in Hom(B, Y)$. *Then one has the following equality of sets:*

$$(L_3 \otimes L_4) \circ (L_1 \otimes L_2) = (L_3 \circ L_1) \otimes (L_4 \circ L_2). \tag{2.39}$$

(2) $id_M \otimes id_N = id_{M\otimes N}$ *for any object* M *and* N.

(3) $(M \otimes A) \otimes X = M \otimes (A \otimes X)$ *for any objects* M, A *and* X.

(4) $(L_1 \otimes L_2) \otimes L_3 = L_1 \otimes (L_2 \otimes L_3)$ *for any arrows* $L_1 \in Hom(M, A)$, $L_2 \in Hom(N, B)$ *and* $L_3 \in Hom(P, C)$.

(5) $\{*\} \otimes A \simeq A \simeq A \otimes \{*\}$ *for all object* A *and* $id_{\{*\}} \otimes L \simeq L \simeq L \otimes id_{\{*\}}$ *for all arrows* L, *where* $A \simeq B$ *means that the two sets* A *and* B *are in bijection.*

Proof. (1)

$$\begin{aligned}
I &= (L_3 \otimes L_4) \circ (L_1 \otimes L_2) \\
&= \pi\Big(((L_1 \otimes L_2) \times (L_3 \otimes L_4)) \cap \Big(\overline{N \times M} \times \Delta_{A\times B} \times X \times Y\Big)\Big) \\
&= \big\{\big(m, n, \tilde{x}, \tilde{y}\big) : \exists (a, b) \in A \times B \text{ s.t. } (m, n, a, b) \in L_1 \otimes L_2, \quad (a, b, x, y) \in L_3 \otimes L_4\big\} \\
&= \big\{\big(m, n, \tilde{x}, \tilde{y}\big) : \exists a \in A,\ (m, a) \in L_1, \quad (a, x) \in L_3 \exists b \in B,\ (n, b) \in L_2 \quad (b, y) \in L_4\big\} \\
&= (L_3 \circ L_1) \otimes (L_4 \circ L_2)
\end{aligned} \tag{2.40}$$

(2) $\Delta_M \otimes \Delta_N = \{(m, n, m, n) : m \in M \text{ and } n \in N\} = \Delta_{M\otimes N}$.

(3) The associativity between objects is trivial.

(4) For morphisms, we have,

$$\begin{aligned}
L_1 \otimes L_2 &= \{(m, n, a, b) : (m, a) \in L_1, \quad (n, b) \in L_2\}, \\
(L_1 \otimes L_2) \otimes L_3 &= \big\{\big(m, n, p, a, b, c\big) : (m, a) \in L_1, (n, b) \in L_2, \big(p, c\big) \in L_3\big\}, \\
L_2 \otimes L_3 &= \big\{\big(n, p, b, c\big) : (n, b) \in L_2, \quad \big(p, c\big) \in L_3\big\}, \\
L_1 \otimes (L_2 \otimes L_3) &= \big\{\big(m, n, p, a, b, c\big) : (m, a) \in L_1, (n, b) \in L_2, \big(p, c\big) \in L_3\big\}.
\end{aligned} \tag{2.41}$$

(5) is trivial. □

2.3. Lagrangian Operads

If $\mathcal{SYM}$ were a true category, we could consider the endomorphism operad of a symplectic manifold M. However, we may be able to restrict to a subset of Lagrangian submanifolds $\mathcal{O}^n_{\text{rest}}(M) \subset \mathcal{O}^n(M)$ for each $n \geq 0$ such that the composition

$$L_n(L_{k_1},\ldots,L_{k_n}) := L_n \circ (L_{k_1} \otimes \cdots \otimes L_{k_n}), \tag{2.42}$$

yields always a Lagrangian submanifold in $\mathcal{O}^{k_1+\cdots+k_n}_{\text{rest}}(M)$ for every $L_n \in \mathcal{O}^n_{\text{rest}}(M)$ and $L_{k_i} \in \mathcal{O}^{k_i}_{\text{rest}}(M)$, $i = 1,\ldots,n$. For instance, there is always the trivial choice

$$\mathcal{O}^1_{\text{rest}}(M) = \{\Delta_M\}, \quad \mathcal{O}^n_{\text{rest}}(M) = \emptyset, \quad n \neq 1. \tag{2.43}$$

In this way, we may get a true operad $\mathcal{O}_{\text{rest}}(M)$.

The next natural question to ask is the following.

Question. What is a product in a Lagrangian operad over M?

As a first hint, take the situation where the symplectic manifold is a symplectic groupoid G. In this case, we may generate an operad from the multiplication space $G^m \in \mathcal{O}^2(G)$ and the base $G^{(0)} \in \mathcal{O}^0(G)$, the identity being the diagonal $\Delta_G \in \mathcal{O}^1(G)$. Remark that G^m is a product in this operad, that is, that $G^m(G^m,\Delta_G) = G^m(\Delta_G, G^m)$. Notice that the inverse of the symplectic groupoid does not play any role in this construction.

We will answer this question completely for the case were the symplectic manifold is $T^*\mathbb{R}^d$ and will try to develop a deformation theory for the product in this case.

Local Cotangent Lagrangian Operads

Remember that $T^*\mathbb{R}^d$ has always a structure of a symplectic groupoid over $\mathbb{R}^d$: the trivial one. The multiplication space is given in this case by

$$\Delta_2 = \left\{(p_1,x),(p_2,x),(p_1+p_2,x) : p_1,p_2 \in \mathbb{R}^{d*},\ x \in \mathbb{R}^d\right\}. \tag{2.44}$$

The base is

$$\Delta_0 = \left\{(0,x) : x \in \mathbb{R}^d\right\}. \tag{2.45}$$

If we set further

$$\Delta_n := \left\{(p_1,x),\ldots,(p_n,x),(p_1+\ldots+p_n,x) : p_i \in \mathbb{R}^{d*}, x \in \mathbb{R}^d\right\}, \tag{2.46}$$

it immediate to see that the operad generated by Δ_0 and Δ_2 is exactly

$$\mathcal{O}^n_\Delta\left(T^*\mathbb{R}^d\right) = \{\Delta_n\}, \tag{2.47}$$

and that Δ_2 is a product in it.

Following [3], we will call this operad the *cotangent Lagrangian operad* over $T^*\mathbb{R}^d$. It is the exact analog of the product operad in a monoidal category, the only difference is that there is no true endomorphism operad to embed $\mathcal{O}_\Delta(T^*\mathbb{R}^d)$ into. The idea now is to enlarge the cotangent Lagrangian operad, that is, by considering Lagrangian submanifolds close enough to Δ_n for each $n \in \mathbb{N}$ in order to have still an operad.

Notice at this point that the Δ_n's are given by generating functions. Namely, we may identify $\overline{(T^*\mathbb{R}^d)^n} \times T^*\mathbb{R}^d$ with T^*B_n, where $B_n := (\mathbb{R}^{d*})^n \times \mathbb{R}^d$. Then,

$$\Delta_n = \left\{ \left(\left(p_1, \frac{\partial S_0^n}{\partial p_1}(z) \right), \dots, \left(p_n, \frac{\partial S_0^n}{\partial p_n}(z) \right), \left(\frac{\partial S_0^n}{\partial x}(z), x \right) \right) : z = (p_1, \dots, p_n, x) \in B_n \right\}, \tag{2.48}$$

where S_0^n is the function on B_n defined by (In the sequels, we will use the shorter notation $(p_1 + \cdots + p_n)x$ instead of $\sum_{i=1}^d (p_1^i + \cdots + p_n^i)x_i$.)

$$S_0^n(p_1, \dots, p_n, x) = \sum_{i=1}^d \left(p_1^i + \cdots + p_n^i \right) x_i. \tag{2.49}$$

The cotangent Lagrangian operad may then be identified with

$$\mathcal{O}_\Delta^n = \left\{ S_0^n \right\}, \qquad \mathcal{O}_\Delta^0 = \{0\}. \tag{2.50}$$

In order to define a deformation operad for S, a natural idea would be to consider Lagrangian submanifolds whose generating functions are of the form

$$F = S_0^n + \widetilde{F}, \tag{2.51}$$

where $\widetilde{F} \in C^\infty(B_n)$. The Lagrangian submanifold associated to F is

$$L_F := \text{graph } dF. \tag{2.52}$$

As such, the idea does not work in general. In fact, we have to consider generating functions only defined in some neighborhood. Let us be more precise.

We introduce the following notation:

$$B_n^0 = \{0\} \times \mathbb{R}^d \subset B_n, \tag{2.53}$$

$V(B_n^0)$ will stand for the set of all neighborhoods of B_n^0 in B_n.

Definition 2.13. We define $\mathcal{O}_{\text{loc}}^n(T^*\mathbb{R}^d)$ to be the space of germs at B_n^0 of smooth functions $\widetilde{F}$ (defined on an open neighborhood $U_{\widetilde{F}} \subset B_n$ of B_n^0) which satisfy $\widetilde{F}(0, x) = 0$ and $\nabla_p \widetilde{F}(0, x) = 0$. Note that the composition will always be understood in terms of composition of germs.

Proposition 2.14. *Let $F \in \mathcal{O}_{\Delta}^{n} + \mathcal{O}_{\mathrm{loc}}^{n}$ and $G_i \in \mathcal{O}_{\Delta}^{k_i} + \mathcal{O}_{\mathrm{loc}}^{k_i}$ for $i = 1, \ldots, n$. Consider the function ϕ defined by the formula*

$$\phi(p_G, x_F) = G_1 \cup \cdots \cup G_n(p_G, x_G) + F(p_F, x_F) - x_G p_F, \tag{2.54}$$

$$\begin{gathered} p_F = \nabla_x G_1 \cup \ldots \cup G_n(p_G, x_G), \\ x_G = \nabla_p F(p_F, x_F), \end{gathered} \tag{2.55}$$

where

$$G_1 \cup \cdots \cup G_n(p_G, x_G) := G_1(p_{G_1}, x_{G_1}) + \cdots + G_n(p_{G_n}, x_{G_n}) \tag{2.56}$$

and $p_G = (p_{G_1}, \ldots, p_{G_n})$, $p_{G_i} \in (\mathbb{R}^{d*})^{k_i}$, $x_{G_i} \in \mathbb{R}^d$ and $(p_{G_i}, x_{G_i}) \in U_{G_i}$, for $i = 1, \ldots, n$.

Then,

$$\phi \in \mathcal{O}_{\Delta}^{k_1+\cdots+k_n} + \mathcal{O}_{\mathrm{loc}}^{k_1+\cdots+k_n}, \quad \text{and} \quad L_\phi = L_F(L_{G_1}, \ldots, L_{G_n}). \tag{2.57}$$

In other words, $\mathcal{O}_{\Delta} + \mathcal{O}_{\mathrm{loc}}$ together with the product

$$\phi = F(G_1, \ldots, G_n) \tag{2.58}$$

is an operad.

Moreover, the induced operad structure on $\mathcal{O}_{\mathrm{loc}}$ is given by

$$R\left(\widetilde{F}; \widetilde{G}_1, \ldots, \widetilde{G}_n\right) = H, \tag{2.59}$$

where H is the function $H \in \mathcal{O}_{\mathrm{loc}}^{k_1+\cdots+k_n}$ defined by

$$\begin{gathered} H(p_G, x_F) = \widetilde{G}(p_G, x_G) + \widetilde{F}(p_F, x_F) - \nabla_p \widetilde{F}(p_F, x_F) \nabla_x \widetilde{G}(p_G, x_G), \\ p_F = p_F^0 + \nabla_x \widetilde{G}(p_G, x_G), \quad p_F^0 := \left(p_{G_1}^{\Sigma}, \ldots, p_{G_n}^{\Sigma}\right), \\ x_G = x_G^0 + \nabla_p \widetilde{F}(p_F, x_F), \quad x_G^0 := (x_F, \ldots, x_F). \end{gathered} \tag{2.60}$$

Remark 2.15 (Saddle point formula). Formula (2.54) for Φ can be interpreted in terms of saddle point evaluation for $\hbar \to 0$ of the following integral:

$$\begin{aligned} &\int \mathrm{e}^{(i/\hbar)[F(p^1,\ldots,p^k,x)+\sum_{i=1}^{k}(G_i(\pi^{i1},\ldots,\pi^{il_i},y_i)-p^i\cdot y_i)]} \prod_{i=1}^{k} \frac{\mathrm{d}^n p^i \mathrm{d}^n y_i}{(2\pi\hbar)^n} \\ &\quad = \mathrm{e}^{(\mathrm{i}/\hbar)\Phi(\pi^{11},\ldots,\pi^{1l_1},\pi^{21},\ldots,\pi^{2l_2}\ldots\pi^{k1},\ldots,\pi^{kl_k},x)} (C + O(\hbar)), \end{aligned} \tag{2.61}$$

where C is some constant.

Proof of Proposition 2.14. To simplify the computations, we identify $(T^*\mathbb{R}^d)^n$ with $T^*(\mathbb{R}^{dn})$ and $(T^*\mathbb{R}^d)^{k_i}$ with $T^*(\mathbb{R}^{dk_i})$. With this identifications the graphs of F and G_i, $i = 1,\dots,n$ may be written as

$$L_F = \{((p_F, \nabla_p F(p_F, x_F)), (\nabla_x F(p_F, x_F), x_F)) : (p_F, x_F) \in U_F\} \subset T^*\left(\mathbb{R}^{dn}\right) \times T^*\mathbb{R}^d,$$

$$L_{G_i} = \{((p_{G_i}, \nabla_p G_i(p_{G_i}, x_{G_i})), (\nabla_x G_i(p_{G_i}, x_{G_i}), x_{G_i})) : (p_{G_i}, x_{G_i}) \in U_{G_i}\} \subset T^*\left(\mathbb{R}^{dk_i}\right) \times T^*\mathbb{R}^d, \tag{2.62}$$

where $U_F \in V(B_n^0)$ and $U_{G_i} \in V(B_{k_i}^0)$ for $i = 1,\dots,n$.

Consider now the composition,

$$L_F(L_{G_1},\dots,L_{G_n}) = L_F \circ (L_{G_1} \otimes \cdots \otimes L_{G_n}). \tag{2.63}$$

First of all, observe that,

$$\begin{aligned} L_G &:= L_{G_1} \otimes \cdots \otimes L_{G_n} \\ &= \{((p_G, \nabla_p G(p_G, x_G)), (\nabla_x G(p_G, x_G), x_G)) : (p_{G_i}, x_{G_i}) \in U_{G_i}\} \\ L_G &\subset T^*\left(\mathbb{R}^{d(k_1+\cdots+k_n)}\right) \times T^*\left(\mathbb{R}^{dn}\right). \end{aligned} \tag{2.64}$$

Thus,

$$\begin{aligned} L_F \circ L_G &= \pi\Big((L_G \times L_F) \cap \Big(T^*\mathbb{R}^{d(k_1+\cdots+k_n)} \times \Delta_{T^*\mathbb{R}^{dn}} \times T^*\mathbb{R}^d\Big)\Big) \\ &= \Big\{((p_G, \nabla_p G(p_G, x_G)), (\nabla_x F(p_F, x_F), x_F)) : x_G = \nabla_p F(p_F, x_F), \\ &\qquad p_F = \nabla_x G(p_G, x_G),\ (p_G, x_F) \in \widetilde{U}\Big\} \\ L_F \circ L_G &\subset T^*\left(\mathbb{R}^{d(k_1+\cdots+k_n)}\right) \times T^*\mathbb{R}^d, \end{aligned} \tag{2.65}$$

where $\widetilde{U}$ is the subset of $(p_G, x_F) \in B_{k_1+\cdots+k_n}$ such that the system,

$$\begin{aligned} p_F &= \nabla_x G(p_G, x_G), \\ x_G &= \nabla_p F(p_F, x_F), \end{aligned} \tag{2.66}$$

has a unique solution (p_F, x_G) and such that $(p_{G_i}, x_{G_i}) \in U_{G_i}$, $i = 1,\dots,n$, and $(p_F, x_F) \in U_F$. Let us check that $\widetilde{U}$ always exists and is a neighborhood of $B_{k_1+\cdots+k_n}^0$. To begin with, observe that for any $(0, x_F) \in B_n^0$ this system has the unique solution $(0, \nabla_p F(0, x_F))$. Set now,

$$H(p_G, x_F, p_F, x_G) = \begin{pmatrix} pF - \nabla_x G(pG, xG) \\ xF - \nabla_p F(pF, xF) \end{pmatrix}. \tag{2.67}$$

Thanks to the fact that $G(0,x) = \sum_{i=1}^{n} G_i(0,x) = 0$ we get that the Jacobi matrix

$$D_{p_F,x_G} H(0,x_f,0,\nabla_p F(0,x_F)) = \begin{pmatrix} \mathrm{id} & 0 \\ -\nabla_p\nabla_p F(0,x_F) & \mathrm{id} \end{pmatrix} \tag{2.68}$$

is invertible.

Thus, the implicit function theorem gives us the desired neighborhood $\widetilde{U}$ of $B^0_{k_1+\cdots+k_n}$.

Now, take ϕ as defined in (2.54). The previous considerations tell us that ϕ is exactly defined on $\widetilde{U}$. Let us compute its graphs,

$$L_\phi = \left\{ \left((p_G, \nabla_p \Phi(p_G,x_F)), (\nabla_x \Phi(p_G,x_F), x_F) : (p_G,x_F) \in \widetilde{U} \right\}. \tag{2.69}$$

We have that

$$\begin{aligned} \nabla_p \phi(p_G,x_F) &= \nabla_p G(p_G,x_G) + \nabla_x G(p_G,x_G)\frac{dx_G}{dp} + \nabla_p F(p_F,x_F)\frac{dp_F}{dp} - p_F\frac{dx_G}{dp} - \frac{dp_F}{dp}x_G \\ &= \nabla_p G(p_G,x_G). \end{aligned} \tag{2.70}$$

Similarly, $\nabla_x \phi(p_G,x_F) = \nabla_x F(p_F,x_F)$. Thus, $L_\phi = L_F \circ L_G$.

At last, let us check that $\phi \in \mathcal{O}^{k_1+\cdots+k_n}_{\mathrm{loc}}$. First of all, remember that

$$\begin{aligned} F(p_F,x_F) &= p_F^{\Sigma} x_F + \widetilde{F}(p_F,x_F), \\ G(p_G,x_F) &= \sum_{i=1}^{n} p^{\Sigma}_{G_i} x_{G_i} + \widetilde{G}(p_G,x_G). \end{aligned} \tag{2.71}$$

Thus, we obtain immediately that

$$\phi(p_G,x_F) = p_G^{\Sigma} x_F + H(p_G,x_F), \tag{2.72}$$

where H is a function only defined on $\widetilde{U}$ by the equations

$$\begin{aligned} H(p_G,x_F) &= \widetilde{G}(p_G,x_G) + \widetilde{F}(p_F,x_F) - \nabla_p \widetilde{F}(p_F,x_F)\nabla_x \widetilde{G}(p_G,x_G), \\ p_F &= p_F^0 + \nabla_x \widetilde{G}(p_G,x_G), \quad p_F^0 := \left(p^{\Sigma}_{G_1},\ldots,p^{\Sigma}_{G_n}\right), \\ x_G &= x_G^0 + \nabla_p \widetilde{F}(p_F,x_F), \quad x_G^0 := (x_F,\ldots,x_F). \end{aligned} \tag{2.73}$$

But now, if we set $p_G = 0$ then $p_F = 0$, $x_G = x_G^0 + \nabla_p \widetilde{F}(0,x_F)$ and $H(0,x_F) = 0$. Similarly, one easily checks that $\nabla_p H(0,x_F) = 0$ □

We will call the operad $\mathcal{O}_\Delta + \mathcal{O}_{\text{loc}}$ *local cotangent Lagrangian operad* over $T^*\mathbb{R}^d$ or for short the local Lagrangian operad when no ambiguities arise. The induced operad $\mathcal{O}_{\text{loc}}$ will be called the *local deformation operad of* $\mathcal{O}_\Delta$.

Associative Products in the Local Deformation Operad

We say that a generating function $S \in C^\infty(B_2)$ satisfies the *Symplectic Groupoid Associativity equation* if for a point $(p_1, p_2, p_3, x) \in B_3$ sufficiently close to B_3^0 the following implicit system for $\overline{x}, \overline{p}, \widetilde{x}$ and $\widetilde{p}$:

$$\begin{aligned} \overline{x} &= \nabla_{p_1} S(\overline{p}, p_3, x), \quad \overline{p} = \nabla_x S(p_1, p_2, \overline{x}), \\ \widetilde{x} &= \nabla_{p_2} S(p_1, \widetilde{p}, x), \quad \widetilde{p} = \nabla_x S(p_2, p_3, \widetilde{x}) \end{aligned} \tag{2.74}$$

has a unique solution and if the following additional equation holds:

$$S(p_1, p_2, \overline{x}) + S(\overline{p}, p_3, x) - \overline{x}\,\overline{p} = S(p_2, p_3, \widetilde{x}) + S(p_1, \widetilde{p}, x) - \widetilde{x}\widetilde{p}. \tag{2.75}$$

If S also satisfies the *Symplectic Groupoid Structure conditions*, that is, if

$$S(p, 0, x) = S(0, p, x) = px, \qquad S(p, -p, x) = 0 \tag{2.76}$$

then S generates a Poisson structure

$$\alpha(x) = 2\left(\nabla_{p_k^1} \nabla_{p_l^2} S(0, 0, x)\right)_{k,l=1}^d \tag{2.77}$$

on $\mathbb{R}^d$ together with a local symplectic groupoid integrating it, whose structure maps are given by

$$\begin{aligned} \epsilon(x) &= (0, x) \quad \text{unit map}, \\ i(p, x) &= (-p, x) \quad \text{inverse map}, \\ s(p, x) &= \nabla_{p_2} S(p, 0, x) \quad \text{source map}, \\ t(p, x) &= \nabla_{p_1} S(0, p, x) \quad \text{target map}. \end{aligned} \tag{2.78}$$

In this case, we call S a generating function of the Poisson structure α or a generating function of the local symplectic groupoid. See [2, 4, 10] for proofs and explanations about generating functions of Poisson structures.

The following proposition explains what is a product in the local cotangent Lagrangian operad.

Proposition 2.16. *$\widetilde{S} \in \mathcal{O}^2_{\text{loc}}$ is a product in $\mathcal{O}_{\text{loc}}$ if and only if $S = S_0^2 + \widetilde{S}$ satisfies the Symplectic Groupoid Associativity equation.*

Proof. We know that $\widetilde{S}$ is a product in $\mathcal{O}_{\text{loc}}$ if and only if $S = S_0^2 + \widetilde{S}$ is a product in $\mathcal{O}_\Delta + \mathcal{O}_{\text{loc}}$, that is, if and only if $S(S, I) = S(I, S)$. Let us compute.

$$\begin{aligned} S(S,I)(p_1,p_2,p_3,x) &= S\cup I(p_1,p_2,p_3,\overline{x}_1,\overline{x}_2) + S(\overline{p}_1,\overline{p}_2,x) - \overline{x}_1\overline{p}_1 - \overline{p}_2\overline{x}_2 \\ &= S(p_1,p_2,\overline{x}_1) + p_3\overline{x}_2 + S(\overline{p}_1,\overline{p}_2,x) - \overline{p}_1\overline{x}_1 - \overline{p}_2\overline{x}_2 \end{aligned} \tag{2.79}$$

with

$$\begin{aligned} \overline{p}_1 &= \nabla_{x_1} S\cup I(p_1,p_2,p_3,\overline{x}_1,\overline{x}_2) = \nabla_x G(\overline{x}), \\ \overline{p}_2 &= \nabla_{x_2} S\cup I(p_1,p_2,p_3,\overline{x}_1,\overline{x}_2) = p_3, \\ \overline{x}_1 &= \nabla_{p_1} S(\overline{p}_1,\overline{p}_2,x), \\ \overline{x}_2 &= \nabla_{p_2} S(\overline{p}_1,\overline{p}_2,x). \end{aligned} \tag{2.80}$$

Then we get

$$\begin{aligned} S(S,I) &= S(p_1,p_2,\overline{x}) + S(\overline{p},p_3,x) - \overline{px}, \\ \overline{x} &= \nabla_{p_1} S(\overline{p},p_3,x), \\ \overline{p} &= \nabla_x S(p_1,p_2,\overline{x}). \end{aligned} \tag{2.81}$$

Similarly, we get

$$\begin{aligned} S(I,S) &= S(p_2,p_3,\widetilde{x}) + S(p_1,\widetilde{p},x) - \widetilde{p}\widetilde{x}, \\ \widetilde{x} &= \nabla_{p_2} S(p_1,\widetilde{p},x) \\ \widetilde{p} &= \nabla_x S(p_2,p_3,\widetilde{x}). \end{aligned} \tag{2.82}$$

Hence, $\widetilde{S} \in \mathcal{O}^2_{\text{loc}}(T^*\mathbb{R}^d)$ is a product if and only if $S_0^2 + \widetilde{S}$ satisfies the SGA equation. □

At this point, we may still introduce the Gerstenhaber bracket as in (2.12) and the product equation in terms of the bracket would still be $(1/2)[\widetilde{S}, \widetilde{S}] = 0$. We may also still write a formula for the coboundary operator. But, as this time the compositions in $\mathcal{O}_\Delta + \mathcal{O}_{\text{loc}}$ are not multilinear, we cannot develop the expression $(1/2)[\widetilde{S}, \widetilde{S}]$ in terms of the coboundary operator. Nevertheless, in Section 4, we will develop the bracket with help of Taylor's expansion and recover a form very close to (2.23) in the additive category case.

Equivalence of Associative Products

To each $F \in \mathcal{O}^1_\Delta + \mathcal{O}^1_{\text{loc}}$, we may associate a symplectomorphism ψ_F which is defined only on a neighborhood U_F of B^0_1 in $T^*\mathbb{R}^d$ and which fixes B^1_0. The composition of two such ψ_G and ψ_F, which may always be defined on a possibly smaller neighborhood $\widetilde{U} \subset U_G$ of B^0_1, is exactly $\psi_{F(G)}$ where $F(G)$ is the composition of F by G in the local Lagrangian operad.

We denote by $F^{-1} \in \mathcal{O}^1_\Delta + \mathcal{O}^1_{\text{loc}}$ the generating function of the $(\psi_F)^{-1}$, that is, the generating function such that $F(F^{-1}) = F^{-1}(F) = I$. Two associative products S and $\widetilde{S}$ will be called equivalent if

$$\widetilde{S} = F(S)\left(F^{-1}, F^{-1}\right) \tag{2.83}$$

for a certain $F \in \mathcal{O}^1_\Delta + \mathcal{O}^1_{\text{loc}}$. It is clear that if $S \in \mathcal{O}^1_\Delta + \mathcal{O}^1_{\text{loc}}$ is an associative product, then $\widetilde{S}$ also is. The following questions naturally arises.

Questions

If S generates a local symplectic groupoid, does $\widetilde{S}$ also generate one? Are these two local groupoids isomorphic?

In fact, two equivalent associative products, which are also generating functions of local symplectic groupoids, induce isomorphic local symplectic groupoids. The isomorphism is given explicitly by ψ_F. As a consequence the induced Poisson structures on the base are the same, that is,

$$\alpha(x) = \nabla_{p_1}\nabla_{p_2} S(0,0,x) = \nabla_{p_1}\nabla_{p_2}\widetilde{S}(0,0,x). \tag{2.84}$$

The following two Propositions prove these statements.

Proposition 2.17. *Let $F \in \mathcal{O}^1_\Delta + \mathcal{O}^1_{\text{loc}}$. The following implicit equations:*

$$\begin{aligned} x_1 &= \nabla_p F(p_1, x_2), \\ p_2 &= \nabla_x F(p_1, x_2), \end{aligned} \tag{2.85}$$

define a symplectomorphism $\psi_F(p_1, x_1) = (p_2, x_2)$ on a neighborhood U_F of $B^0_1 = \{(0,x) : x \in \mathbb{R}^d\}$ in $T^*\mathbb{R}^d$ which fixes B^0_1 and which is close to the identity in the sense that $F(p,x) = px + \widetilde{F}(p,x)$ induces the identity if $\widetilde{F} = 0$. Consider now ψ_F and ψ_G defined, respectively, on U_F and U_G for $F, G \in \mathcal{O}^1_\Delta + \mathcal{O}^1_{\text{loc}}$. Then one has that $\psi_G \circ \psi_F = \psi_{F(G)}$ on $U_{F(G)}$.

Proof. (1) Let us check that the system (2.85) generates a diffeomorphism around B^0_1. Namely, one verifies that $(\overline{p}_1, \overline{x}_1, \overline{p}_2, \overline{x}_2) := (0, \nabla_p F(0, x_2), 0, x_2)$ is a solution of the system. Set now

$$H(p_1, x_1, p_2, x_2) := \begin{pmatrix} x_1 - \nabla_p F(p_1, x_2) \\ p_2 - \nabla_x F(p_1, x_2) \end{pmatrix}. \tag{2.86}$$

As

$$D_{p_1,x_1}H(\overline{p}_1,\overline{x}_2,\overline{p}_2,\overline{x}_2)=\begin{pmatrix}-\nabla_p\nabla_pF(0,\overline{x}_2) & \mathrm{id}\\ \nabla_x\nabla_pF(0,\overline{x}_2) & 0\end{pmatrix},$$
$$D_{p_2,x_2}H(\overline{p}_1,\overline{x}_2,\overline{p}_2,\overline{x}_2)=\begin{pmatrix}0 & \nabla_x\nabla_pF(0,\overline{x}_2)\\ \mathrm{id} & 0\end{pmatrix}, \tag{2.87}$$

the implicit function theorem gives us the result. Let us call $\widetilde{U}$ the neighborhood of B_1^0 where ψ_F is defined.

(2) We check now that ψ_F is symplectic. From (2.85) we get the relation

$$\frac{\partial p_l^2}{\partial p_k^1}=\frac{\partial x_k^1}{\partial x_l^2}, \tag{2.88}$$

which directly implies that $d\psi_F J(d\psi_F)^*=J$ where

$$J=\begin{pmatrix}0 & \mathrm{id}\\ -\mathrm{id} & 0\end{pmatrix}. \tag{2.89}$$

(3) Let us see that $\psi_F(0,x)=(0,x)$. We have already noticed that $(0,\nabla_pF(0,x_2),0,x_2)$ is a solution of the system (2.85). But $F(p,x)=px+\widetilde{F}(p,x)$ with $\nabla_p\widetilde{F}(0,p)=0$ and then $\nabla_x\nabla_pF(0,x_2)=x_2$.

(4) Clearly $F(p,x)=px$ generates the identity.

(5) Recall that

$$L_G=\{(p_1,\nabla_pG(p_1,x_2),\nabla_xG(p_1,x_2),x_2):(p_1,x_2)\in U_G\},$$
$$L_F=\{(p_2,\nabla_pF(p_2,x_3),\nabla_xF(p_2,x_3),x_3):(p_2,x_3)\in U_F\}. \tag{2.90}$$

Thus, $L_G=\operatorname{graph}\psi_G$ and $L_F=\operatorname{graph}\psi_F$. The composition of these two canonical relations yields that $L_F\circ L_G=\operatorname{graph}\psi_F\circ\psi_G$. On the other hand, $L_F\circ L_G=L_{F(G)}=\operatorname{graph}\psi_{F(G)}$. Taking care on the domain of definitions, we have that $\psi_F\circ\psi_G=\psi_{F(G)}$ on $U_{F(G)}$. □

Proposition 2.18. *Let $S\in\mathcal{O}_\Delta^2+\mathcal{O}_{\mathrm{loc}}^2$ be a generating function of a symplectic groupoid, that is,*

$$S(S,I)=S(I,S),\qquad S(p,0,x)=S(0,p,x)=px,\qquad S(p,-p,x)=0. \tag{2.91}$$

Let $F\in\mathcal{O}_\Delta^1+\mathcal{O}_{\mathrm{loc}}^1$ such that $F(-p,x)=-F(p,x)$. Then,

$$\widetilde{S}:=(F(S))\left(F^{-1},F^{-1}\right) \tag{2.92}$$

is also a generating function of a symplectic groupoid. The subset of odd function in p forms a subgroup of $\mathcal{O}_\Delta^1+\mathcal{O}_{\mathrm{loc}}^1$. Moreover, ψ_F is a groupoid isomorphism between the local symplectic groupoid generated *by S and the one generated by $\widetilde{S}$. As a consequence S and $\widetilde{S}$ induce the same Poisson structure on the base.*

Proof. To simplify the notation, we set $G = F^{-1}$. A straightforward computation gives that

$$\begin{aligned} F(S)(G,G)(p_1,p_2,x) &= S(\overline{p},\widetilde{p},\dot{x}) + F(\dot{p},x) + G(p_1,\overline{x}) + G(p_2,\widetilde{x}) - \overline{p}\,\overline{x} - \widetilde{p}\widetilde{x} - \dot{x}\dot{p}, \\ \dot{x} = \nabla_p F(\dot{p},x), \qquad & \overline{x} = \nabla_{p_1} S(\overline{p},\widetilde{p},\dot{x}), \qquad \widetilde{x} = \nabla_{p_2} S(\overline{p},\widetilde{x},\dot{x}), \\ \dot{p} = \nabla_x S(\overline{p},\widetilde{p},\dot{x}), \qquad & \overline{p} = \nabla_x G(p_1,\overline{x}), \qquad \widetilde{p} = \nabla_x G(p_2,\widetilde{x}). \end{aligned} \tag{2.93}$$

(1) Setting $p_1 = p$ and $p_2 = 0$, we have immediately

$$F(S)(G,G)(p,0,x) = G(p,\dot{x}) + F(\dot{p},x) - \dot{x}\dot{p} \tag{2.94}$$

with $\dot{x} = \nabla_p F(\dot{p},x)$ and $\dot{p} = \nabla_x G(p,\dot{x})$. We recognize then that

$$F(S)(G,G)(p,0,x) = F(G)(p,x) = I(p,x) = px. \tag{2.95}$$

The case $p_1 = 0$ and $p_2 = p$ is analog.

(2) One reads directly from the equation

$$px = F^{-1}(p,\dot{x}) + F(\dot{p},x) - \dot{x}\dot{p}, \tag{2.96}$$

where $\dot{x} = \nabla_p F(\dot{p},x)$ and $\dot{p} = \nabla_x F^{-1}(p,\dot{x})$, that if F is odd in p then is also F^{-1} and reciprocally. Similarly, we check directly from the composition formula that $F(G)$ is odd in p if F and G both are. Thus, the odd functions form a subgroup of $\mathcal{O}^1_\Delta + \mathcal{O}^1_{\text{loc}}$.

(3) Suppose now that $p_1 = p$ and $p_2 = -p$. G odd in p implies that $\overline{p} = -\widetilde{p}$. As $S(p,-p,0) = 0$, we get immediately that $\widetilde{x} = \overline{x}$ and $\dot{p} = 0$ which in turns implies that $\dot{x} = x$. Putting everything together, we get that $(F(S))(G,G)(p,-p,x) = 0$

(4) Let us prove now that φ_F is also a groupoid isomorphism. Consider the multiplication space of the symplectic groupoid generated by an generating function S, that is,

$$G^{(m)}(S) = \left\{ (p_1,\nabla_{p_1}S),(p_2,\nabla_{p_2}S),(\nabla_x S,x) : \; p_1,p_2 \in \left(\mathbb{R}^d\right)^*, \quad x \in \mathbb{R}^d \right\}, \tag{2.97}$$

where the partial derivative are evaluated in (p_1,p_2,x).

We have to show that $(\varphi_F \times \varphi_F \times \varphi_F)(G^{(m)}(S)) = G^{(m)}(\widetilde{S})$.

A straightforward computation gives that

$$\begin{aligned} \nabla_{p_1}\widetilde{S}(p_1,p_2,x) &= \nabla_p G(p_1,\overline{x}), \\ \nabla_{p_2}\widetilde{S}(p_1,p_2,x) &= \nabla_p G(p_2,\widetilde{x}), \\ \nabla_x \widetilde{S}(p_1,p_2,x) &= \nabla_x F(\dot{p},x). \end{aligned} \tag{2.98}$$

From this, we check immediately that

$$\psi_G\Big(\big(p_1, \nabla_{p_1}\widetilde{S}(p_1,p_2,x)\big)\Big) = \big(\overline{p}, \nabla_{p_1}S(\overline{p},\widetilde{p},\dot{x})\big),$$
$$\psi_G\Big(\big(p_2, \nabla_{p_2}\widetilde{S}(p_1,p_2,x)\big)\Big) = \big(\widetilde{p}, \nabla_{p_2}S(\overline{p},\widetilde{p},\dot{x})\big), \tag{2.99}$$
$$\psi_F\big((\nabla_x S(\overline{p},\widetilde{p},\dot{x}),\dot{x})\big) = \Big(\nabla_x \widetilde{S}(p_1,p_2,x),x\Big)$$

which ends the proof. □

Remark 2.19. Suppose that S is a generating function of a local symplectic groupoid. Let $F \in \mathcal{O}^1_\Delta + \mathcal{O}^1_{\mathrm{loc}}$ act on S, that is, $\widetilde{S} = (F(S))(F^{-1},F^{-1})$. Then, the condition $S(p,0,x) = S(0,p,x) = px$ is preserved by any $F \in \mathcal{O}^1_\Delta + \mathcal{O}^1_{\mathrm{loc}}$. However, the condition $S(p,-p,0)$ is only preserved by the odd Fs. Observe now that we have imposed the inverse map to be $i(p,x) = (-p,x)$. This implies that

$$\big((-p_2,\nabla_{p_2}S(p_2,p_1,x)),(-p_1,\nabla_{p_1}S(p_2,p_1,x)),(-\nabla_x S(p_2,p_1,x),x)\big) \in G^{(m)}(S), \tag{2.100}$$

and thus, that $S(p_1,p_2,x) = -S(-p_1,-p_2,x)$. From this last equation, we get that S must satisfy $S(p,-p,x) = 0$ and that the induced local symplectic groupoid is a **s**ymmetric one, that is, $t(p,x) = s(-p,x)$. Thus, odd transformations map symmetric groupoids to symmetric groupoids. However, they are not the only ones.

3. The Combinatorics

In this section, we present some tools which will allow us to write down at all orders the perturbative version of the composition, (2.54), in the local cotangent operad. All these compositions have essentially the same form. We will first give an abstract version of the equations describing the compositions, then we will introduce some trees which will help us to keep track of the terms involved in the computations and, at last, we will perform the expansion in the general case.

The tools and methods presented here are essentially the same as those used in the Runge–Kutta theory of ODEs to determine the order conditions of a particular numeric method. We follow approximatively the notations of [5].

3.1. The Equation

Let $F : \mathbb{R}^{n*} \to \mathbb{R}$ and $G : \mathbb{R}^n \to \mathbb{R}$ be two smooth functions. Consider the point $\phi \in \mathbb{R}$ defined by

$$\phi := G(\overline{x}) + F(\overline{p}) - \overline{px}, \tag{3.1}$$

where $\overline{x}$ and $\overline{p}$ are defined by the implicit equations

$$\begin{aligned}\overline{p} &= \nabla_x G(\overline{x}),\\ \overline{x} &= \nabla_p F(\overline{p}).\end{aligned} \tag{3.2}$$

Without any assumptions on F and G, (3.2) may not have a solution at all or the solution may be not unique. Hence, the value ϕ is not always defined. However, if we assume that F and G are formal power series of the form

$$G(x) = p_0 x + \sum_{i=1}^{\infty} \epsilon^i G^{(i)}(x), \qquad F(p) = x_0 p + \sum_{i=1}^{\infty} \epsilon^i F^{(i)}(p), \tag{3.3}$$

equation (3.2) become

$$\overline{p} = p_0 + \sum_{i=1}^{n} \epsilon^i \nabla_x G^{(i)}(\overline{x}), \qquad \overline{x} = x_0 + \sum_{i=1}^{n} \epsilon^i \nabla_p F^{(i)}(\overline{p}), \tag{3.4}$$

which are always recursively uniquely solvable.

Let us compute the first terms of $\overline{p}$, $\overline{x}$ and ϕ to get a feeling of what is happening:

$$\begin{aligned}\overline{p} &= p_0 + \epsilon \nabla_x G^{(1)}(x_0) + \epsilon^2 \nabla_x^{(2)} G^{(1)}(x_0) \nabla_p F^{(1)}(p_0) + \cdots,\\ \overline{x} &= x_0 + \epsilon \nabla_p F^{(1)}(x_0) + \epsilon^2 \nabla_p^{(2)} F^{(1)}(x_0) \nabla_x G^{(1)}(x_0) + \cdots,\\ \phi &= p_0 x_0 + \epsilon\Big(G^{(1)}(x_0) + F^{(1)}(p_0)\Big) + \epsilon^2 2 \nabla_p F^{(1)}(p_0) \nabla_x G^{(1)}(x_0) + \cdots.\end{aligned} \tag{3.5}$$

As we continue the expansion, the terms get more and more involved and, very soon, expressions as such become intractable. One common strategy in physics as in numeric analysis is to introduce some graphs to keep track of the fast growing terms. Let us present these graphs. We mainly take our inspiration from the book [5].

3.2. The Trees

Definition 3.1. We have the following

(1) A *graph* t is given by a set of vertices $V_t = \{1, \ldots, n)$ and a set of edges E_t which is a set of pairs of elements of V_t. We denote the number of vertices by $|t|$. An *isomorphism* between two graphs t and t' having the same number of vertices is a permutation $\sigma \in S_{|t|}$ such that $\{\sigma(v), \sigma(w)\} \in E_{t'}$ if $\{v, w\} \in E_t$. Two graphs are called *equivalent* if there is an isomorphism between them. The *symmetries* of a graph are the automorphisms of the graph. We denote the group of symmetries of a graph t by $sym(t)$.

(2) A *tree* is a graph which has no cycles. Isomorphisms and symmetries are defined the same way as for graphs

Table 1

T	The set of bipartite trees
RT	The set of rooted bipartite trees
$RT_\circ$	The set of elements of RT with white root
$RT_\bullet$	The set of elements of RT with black root

(3) A *rooted tree* is a tree with one distinguished vertex called root. An *isomorphism* of rooted trees is an isomorphism of graphs which sends the root to the root. Symmetries and equivalence are defined correspondingly.

(4) A *bipartite graph* is a graph t together with a map $\omega : V_t \to \{\circ, \bullet\}$ such that $\omega(v) \neq \omega(w)$ if $\{v, w\} \in E_t$. An isomorphism of bipartite trees is an isomorphism of graphs which respects the coloring, that is, $\omega(\sigma(v)) = \omega(v)$.

(5) A *weighted graph* is a graph t together with a weight map $L : V_t \to \mathbb{N} \setminus \{0\}$. An isomorphism of weighted graph is an isomorphism of graph σ which respects the weights, that is, $\sigma(L(v)) = L(\sigma(v))$. We denote by $\|t\|$ the sum of the weights on all vertices of t.

Table 1 summarizes some notations we will use in the sequel.

We will give the name *Cayley trees* to trees in T.

We denote by $[A]$ the set of equivalence classes of graphs in A (ex: $[RT]$). They are called *topological* "A" trees. Moreover, we denote by A_∞ the weighted version of graphs in A. Notice that we will use the notation $[A]_\infty$ instead of the more correct $[A_\infty]$.

The elements of $[RT]_\infty$ can be described recursively as follows:

(1) $\circ_i, \bullet_j \in [RT]_\infty$ where $i = L(\circ_i)$ and $j = L(\bullet_j)$;

(2) if $t_1, \ldots, t_m \in [RT_\circ]_\infty$, then the tree $[t_1, \ldots, t_m]_{\bullet_i} \in [RT]_\infty$ where $[t_1, \ldots, t_m]_{\bullet_i}$ is defined by connecting the roots of $t_1, \ldots, t_m$ with the weighted vertex $\bullet_i$ and declaring that $\bullet_i$ is the new root. And the same if we interchange $\circ$ and $\bullet$.

Now, let us describe in terms of trees the expressions arising in the expansions of Section 3.1.

Definition 3.2. Given two collections of functions $F = \{F^{(i)}\}_{i=1}^\infty$ and $G = \{G^{(j)}\}_{j=1}^\infty$, where $F_i : \mathbb{R}^{n*} \to \mathbb{R}$ and $G_j : \mathbb{R}^n \to \mathbb{R}$ are smooth functions, we may associate to any rooted tree $t \in [RT]_\infty$ a vector field on $T^*\mathbb{R}^d$, $DC_t(F, G) \in \mathrm{Vect}(T^*\mathbb{R}^d)$, called the *elementary differential* and a function on $T^*\mathbb{R}^d$, $C_t(F, G) \in C^\infty(T^*\mathbb{R}^d)$, called the *elementary function*.

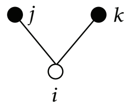
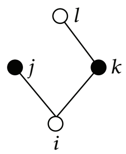
(1) The *elementary differential* $DC_t(F, G)$ is recursively defined as follows:

(a) $DC_{\circ_i}(F, G)(p, x) = \nabla_x G^{(i)}(x)$, $DC_{\bullet_j}(F, G)(p, x) = \nabla_p F^{(j)}(p)$;

(b) $DC_t(F, G) = \nabla_x^{(m+1)} G^{(i)}(DC_{t_1}(F, G), \ldots, DC_{t_m}(F, G))$ if $t = [t_1, \ldots, t_m]_{\circ_i}$;

(c) $DC_t(F, G) = \nabla_p^{(m+1)} F^{(j)}(DC_{t_1}(F, G), \ldots, DC_{t_m}(F, G))$ if $t = [t_1, \ldots, t_m]_{\bullet_j}$.

Table 2

Diagram	Elementary differential	Elementary function
	$\nabla_x^{(2)} G^{(i)} \nabla_p F^{(j)}$	$\nabla_x G^{(i)} \nabla_p F^{(j)}$
	$\nabla_p^{(3)} F^{(i)} (\nabla_x G^{(j)}, \nabla_x G^{(k)})$	$\nabla_p^{(2)} F^{(i)} (\nabla_x G^{(j)}, \nabla_x G^{(k)})$
	$\nabla_x^{(3)} G^{(i)} (\nabla_p F^{(j)}, \nabla_p^{(2)} F^{(k)} \nabla_x G^{(l)})$	$\nabla_x^{(2)} G^{(i)} (\nabla_p F^{(j)}, \nabla_p^{(2)} F^{(k)} \nabla_x G^{(l)})$

(2) The *elementary function* $C_t(F,G)$, are recursively defined as follows:

(a) $C_{\circ_i}(F,G)(p,x) = G^{(i)}(x)$, $C_{\bullet_j}(F,G)(p,x) = F^{(j)}(p)$;

(b) $C_t(F,G) = \nabla_x^{(m)} G^{(i)} (DC_{t_1}(F,G), \dots, DC_{t_m}(F,G))$ if $t = [t_1, \dots, t_m]_{\circ_i}$;

(c) $C_t(F,G) = \nabla_p^{(m)} F^{(j)} (DC_{t_1}(F,G), \dots, DC_{t_m}(F,G))$ if $t = [t_1, \dots, t_m]_{\bullet_j}$.

The notation $\nabla_x^{(m)}$ (resp. $\nabla_p^{(m)}$) stands for the mth derivative in the direction x (resp. p).

Some examples are given in Table 2.

Remark that for elementary functions it is not important which vertex is the root. This is not the case for elementary differentials.

Definition 3.3. Let $u = [u_1, \dots, u_k], v = [v_1, \dots, v_l] \in [RT]$ (resp. $\in [RT]_\infty$). Following [5], we define the *Butcher product* as follows:

$$u \circ v := [u_1, \dots, u_k, [v_1, \dots, v_l]]. \tag{3.6}$$

We have not written the obvious conditions on the u_i's and the v_i's so that the product remains bipartite (resp., weighted bipartite).

Definition 3.4 (*equivalence relation on (weighted) rooted topological trees*). Recall that an equivalence relation on a set A is a special subset R of $A \times A$. The equivalence relations on A are moreover ordered by inclusion. It makes then sense to consider the minimal equivalence on A containing a certain subset $U \subset A$.

We consider here the minimal equivalence relation on $[RT]$ (resp. on $[RT]_\infty$)) such that $u \circ v \sim v \circ u$.

Properties of this Relation

The following is clear that.

(1) Two topological rooted trees are equivalent if it is possible to pass from one to the other by changing the root. More precisely: $t, t' \in [RT]_{(\infty)}$, $t \sim t'$ if and only if there

exists a representative (E, V, r) of t and a representative (E', V', r') of t' and a vertex $r'' \in V$ such that (E, V, r'') and (E', V', r') are isomorphic (weighted) rooted trees.

(2) The quotient of $[RT]_{(\infty)}$ by this equivalence relation is exactly $[T]_{(\infty)}$.

(3) It follows immediately from the definition that $C_t(F, G) = C_{t'}(F, G)$ if $t \sim t'$ for $i = 1, 2$.

Then, it makes sense to define the elementary functions on bipartite trees.

At last, we introduce some important functions on trees: the symmetry coefficients.

Definition 3.5. Let $t = [t_1, \ldots, t_m] \in [RT]_\infty$. Consider the list $\widetilde{t}_1, \ldots, \widetilde{t}_k$ of all nonisomorphic trees appearing in $t_1, \ldots, t_m$. Define μ_i as the number of time the tree $\widetilde{t}_i$ appears in $t_1, \ldots, t_m$. Then we introduce the *symmetry coefficient* $\sigma(t)$ of t by the following recursive definition:

$$\sigma(t) = \mu_1!\mu_2! \ldots \sigma\left(\widetilde{t}_1\right) \ldots \sigma\left(\widetilde{t}_k\right) \tag{3.7}$$

and initial condition $\sigma(\circ_i) = \sigma(\bullet_j) = 1$.

It is clear that $\sigma(t)$ is the number of symmetries for each representative of t (i.e. $\sigma(t) = |Sym(t')|$ for all $t' \in t$).

3.3. The Expansion

We give now a power series expansion for (3.1).

Proposition 3.6. *Suppose that we are given the following formal power series in ϵ,*

$$G(x) = p_0 x + \sum_{i=1}^{\infty} \epsilon^i G^{(i)}(x), \qquad F(p) = x_0 p + \sum_{j=1}^{\infty} \epsilon^j F^{(j)}(p), \tag{3.8}$$

where $G^{(i)} : \mathbb{R}^n \to \mathbb{R}^{n*}$ and $F^{(j)} : \mathbb{R}^{n*} \to \mathbb{R}^n$ are smooth functions for $i, j > 0$. Define $\phi(p_0, x_0) \in \mathbb{R}[[\epsilon]]$ as

$$\phi(p_0, x_0) := G(\overline{x}) + F(\overline{p}) - \overline{p}\,\overline{x}, \tag{3.9}$$

where the formal power series $\overline{x}(\epsilon)$ and $\overline{p}(\epsilon)$ are uniquely determined by the implicit equations

$$\overline{p} = p_0 + \sum_{i=1}^{\infty} \epsilon^i \nabla_x G^{(i)}(\overline{x}), \qquad \overline{x} = x_0 + \sum_{j=1}^{\infty} \epsilon^j \nabla_p F^{(j)}(\overline{p}). \tag{3.10}$$

Then, one has that

$$\phi(p_0, x_0) = p_0 x_0 + \sum_{t \in T_\infty} \frac{\epsilon^{\|t\|}}{|t|!} C_t(F, G)(p_0, x_0). \tag{3.11}$$

The proof of Proposition 3.6 is broken into several lemmas.

The method used is essentially the same as in numerical analysis when one wants to express the Taylor series of the numerical flow of a Runge–Kutta method. Namely, the defining equations for $\overline{p}(\epsilon)$ and $\overline{x}(\epsilon)$ have a form very close to the partitioned implicit Euler method (see [5]).

Lemma 3.7. *There exist unique formal power series for $\overline{x}(\epsilon)$ and for $\overline{p}(\epsilon)$ which satisfy* (3.2). *They are given by*

$$\overline{x}(\epsilon) = x_0 + \sum_{t\in[RT_\bullet]_\infty} \frac{\epsilon^{\|t\|}}{\sigma(t)} DC_t(F,G), \tag{3.12}$$

$$\overline{p}(\epsilon) = p_0 + \sum_{t\in[RT_\circ]_\infty} \frac{\epsilon^{\|t\|}}{\sigma(t)} DC_t(F,G). \tag{3.13}$$

Proof. Uniqueness is trivial. Let us check that we have the right formal series. We only check (3.12). The other computation is similar.

$$\begin{aligned}
\overline{x}(\epsilon) &= x_0 + \sum_{i\geq 1} \epsilon^i \nabla_p F^{(i)}(\overline{p}) \\
&= x_0 + \sum_{i\geq 1} \epsilon^i \sum_{m\geq 0} \frac{1}{m!} \nabla_p^{(m+1)} F^{(i)} \left(\sum_{t\in[RT_\circ]_\infty} \frac{\epsilon^{\|t\|}}{\sigma(t)} DC_t(F,G), \dots, \sum_{t\in[RT_\circ]_\infty} \frac{\epsilon^{\|t\|}}{\sigma(t)} DC_t(F,G) \right) \\
&= x_0 + \sum_{i\geq 1} \sum_{m\geq 0} \sum_{t_1\in[RT_\circ]_\infty} \cdots \sum_{t_m\in[RT_\circ]_\infty} \frac{\epsilon^{i+\|t_1\|+\cdots+\|t_m\|}}{m!\sigma(t_1)\dots\sigma(t_m)} \\
&\quad \times \nabla_p^{(m+1)} F^{(i)}(DC_{t_1}(F,G), \cdots, DC_{t_m}(F,G)) \\
&= x_0 + \sum_{i\geq 1} \sum_{m\geq 0} \sum_{t_1} \dots \sum_{t_m} \frac{\epsilon^{\|t\|}}{m!\sigma(t)} (\mu_1!\mu_2!\dots) DC_t(F,G)\ , \quad \text{with } t = [t_1,\dots,t_m]_{\bullet_i} \\
&= x_0 + \sum_{t\in[RT_\bullet]_\infty} \frac{\epsilon^{\|t\|}}{\sigma(t)} DC_t(F,G).
\end{aligned} \tag{3.14}$$

□

Lemma 3.8. *One has the following expansion for $\phi(p_0,x_0)$:*

$$\phi(p_0,x_0) = p_0x_0 + \sum_{t\in[RT]_\infty} \frac{\epsilon^{\|t\|}}{\sigma(t)} C_t(F,G) - \left(\sum_{t\in[RT_\circ]_\infty} \frac{\epsilon^{\|t\|}}{\sigma(t)} DC_t(F,G) \right) \left(\sum_{t\in[RT_\bullet]_\infty} \frac{\epsilon^{\|t\|}}{\sigma(t)} DC_t(F,G) \right). \tag{3.15}$$

Proof. We compute the different terms arising in $G(\overline{x}) + F(\overline{p}) - \overline{p}\,\overline{x}$ in terms of trees.

$$\begin{aligned} G(\overline{x}) &= p_0\overline{x} + \sum_{i\geq 1}\epsilon^i \sum_{m\geq 0}\frac{1}{m!}\nabla_x^{(m)} G^{(i)}\left(\sum_{t\in[RT_\bullet]_\infty}\frac{\epsilon^{\|t\|}}{\sigma(t)}DC_t(F,G),\ldots,\sum_{t\in[RT_\bullet]_\infty}\frac{\epsilon^{\|t\|}}{\sigma(t)}DC_t(F,G)\right) \\ &= p_0\overline{x} + \sum_{i\geq 1}\sum_{m\geq 0}\sum_{t_1\in[RT_\bullet]_\infty}\cdots\sum_{t_m\in[RT_\bullet]_\infty}\frac{\epsilon^{\|t\|}}{m!\sigma(t)}(\mu_1!\mu_2!\ldots) \\ &\qquad \times \nabla_x^{(m)} G^{(i)}(DC_{t_1}(F,G),\ldots,DC_{t_m}(F,G)),\quad \text{with } t = [t_1,\ldots,t_m]_{\bullet_i} \\ &= p_0\overline{x} + \sum_{t\in[RT_\circ]_\infty}\frac{\epsilon^{\|t\|}}{\sigma(t)}C_t(F,G). \end{aligned} \tag{3.16}$$

By the same sort of computations we obtain

$$F(\overline{p}) = x_0\overline{p} + \sum_{t\in[RT_\bullet]_\infty}\frac{\epsilon^{\|t\|}}{\sigma(t)}C_t(F,G). \tag{3.17}$$

Finally, we get the desired result as

$$p_0\overline{x} + x_0\overline{p} - \overline{p}\,\overline{x} = p_0x_0 - \left(\sum_{t\in[RT_\circ]_\infty}\frac{\epsilon^{\|t\|}}{\sigma(t)}DC_t(F,G)\right)\left(\sum_{t\in[RT_\bullet]_\infty}\frac{\epsilon^{\|t\|}}{\sigma(t)}DC_t(F,G)\right). \tag{3.18}$$

□

Thus, $\phi(p_0, x_0)$ is expressed as sums over topological weighted rooted bipartite trees. We would like now to regroup the terms of the formula in the previous lemma. To do so, we express all terms in terms of topological trees (no longer rooted).

Lemma 3.9. *Let $u \in [RT_\circ]_\infty$ and $v \in [RT_\bullet]_\infty$. Then,*

$$DC_u(F,G)DC_v(F,G) = C_{u\circ v}(F,G) = C_{v\circ u}(F,G). \tag{3.19}$$

Proof. Suppose $u = [u_1,\ldots,u_m]_{\circ_i}$, $v = [v_1,\ldots,v_l]_{\bullet_j}$, then we get

$$\begin{aligned} A &= DC_u(F,G)DC_v(F,G) \\ &= \nabla_x^{(m+1)} G^{(i)}(DC_{u_1}(F,G),\ldots,DC_{u_m}(F,G))\cdot DC_v(F,G) \\ &= \nabla_x^{(m+1)} G^{(i)}(DC_{u_1}(F,G),\ldots,DC_{u_m}(F,G),DC_v(F,G)) \\ &= C_{u\circ v}(F,G). \end{aligned} \tag{3.20}$$

□

Lemma 3.10. *Let $t = (V_t, E_t) \in T_\infty$. For all $v \in V_t$ let t_v be the bipartite rooted tree $(V_t, E_t, v) \in RT_\infty$. For $v \in V_t$ and $e = \{u, v\} \in E_t$ one has*

$$\frac{|sym(t)|}{|sym(t_v)|} = \left|\left\{v' \in \frac{V_t}{t_{v'} \text{ is isomorphic to } t_v}\right\}\right|,$$
$$\frac{|sym(t)|}{|sym(t_u)||sym(t_v)|} = \left|\left\{e' \in \frac{E_t}{t_{u'} \sqcup t_{v'} \text{ is isomorphic to } t_u \sqcup t_v}\right\}\right|. \tag{3.21}$$

Proof. Consider the induced action of the symmetry group of the tree on the set of vertices. Notice that two vertices v and w are in the same orbit if and only if t_v is isomorphic to t_w. Then the number of vertices of t which lead to rooted tree isomorphic to t_v is exactly the cardinality of the orbit of v, which is exactly $|sym(t)|$ divided by the cardinality of the isotropy subgroup which fixes v. But the latter is $|sym(t_v)|$ by definition. We then get the first statement.

For the second statement we have to consider the induced action on the edges and apply the same type of argument. □

Lemma 3.11. *We get*

$$\phi(p_0, x_0) = p_0 x_0 + \sum_{t \in T_\infty} \frac{\epsilon^{\|t\|}}{|t|!} C_t(F, G). \tag{3.22}$$

Proof. Let us perform the last computation.

$$\begin{aligned}
\phi(p_0, x_0) &= p_0 x_0 + \sum_{t \in [RT]_\infty} \frac{\epsilon^{\|t\|}}{\sigma(t)} C_t(F, G) - \sum_{u \in [RT_\circ]_\infty} \sum_{v \in [RT_\bullet]_\infty} \frac{\epsilon^{\|u\|+\|v\|}}{\sigma(u)\sigma(v)} DC_u(F, G) DC_v(F, G) \\
&= p_0 x_0 + \sum_{\bar{t} \in [T]_\infty} \epsilon^{|\bar{t}|} C_{\bar{t}}(F, G) \left\{ \sum_{t \in \bar{t}} \frac{1}{|sym(t)|} - \sum_{u \in [RT_\bullet]_\infty, v \in [RT_\circ]_\infty u \circ v \in \bar{t}} \frac{1}{|sym(u) \parallel sym(v)|} \right\} \\
&= p_0 x_0 + \sum_{t \in T_\infty} \frac{\epsilon^{\|t\|}}{|t|!} C_t(F, G) \left\{ \sum_{v \in V_t} \frac{|sym(t)|}{|sym(t_v)|} \frac{1}{k(t, v)}, \right. \\
&\qquad \left. - \sum_{e = \{u, v\} \in E_t} \frac{|sym(t)|}{|sym(t_u) \parallel sym(t_v)|} \frac{1}{l(t, e)} \right\}
\end{aligned} \tag{3.23}$$

where $k(t, v) = |\{v' \in V_t / t_{v'}$ is isomorphic to $t_v\}|$ and $l(t, e) = |\{e' \in E_t / t_{u'} \sqcup t_{v'}$ is isomorphic to $t_u \sqcup t_v)|$. Using Lemma 3.10 and the fact that for a tree the difference between the number of vertices and the number of edges is equal to 1 we get the desired result. □

Using now the fact that S is a formal power series we immediately get Proposition 3.6.

4. Deformation of a Nonlinear Structure

4.1. The Formal Cotangent Lagrangian Operad

The formal cotangent Lagrangian operad on $T^*\mathbb{R}^d$ is the perturbative/formal version of the local cotangent operad on $T^*\mathbb{R}^d$. Recall that in the latter the product for $F \in \mathcal{O}^n_\Delta + \mathcal{O}^n_{\text{loc}}$ and $G_i \in \mathcal{O}^n_\Delta + \mathcal{O}^{k_i}_{\text{loc}}$, $i = 1, \dots n$ was expressed as in Proposition 2.14:

$$\begin{aligned} F(G_1, \dots, G_n)(p_G, x_F) &= G_1 \cup \cdots \cup G_n(p_G, x_G) + F(p_F, x_F) - p_F \cdot x_G, \\ p_F &= \nabla_x G_1 \cup \cdots \cup G_n(p_G, x_G), \\ x_F &= \nabla_p F(p_F, x_F). \end{aligned} \tag{4.1}$$

If we consider p_G and x_F as parameters in the previous equations, we have then that

$$G(p_G, \cdot) : \mathbb{R}^{nd} \longrightarrow \mathbb{R}, \qquad F(\cdot, x_F) : \left(\mathbb{R}^{nd}\right)^* \longrightarrow \mathbb{R}. \tag{4.2}$$

Suppose now that the F and G_i, $i = 1, \dots, n$, are formal series of the form

$$\begin{aligned} F(p_F, x_F) &= p_F^\Sigma \cdot x_F + \sum_{i=1}^\infty \epsilon^i F^{(i)}(p_F, x_F), \\ G_l(p_{G_l}, x_{G_l}) &= p_{G_l}^\Sigma \cdot x_{G_l} + \sum_{i=1}^\infty \epsilon^i G_l^{(i)}(p_{G_l}, x_{G_l}), \end{aligned} \tag{4.3}$$

where

$$p^\Sigma := \sum_{i=1}^n p_i \quad \text{for } p = (p_1, \dots, p_n) \in \left(\mathbb{R}^{dn}\right)^*. \tag{4.4}$$

We may rewrite F and G as

$$\begin{aligned} F(p_F, x_F) &= x_0^F p_F + \sum_{i=1}^\infty \epsilon^i F^{(i)}(p_F, x_F), \\ G(p_G, x_G) &= p_0^G x_G + \sum_{i=1}^\infty \epsilon^i G^{(i)}(p_G, x_G), \end{aligned} \tag{4.5}$$

where $x_0^F = (x_F, \dots, x_F) \in \mathbb{R}^{dn}$ and $p_0^G = (p_{G_1}^\Sigma, \dots, p_{G_n}^\Sigma) \in (\mathbb{R}^{nd})^*$ for $x_G \in \mathbb{R}^{dn}$ and $p_F \in (\mathbb{R}^{dn})^*$.

Applying now Proposition 3.6, we obtain for the compositions the following expansion:

$$F(G_1, \dots, G_n)(p_G, x_G) = p_G^\Sigma \cdot x_F + \sum_{t \in T_\infty} \frac{\epsilon^{\|t\|}}{|t|!} C_t\big(F(\cdot, x_F), G_1 \cup \ldots \cup G_n(p_G, \cdot)\big)\left(p_0^G, x_0^F\right). \tag{4.6}$$

This motivates to define the formal deformation space of the cotangent Lagrangian operad $\mathcal{O}_\Delta(T^*\mathbb{R}^d)$ as

$$\mathcal{O}^n_{\text{form}}\left(T^*\mathbb{R}^d, \Delta\right) := \left\{\sum_{i=1}^{\infty} \epsilon^i F^{(i)} : F^{(i)} \in P^n_i\left(T^*\mathbb{R}^d\right)\right\}, \tag{4.7}$$

where $P^n_i(T^*\mathbb{R}^d)$ stands for the vector space of functions $F : B_n \to \mathbb{R}$ such that

(1) $F(p,x)$ is a polynomial in the variables $p = (p_1, \ldots, p_n)$,

(2) $F(\mu p, x) = \mu^{i+1} F(p,x)$.

One may think of $\mathcal{O}_{\text{loc}} + \mathcal{O}_{\text{form}}$ as the Taylor series of functions in $\mathcal{O}_\Delta + \mathcal{O}_{\text{loc}}$. The compositions are given by formula (4.6), which also tells us that $\mathcal{O}_\Delta + \mathcal{O}_{\text{loc}}$ is an operad. The unit is

$$I(p,x) = px, \quad I \in \mathcal{O}_\Delta + \mathcal{O}^1_{\text{form}}. \tag{4.8}$$

The induced operad structure on $\mathcal{O}_{\text{form}}$ is then given by

$$\begin{gathered} I \in \mathcal{O}^1_{\text{form}}, \quad I(p,x) = 0, \\ \mathcal{O}^n_{\text{form}} = \left\{\sum_{i=1}^{\infty} \epsilon^i F^{(i)} : F^{(i)} \in P^n_i\left(T^*\mathbb{R}^d\right)\right\}, \\ F(G_1, \ldots, G_n)(p_G, x_F) = \sum_{t \in T_\infty} \frac{\epsilon^{\|t\|}}{|t|!} C_t(F, G_1 \cup \ldots \cup G_n). \end{gathered} \tag{4.9}$$

This operad will be called *the formal deformation operad* of the cotangent Lagrangian operad $\mathcal{O}_\Delta$.

4.2. Product in the Formal Deformation Operad

Exactly as for the local deformation operad, S_ϵ is a product in $\mathcal{O}_{\text{form}}$ if and only if $S^2_0 + S_\epsilon$ satisfies formally the SGA equation. Moreover, if $S^2_0 + S_\epsilon$ satisfies the SGS conditions, then $S^2_0 + S_\epsilon$ is the generating function of a formal symplectic groupoid over $\mathbb{R}^d$.

Again, the zero of $\mathcal{O}^2_{\text{form}}$ is a product in $\mathcal{O}_{\text{form}}$. We will stick to the conventions introduced for $\mathcal{O}_{\text{loc}}$. Namely, 0_1 will stand for the zero of $\mathcal{O}^1_{\text{form}}$, which is also the identity of the operad and 0_2 will stand for the zero of $\mathcal{O}^2_{\text{form}}$, which is the trivial product of the operad.

Thanks to the composition formula (4.6), we are now able to rewrite the product equation in $\mathcal{O}_{\text{form}}$ as a cohomological equation, exactly as the deformation equation of a product in an additive category. Note that the Taylor expansion plays the same role as the linear expansion played in the additive case.

Let us define the Gerstenhaber bracket in $\mathcal{O}_{\text{form}}$ as follows:

$$[F, G] = F \circ G - (-1)^{(k-1)(l-1)} G \circ F, \tag{4.10}$$

where

$$F \circ G = \sum_{i=1}^{k} (-1)^{(i-l)(l-1)} F \left\{ 0_1, \ldots, 0_1, \underbrace{G}_{i\text{th}}, 0_1, \ldots, 0_1 \right), \tag{4.11}$$

for $F \in \mathcal{O}^k_{\text{form}}$ and $G \in \mathcal{O}^l_{\text{form}}$.

We are now able to define a true coboundary operator.

Proposition 4.1. *Consider $d : \mathcal{O}^n_{\textit{form}} \to \mathcal{O}^{n+1}_{\textit{form}}$*

$$dF := [0_2, F]. \tag{4.12}$$

Then, d may be written as

$$\begin{aligned} dF(p_1, \ldots, p_{n+1}) &= F(p_1, \ldots, p_n, x) \\ &\quad + \sum_{j=1}^{n} (-1)^{n+j-1} F(p_1, \ldots, p_j + p_{j+1}, \ldots, p_n, x) + (-1)^{n-1} F(p_2, \ldots, p_{n+1}, x). \end{aligned} \tag{4.13}$$

Moreover, d is linear and $d^2 = 0$.

Proof. For more clarity, let us break our convention and write $\widetilde{I}$ instead of 0_1 and $\widetilde{S}$ instead of 0_2. We have that $[\widetilde{S}, F] = \widetilde{S} \circ F - (-1)^{n-1} F \circ \widetilde{S}$. As $\widetilde{S} = 0$, only the trees $\circ_i$ and $\bullet_j$ will contribute to the product. Then we have,

$$\begin{aligned} I_1 &= \widetilde{S} \circ F(p_1, \ldots, p_{n+1}, x) \\ &= \sum_{i \geq 1} \epsilon^i \Bigg(C_{\bullet_i}\left(\widetilde{S}(\cdot, x), F \cup \widetilde{I}(p, \cdot)\right)\left(\left(\sum_{1}^{n} p_l, p_{n+1}\right), (x, x)\right) \\ &\qquad + (-1)^{n-1} C_{\bullet_i}\left(\widetilde{S}(\cdot, x), \widetilde{I} \cup F(p, \cdot)\right)\left(\left(p_1, \sum_{2}^{n+1} p_l\right), (x, x)\right)\Bigg) \\ &= \sum_{i \geq 1} \epsilon^i \left(F^{(i)}(p_1, \ldots, p_n, x) + (-1)^{n-1} F^{(i)}(p_2, \ldots, p_{n+1}, x)\right), \\ I_2 &= F \circ \widetilde{S}(p_1, \ldots, p_{n+1}) \\ &= \sum_{j=1}^{n} (-1)^{j-1} \sum_{i \geq 1} \epsilon^i C_{\circ_i}\Bigg(F(\cdot, x), \Bigg(\widetilde{I} \cup \cdots \widetilde{I} \cup \underbrace{\widetilde{S}}_{j\text{th}} \cup \widetilde{I} \cdots \\ &\qquad \cdots \cup \widetilde{I}\Bigg)(p, \cdot)\Bigg) \times ((p_1, \ldots, p_j + p_{j+1}, \ldots, p_{n+1}), (x, \ldots, x)) \end{aligned}$$

$$= \sum_{j=1}^{n}(-1)^{j-1}\sum_{i\geq 1}\epsilon^{i}F^{(i)}(p_1,\dots,p_j+p_{j+1},\dots,p_{n+1},x), \tag{4.14}$$

which gives the desired formula. The check that $d^2 = 0$ is straightforward. □

We have then a complex

$$\left(C^{\bullet} = \bigoplus\nolimits_{n\geq 0}\mathcal{O}^{n}_{\text{form}}, d\right). \tag{4.15}$$

This complex is exactly the Hochschild complex of (formal) multidifferential operators lifted on the level of symbols (see for instance [11]). This remark gives us the cohomology of the complex

$$H^{n}(C^{\bullet}) \simeq \epsilon\mathcal{V}^{n}\left(\mathbb{R}^{d}\right)[[\epsilon]], \tag{4.16}$$

where $\mathcal{V}^n(\mathbb{R}^d)$ is the space of n-multi-vector fields on $\mathbb{R}^d$.

We come now to the question of finding a product S_ϵ in the formal deformation operad of $\mathcal{O}_\Delta$. This is exactly the same problem as deforming the trivial generating function S_0^2 in $\mathcal{O}_\Delta + \mathcal{O}_{\text{form}}$. We are thus looking for an element $S_\epsilon \in \mathcal{O}^2_{\text{form}}$ of the form

$$S_\epsilon = \epsilon S_1 + \epsilon^2 S_2 + \cdots \tag{4.17}$$

such that

$$[S_\epsilon, S_\epsilon] = 0. \tag{4.18}$$

Equation (4.18) becomes, on the level of trees,

$$\sum_{t\in T_\infty}\frac{\epsilon^{\|t\|}}{|t|!}\left(C_t(S_\epsilon, S_\epsilon\cup I) - C_t(S_\epsilon, I\cup S_\epsilon)\right) = 0. \tag{4.19}$$

One sees immediately that this equation is equivalent to the following infinite set of recursive equations:

$$dS_n + H_n(S_{n-1},\dots,S_1) = 0, \tag{4.20}$$

where

$$H_n(S_{n-1},\dots,S_1) = \sum_{t\in T_\infty^{k,n} 2\leq|k|\leq n}\frac{1}{|t|!}\left(C_t(S_\epsilon, S_\epsilon\cup I) - C_t(S_\epsilon, I\cup S_\epsilon)\right), \tag{4.21}$$

where $T_\infty^{k,n}$ is the subset of trees in $T_\infty^{k,n}$ with k vertices and such that $\|t\| = n$. These recursive equations are the exact analog of (2.23).

4.3. Formal Symplectic Groupoid Generating Function

We restate now the main theorem of [2], Theorem 4.2, in terms of the new structures defined in this paper.

Theorem 4.2. *For each Poisson structure α on $\mathbb{R}^d$, one has that*

$$S_\epsilon(\alpha) = \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \sum_{\Gamma \in T_{n,2}} W_\Gamma \widehat{B}_\Gamma(\alpha) \tag{4.22}$$

is a product in the formal deformation operad $\mathcal{O}_{form}(T^*\mathbb{R}^d, \Delta)$ of the cotangent Lagrangian operad $\mathcal{O}_\Delta(T^*\mathbb{R}^d)$. Moreover, $S_\epsilon(\alpha)$ is the unique natural product in $\mathcal{O}_{form}(T^*\mathbb{R}^d, \Delta)$ whose first order is $\epsilon\alpha$.

In the above theorem, the $T_{n,2}$ stand for the set of Kontsevich trees of type $(n, 2)$, W_Γ is the Kontsevich weight of Γ and $\widehat{B}_\Gamma$ is the symbol of the bidifferential operator B_Γ associated to Γ. We refer the reader to [2] for exact definitions of Kontsevich trees, weights, operators, and naturallity.

We called $S_\epsilon(\alpha)$ the (formal) *symplectic groupoid generating function* because, as shown in [2], it generates a "geometric object", a (formal) symplectic groupoid over $\mathbb{R}^d$ associated to the Poisson structure α whose structure maps are explicitly given by

$$\begin{aligned} \epsilon_\epsilon(x) &= (0, x) \quad \text{unit map}, \\ i_\epsilon(p, x) &= (-p, x) \quad \text{inverse map}, \\ s_\epsilon(p, x) &= x + \nabla_{p_2} S_\epsilon(\alpha)(p, 0, x) \quad \text{sourcex map}, \\ t_\epsilon(p, x) &= x + \nabla_{p_1} S_\epsilon(\alpha)(0, p, x) \quad \text{target map}. \end{aligned} \tag{4.23}$$

This exhibits a strong relationship between star products and symplectic groupoids already foreseen by Costes, et al., Karasëv and Zakrzewski in respectively [12–14]. Recently and from a completely different point of view, Karabegov in [15] went still a step further by showing how to associate a kind of "formal symplectic groupoid" to any star product.

In [4, 10], we prove that the product $S_\epsilon(\alpha)$ has a nonzero convergence radius provided that the Poisson structure α is analytic. In this case, the generated formal symplectic groupoid is the local one. We also compared this local symplectic groupoid with the one constructed by Karasëv and Maslov in [13], and we proved that this two local symplectic groupoids are not only isomorphic as they should but exactly identical.

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