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Research Article

Group Divisible Designs with Two Associate Classes and $\lambda_2 = 1$

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The original classiffcation of PBIBDs defined group divisible designs GDD($v = v_1 + v_2 + \cdots + v_g, g, k, \lambda_1, \lambda_2$) with $\lambda_1 \neq 0$. In this paper, we prove that the necessary conditions are suffcient for the existence of the group divisible designs with two groups of unequal sizes and block size three with $\lambda_2 = 1$.

1. Introduction

A *pairwise balanced design* is an ordered pair $(S, \mathcal{B})$, denoted PBD$(S, \mathcal{B})$, where S is a finite set of symbols, and $\mathcal{B}$ is a collection of subsets of S called *blocks*, such that each pair of distinct elements of S occurs together in exactly one block of $\mathcal{B}$. Here $|S| = v$ is called the *order* of the PBD. Note that there is no condition on the size of the blocks in $\mathcal{B}$. If all blocks are of the same size k, then we have a *Steiner system* S (v, k). A PBD with index λ can be defined similarly; each pair of distinct elements occurs in λ blocks. If all blocks are same size, say k, then we get a balanced incomplete block design BIBD(v, b, r, k, λ). In other words, a BIBD(v, b, r, k, λ) is a set S of v elements together with a collection of b k-subsets of S, called blocks, where each point occurs in r blocks, and each pair of distinct elements occurs in exactly λ blocks (see [1–3]).

Note that in a BIBD(v, b, r, k, λ), the parameters must satisfy the necessary conditions

(1) $vr = bk$ and

(2) $\lambda(v-1) = r(k-1)$.

With these conditions, a BIBD(v, b, r, k, λ) is usually written as BIBD(v, k, λ).

A *group divisible design* GDD($v = v_1 + v_2 + \cdots + v_g, g, k, \lambda_1, \lambda_2$) is an ordered triple $(V, G, \mathcal{B})$, where V is a v-set of symbols, G is a partition of V into g sets of size $v_1, v_2, \ldots, v_g$, each set being called *group*, and $\mathcal{B}$ is a collection of k-subsets (called *blocks*) of V, such that each pair of symbols from the same group occurs in exactly λ_1 blocks, and each pair of symbols from different groups occurs in exactly λ_2 blocks (see [1, 2, 4]). Elements occurring together in the same group are called *first associates*, and elements occurring in different groups we called *second associates*. We say that the GDD is defined on the set V. The existence of such GDDs has been of interest over the years, going back to at least the work of Bose and Shimamoto in 1952 who began classifying such designs [5]. More recently, much work has been done on the existence of such designs when $\lambda_1 = 0$ (see [6] for a summary), and the designs here are called partially balanced incomplete block designs (PBIBDs) of group divisible type in [6]. The existence question for $k = 3$ has been solved by Fu and Rodger [1, 2] when all groups are the same size.

In this paper, we continue to focus on blocks of size 3, solving the problem when the required designs having two groups of unequal size, namely, we consider the problem of determining necessary conditions for an existence of GDD($v = m + n, 2, 3, \lambda_1, \lambda_2$) and prove that the conditions are sufficient for some infinite families. Since we are dealing on GDDs with two groups and block size 3, we will use GDD($m, n; \lambda_1, \lambda_2$) for GDD($v = m+n, 2, 3, \lambda_1, \lambda_2$) from now on, and we refer to the blocks as triples. We denote $(X, Y; \mathcal{B})$ for a GDD($m, n; \lambda_1, \lambda_2$) if X and Y are m-set and n-set, respectively. Chaiyasena, et al. [7] have written a paper in this direction. In particular, they have completely solved the problem of determining all pairs of integers (n, λ) in which a GDD($1, n; 1, \lambda$) exists. We continue to investigate in this paper all triples of integers (m, n, λ) in which a GDD($m, n; \lambda, 1$) exists. We will see that necessary conditions on the existence of a GDD($m, n; \lambda_1, \lambda_2$) can be easily obtained by describing it graphically as follows.

Let λK_v denote the graph on v vertices in which each pair of vertices is joined by λ edges. Let G_1 and G_2 be graphs. The graph $G_1 \vee_\lambda G_2$ is formed from the union of G_1 and G_2 by joining each vertex in G_1 to each vertex in G_2 with λ edges. A *G-decomposition* of a graph H is a partition of the edges of H such that each element of the partition induces a copy of G. Thus the existence of a GDD($m, n; \lambda_1, \lambda_2$) is easily seen to be equivalent to the existence of a K_3-decomposition of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$. The graph $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$ is of order $m + n$ and size $\lambda_1[\binom{m}{2} + \binom{n}{2}] + \lambda_2 mn$. It contains m vertices of degree $\lambda_1(m-1) + \lambda_2 n$ and n vertices of degree $\lambda_1(n-1) + \lambda_2 m$. Thus the existence of a K_3-decomposition of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$ implies

(1) $3 \mid \lambda_1[\binom{m}{2} + \binom{n}{2}] + \lambda_2 mn$, and

(2) $2 \mid \lambda_1(m-1) + \lambda_2 n$ and $2 \mid \lambda_1(n-1) + \lambda_2 m$.

2. Preliminary Results

We will review some known results concerning triple designs that will be used in the sequel, most of which are taken from [3].

Theorem 2.1. *Let v be a positive integer. Then there exists a* BIBD($v, 3, 1$) *if and only if* $v \equiv 1$ *or* 3 (mod 6).

A BIBD($v, 3, 1$) is usually called *Steiner triple system* and is denoted by STS(v). Let $(V, \mathcal{B})$ be an STS(v). Then the number of triples $b = |\mathcal{B}| = v(v-1)/6$. A *parallel class* in an STS(v) is a set of disjoint triples whose union is the set V. A parallel class contains $v/3$ triples, and, hence, an STS(v) having a parallel class can exist only when $v \equiv 3 \pmod 6$. When the set $\mathcal{B}$ can be partitioned into parallel classes, such a partition $\mathcal{R}$ is called a *resolution* of the STS(v), and the STS(v) is called *resolvable*. If $(V, \mathcal{B})$ is an STS(v), and $\mathcal{R}$ is a resolution of it, then $(V, \mathcal{B}, \mathcal{R})$ is called a *Kirkman triple system*, denoted by KTS(v), with $(V, \mathcal{B})$ as its *underlying* STS. It is well known that a KTS(v) exists if and only if $v \equiv 3 \pmod 6$. Thus if $(V, \mathcal{B}, \mathcal{R})$ is a KTS(v), then $\mathcal{R}$ contains $(v-1)/2$ parallel classes.

Theorem 2.2. *There exists a* PBD$(6k+5)$ *with one block of size 5 and* $6k^2 + 9k$ *blocks of size 3.*

Example 2.3. Let $S = \{1, 2, 3, \dots, 11\}$. Then PBD(11) is an ordered pair $(S, \mathcal{B})$, where $\mathcal{B}$ contains the following blocks:

$$\begin{matrix} \{1,2,3,4,5\} & \{2,6,9\} & \{3,7,8\} & \{4,8,11\} \\ \{1,6,7\} & \{2,7,11\} & \{3,9,10\} & \{5,6,8\} \\ \{1,8,9\} & \{2,8,10\} & \{4,6,10\} & \{5,7,10\} \\ \{1,10,11\} & \{3,6,11\} & \{4,7,9\} & \{5,9,11\}. \end{matrix} \tag{2.1}$$

A *factor* of a graph G is a spanning subgraph. An *r-factor* of a graph is a spanning r-regular subgraph, and an *r-factorization* is a partition of the edges of the graph into disjoint r-factors. A graph G is said to be *r-factorable* if it admits an r-factorization. In particular, a 1-factor is a perfect matching, and a 1-factorization of an r-regular graph G is a set of 1-factors which partition the egde set of G. The following results are well known.

Theorem 2.4. *The complete graph* K_{2n} *is 1-factorable,* K_{2n+1} *is 2-factorable, and* K_{3n+1} *is 3-factorable.*

The following results on existence of λ-fold triple systems are well known (see, e.g., [3]).

Theorem 2.5. *Let n be a positive integer. Then a* BIBD$(n, 3, \lambda)$ *exists if and only if λ and n are in one of the following cases:*

(a) $\lambda \equiv 0 \pmod 6$ *and* $n \neq 2$,

(b) $\lambda \equiv 1$ *or* $5 \pmod 6$ *and* $n \equiv 1$ *or* $3 \pmod 6$,

(c) $\lambda \equiv 2$ *or* $4 \pmod 6$ *and* $n \equiv 0$ *or* $1 \pmod 3$, *and*

(d) $\lambda \equiv 3 \pmod 6$ *and n is odd.*

The results of Chaiyasena, et al. [7] will be useful, and we will state their results as follows.

Theorem 2.6. *Let v be a positive integer with $v \geq 3$. The spectrum of λ, denoted $S_{1,v}$ is defined as*

$$S_{1,v} = \{\lambda : \text{a GDD}(1, v; 1, \lambda) \text{ exists}\}. \tag{2.2}$$

Then

(a) $S_{1,v} = \{1, 3, 5, \ldots, v-1\}$ if $v \equiv 0 \pmod 6$,

(b) $S_{1,v} = \{6, 12, 18, \ldots, v-1\}$ if $v \equiv 1 \pmod 6$,

(c) $S_{1,v} = \{1, 7, 13, \ldots, v-1\}$ if $v \equiv 2 \pmod 6$,

(d) $S_{1,v} = \{2, 4, 6, \ldots, v-1\}$ if $v \equiv 3 \pmod 6$,

(e) $S_{1,v} = \{3, 9, 15, \ldots, v-1\}$ if $v \equiv 4 \pmod 6$, and

(f) $S_{1,v} = \{4, 10, 16, \ldots, v-1\}$ if $v \equiv 5 \pmod 6$.

The following notations will be used throughout the paper for our constructions.

(1) Let $T = \{x, y, z\}$ be a triple and $a \notin T$. We use $a * T$ for three triples of the form $\{a, x, y\}, \{a, x, z\}, \{a, y, z\}$. If $\mathcal{T}$ is a set of triples, then $a * \mathcal{T}$ is defined as $\{a * T : T \in \mathcal{T}\}$.

(2) Let $G = \langle V(G), E(G) \rangle$ be a graph. If $u, v \in V(G)$, $e = uv \in E(G)$, and $a \notin V(G)$, then we use $a + e$ for the triple $\{a, u, v\}$. We further use $a + E(G)$ for the collection of triples $a + e$ for all $e \in E(G)$. In other words,

$$a + E(G) := \{a + e : e \in E(G)\}. \tag{2.3}$$

In particular, if $\mathcal{F} = \{x_1y_1, x_2y_2, \ldots, x_ny_n\}$ is a 1-factor of K_{2n} and a is not in the vertex set of K_{2n}, then

$$a + \mathcal{F} = \{\{a, x_1, y_1\}, \{a, x_2, y_2\}, \ldots, \{a, x_n, y_n\}\}. \tag{2.4}$$

If $C_m : x_1, x_2, \ldots, x_{m+1} = x_1$ is a cycle in K_n, then

$$a + C_m = \{\{a, x_1, x_2\}, \{a, x_2, x_3\}, \ldots, \{a, x_{m-1}, x_m\}, \{a, x_m, x_1\}\}. \tag{2.5}$$

Also if G is a 2-regular graph and $a \notin V(G)$, then $a + E(G)$ forms a collection of triples such that for each $u \in V(G)$, there are exactly two triples in $a + E(G)$ containing a and u. In general if G is an r-regular graph and $a \notin V(G)$, then $a + E(G)$ forms a collection of triples such that for each $u \in V(G)$, there are exactly r triples in $a + E(G)$ containing a and u.

(3) Let V be a v-set. We use $K(V)$ for the complete graph K_v on the vertex set V.

(4) Let V be a v-set. Let $\mathrm{STS}(V)$ be defined as

$$\mathrm{STS}(V) = \{\mathcal{B} : (V, \mathcal{B}) \text{ is an } \mathrm{STS}(v)\}. \tag{2.6}$$

$\mathrm{KTS}(V)$ and $\mathrm{BIBD}(V, 3, \lambda)$ can be defined similarly, that is,

$$\begin{aligned} \mathrm{KTS}(V) &= \{\mathcal{B} : (V, \mathcal{B}) \text{ is a } \mathrm{KTS}(v)\}, \\ \mathrm{BIBD}(V, 3, \lambda) &= \{\mathcal{B} : (V, \mathcal{B}) \text{ is a } \mathrm{BIBD}(v, 3, \lambda)\}. \end{aligned} \tag{2.7}$$

Let X and Y be disjoint sets of cardinality m and n, respectively. We define $\mathrm{GDD}(X, Y; \lambda_1, \lambda_2)$ as

$$\mathrm{GDD}(X, Y; \lambda_1, \lambda_2) = \{\mathcal{B} : (X, Y; \mathcal{B}) \text{ is a GDD}(m, n; \lambda_1, \lambda_2)\}. \tag{2.8}$$

(5) When we say that $\mathcal{B}$ is a *collection* of subsets (blocks) of a v-set V, $\mathcal{B}$ may contain repeated blocks. Thus, "$\cup$" in our construction will be used for the union of multisets.

3. GDD$(m, n; \lambda, 1)$

Let λ be a positive integer. We consider in this section the problem of determining all pairs of integers (m, n) in which a $\mathrm{GDD}(m, n; \lambda, 1)$ exists. Recall that the existence of $\mathrm{GDD}(m, n; \lambda, 1)$ implies $3 \mid \lambda[m(m-1) + n(n-1)] + 2mn$, $2 \mid \lambda(m-1) + n$ and $2 \mid \lambda(n-1) + m$. Let

$$S(\lambda) := \{(m, n) : \text{a GDD}(m, n; \lambda, 1) \text{ exists}\}. \tag{3.1}$$

By solving systems of linear congruences, we obtain the following necessary conditions.

Lemma 3.1. *Let t be a nonnegative integer.*

(a) *If $(m, n) \in S(6t+1)$, then there exist nonnegative integers h and k such that* $\{m, n\} \in \{\{6k+1, 6h+2\}, \{6k+1, 6h+6\}, \{6k+3, 6h+4\}, \{6k+3, 6h+6\}, \{6k+5, 6h+2\}, \{6k+5, 6h+4\}\}$.

(b) *If $(m, n) \in S(6t+2)$, then there exist nonnegative integers h and k such that* $\{m, n\} \in \{\{6k+6, 6h+4\}, \{6k+6, 6h+6\}\}$.

(c) *If $(m, n) \in S(6t+3)$, then there exist nonnegative integers h and k such that* $\{m, n\} \in \{\{6k+1, 6h+6\}, \{6k+3, 6h+2\}, \{6k+3, 6h+4\}, \{6k+3, 6h+6\}, \{6k+5, 6h+6\}\}$.

(d) *If $(m, n) \in S(6t+4)$, then there exist nonnegative integers h and k such that* $\{m, n\} \in \{\{6k+2, 6h+2\}, \{6k+2, 6h+4\}, \{6k+6, 6h+4\}, \{6k+6, 6h+6\}\}$.

(e) *If $(m, n) \in S(6t+5)$, then there exist nonnegative integers h and k such that* $\{m, n\} \in \{\{6k+1, 6h+6\}, \{6k+3, 6h+4\}, \{6k+3, 6h+6\}\}$.

(f) *If $(m, n) \in S(6t+6)$, then there exist nonnegative integers h and k such that* $\{m, n\} \in \{\{6k+6, 6h+2\}, \{6k+6, 6h+4\}, \{6k+6, 6h+6\}\}$.

In order to obtain sufficient conditions on an existence of $\mathrm{GDD}(m, n; \lambda, 1)$, we first observe the following facts.

(1) Let X and Y be two disjoint sets of size m and n, respectively. Then $\mathrm{STS}(X \cup Y) \neq \emptyset$ if and only if $\mathrm{GDD}(X, Y; 1, 1) \neq \emptyset$.

(2) Let X and Y be two disjoint sets of size m and n; respectively, and let $\lambda \in \{2, 3, 4, 5, 6\}$. Then $\mathrm{GDD}(X, Y; \lambda, 1) \neq \emptyset$ if $\mathrm{STS}(X \cup Y) \neq \emptyset$, $\mathrm{BIBD}(X, 3, \lambda - 1) \neq \emptyset$, and $\mathrm{BIBD}(Y, 3, \lambda - 1) \neq \emptyset$.

Thus, we have the following results.

Lemma 3.2. *Let h and k be nonnegative integers. Then*

(a) $(6k+1,6h+6),(6k+6,6h+1),(6k+3,6h+6),(6k+6,6h+3),(6k+3,6h+4),(6k+4,6h+3),(6k+1,6h+2),(6k+2,6h+1),(6k+5,6h+2),(6k+2,6h+5),(6k+5,6h+4),(6k+4,6h+5) \in S(1)$,

(b) $(6k+1,6h+6),(6k+6,6h+1),(6k+3,6h+6),(6k+6,6h+3),(6k+3,6h+4),(6k+4,6h+3) \in S(3)$, *and*

(c) $(6k+1,6h+6),(6k+6,6h+1),(6k+3,6h+6),(6k+6,6h+3),(6k+3,6h+4),(6k+4,6h+3) \in S(5)$.

Lemma 3.3. *Let h and k be nonnegative integers. Then,*

(a) $(6k+6,6h+6) \in S(2)$ *and*

(b) $(6k+6,6h+4),(6k+4,6h+6) \in S(2)$.

Proof. (a) We first consider an existence of GDD$(6,6;2,1)$, where the groups are $X = \{1,2,3,4,5,6\}$ and $Y = \{a_1,a_2,\ldots,a_6\}$. Let $\mathcal{F} = \{\mathcal{F}_1,\mathcal{F}_2,\ldots,\mathcal{F}_5\}$ be a 1-factorization of $K(X)$. Let $\mathcal{B}_1 = \bigcup_{i=1}^{5}(a_i + \mathcal{F}_i)$, $\mathcal{B}_2 \in \mathrm{STS}(X \cup \{a_6\})$, and $\mathcal{B}_3 \in \mathrm{BIBD}(Y,3,2)$. Then $(X,Y;\mathcal{B})$ forms a GDD$(6,6;2,1)$, where $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. Thus $(6,6) \in S(2)$.

Let X and Y be two sets of size $6k+6$ and $6h+6$, respectively. Suppose that $k \le h$ and $h \ge 1$. Let $a_1,a_2,a_3 \in Y$ and let $Y' = Y - \{a_1,a_2,a_3\}$. Thus, $\mathrm{KTS}(Y') \ne \emptyset$. Let $\mathcal{B}_1 \in \mathrm{KTS}(Y')$ with $\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_{3h+1}$ as its parallel classes. Since $\mathrm{STS}(X \cup Y')$ and $\mathrm{STS}(X \cup \{a_1,a_2,a_3\})$ are not empty, there exist $\mathcal{B}_2 \in \mathrm{STS}(X \cup Y')$ and $\mathcal{B}_3 \in \mathrm{STS}(X \cup \{a_1,a_2,a_3\})$. We now let $\mathcal{B}$ as

$$\left(\bigcup_{i=1}^{3}(a_i * \mathcal{P}_i)\right) \cup \left(\bigcup_{i=4}^{3h+1}\mathcal{P}_i\right) \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \{\{a_1,a_2,a_3\}\}. \tag{3.2}$$

Thus, $(X,Y;\mathcal{B})$ forms a GDD$(6k+6,6h+6;2,1)$ and $(6k+6,6h+6) \in S(2)$.

(b) Let X and Y be two sets of size $6k+6$ and $6h+4$, respectively, $a \in Y$ and let $Y' = Y - \{a\}$. Choose $\mathcal{B}_1 \in \mathrm{KTS}(Y')$ with $\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_{3h+1}$ as its parallel classes. Since $\mathrm{STS}(X \cup Y')$ and $\mathrm{STS}(X \cup \{a\})$ are not empty, there exist $\mathcal{B}_2 \in \mathrm{STS}(X \cup Y')$ and $\mathcal{B}_3 \in \mathrm{STS}(X \cup \{a\})$. We now let $\mathcal{B}$ as

$$\mathcal{B}_2 \cup \mathcal{B}_3 \cup (a * \mathcal{P}_1) \cup \left(\bigcup_{i=2}^{3h+1}\mathcal{P}_i\right). \tag{3.3}$$

Thus, $(X,Y;\mathcal{B})$ forms a GDD$(6k+6,6h+4;2,1)$ and $(6k+6,6h+4) \in S(2)$.

Therefore, the proof is complete. □

Part of the proof of the following lemma is based on an existence of GDD$(4,4;2,3)$ which we now construct. Let $A = \{a,b,c,d\}$ and $B = \{1,2,3,4\}$. Then it is easy to check that $F \in \mathrm{GDD}(A,B;2,3)$, where $F = \{\{1,a,b\},\{1,a,c\},\{1,a,d\},\{2,b,c\},\{2,b,d\},\{2,b,a\},\{3,c,d\},\{3,c,a\},\{3,c,b\},\{4,d,a\},\{4,d,b\},\{4,d,c\},\{a,2,3\},\{a,2,4\},\{a,3,4\},\{b,1,3\},\{b,1,4\},\{b,3,4\},\{c,1,2\},\{c,1,4\},\{c,2,4\},\{d,1,2\},\{d,1,3\},\{d,2,3\}\}$.

Lemma 3.4. *Let h and k be nonnegative integers. Then*

$$(6k+2,6h+3),(6k+3,6h+2),(6k+5,6h+6),(6k+6,6h+5) \in S(3) \tag{3.4}$$

Proof

Case 1. Let X_k be a $(6k+2)$-set containing a_1, a_2, and Y_h be a $(6h+3)$-set containing $1,2,3$.

Subcase 1 ($k = 0$). Let $\mathcal{B}_0 = \{\{1,2,3\},\{1,2,3\},\{1,2,3\},\{1,a_1,a_2\},\{2,a_1,a_2\},\{3,a_1,a_2\}\}$, and we can see that $\mathcal{B}_0 \in \text{GDD}(X_0, Y_0; 3, 1)$. Suppose that $h \geq 1$. Since $X_0 \cup Y_h$ is a set of size $6h+5$, it follows by Theorem 2.2, that there exists a PBD$(6h+5)$, $(X_0 \cup Y_h, \mathcal{B}_1)$, in which $\{1,2,3,a_1,a_2\} \in \mathcal{B}_1$ and $6h^2 + 9h$ triples in $\mathcal{B}_1$. Let $\mathcal{B}_1' = \mathcal{B}_1 - \{\{1,2,3,a,b\}\}$. Since Y_h is a $(6h+3)$-set, it follows, by Theorem 2.5(c), that $\text{BIBD}(Y_h, 3, 2) \neq \emptyset$. Let $\mathcal{B}_2 \in \text{BIBD}(Y_h, 3, 2)$. It is easy to see that $(X_0, Y_h; \mathcal{B})$ forms a GDD$(2, 6h+3; 3, 1)$, where

$$\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1' \cup \mathcal{B}_2. \tag{3.5}$$

Subcase 2 ($k = 1$). A GDD$(8, 6h+3; 3, 1)$ can be constructed as follows. Let $X = A \cup B$, where A and B are sets of size four. It is clear that $\text{STS}(Y_h)$, $\text{STS}(A \cup Y_h)$, and $\text{STS}(B \cup Y_h)$ are not empty. It has been shown above that $\text{GDD}(A, B; 2, 3)$ is not empty. We now choose $\mathcal{B}_1 \in \text{STS}(Y_h)$, $\mathcal{B}_2 \in \text{STS}(A \cup Y_h)$, $\mathcal{B}_3 \in \text{STS}(B \cup Y_h)$, and $\mathcal{B}_4 \in \text{GDD}(A, B; 2, 3)$, and let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$. Then, $(X, Y_h; \mathcal{B})$ form a GDD$(8, 6h+3; 3, 1)$.

Subcase 3 ($k = 2$). We first consider the existence of GDD$(4, 10; 2, 3)$ with $A = \{0, 1, 2, \ldots, 9\}$ and $B = \{a_0, a_1, a_2, a_3\}$. Let $K(A)$ be the complete graph of order 10 with A as its vertex set. It is well known that K_{10} is 1-factorable. In other words, K_{10} can be decomposed as a union of nine edge-disjoint 1-factors. Consequently, K_{10} can be decomposed as a union of three edge-disjoint 3-factors. Also, K_{10} can be decomposed as a union of $10C_3$ and a 3-factor: ten triples $\{\{x, x+1, x+3\} : x = 0, 1, \ldots, 9\}$ and a 3-factor $\mathcal{F}_0$ of K_{10}, where

$$E(\mathcal{F}_0) = \bigcup_{i=0}^{9} \{\{i, i+4\}, \{i, i+5\}, \{i, i+6\}\}, \tag{3.6}$$

reducing arithmetic operations (mod 10). Therefore, $2K_{10}$ can be decomposed as a union of $10C_3$ and four 3-factors.

Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be a 3-factorization of K_{10} and $10C_3$ and $\mathcal{F}_0$ as described above. Then $(A, B; \mathcal{B})$ forms a GDD$(10, 4; 2, 3)$, where the collection

$$\mathcal{B} = \{10C_3\} \cup \bigcup_{i=0}^{3} (a_i + \mathcal{F}_i) \cup \mathcal{B}_1 \tag{3.7}$$

with $\mathcal{B}_1 = \{\{a_0, a_1, a_2\}, \{a_1, a_2, a_3\}, \{a_2, a_3, a_0\}, \{a_3, a_0, a_1\}\}$.

A GDD$(14, 6h+3; 3, 1)$ can be constructed as follows. Let $X = A \cup B$, where A and B are sets of size ten and four, respectively. It is clear that $\text{STS}(Y_h)$, $\text{STS}(A \cup Y_h)$, and $\text{STS}(B \cup Y_h)$ are not empty. It has been shown above that $\text{GDD}(A, B; 2, 3)$ is not empty. We now choose $\mathcal{B}_1 \in \text{STS}(Y_h)$, $\mathcal{B}_2 \in \text{STS}(A \cup Y_h)$, $\mathcal{B}_3 \in \text{STS}(B \cup Y_h)$, and $\mathcal{B}_4 \in \text{GDD}(A, B; 2, 3)$ and let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$. Then $(X, Y_h; \mathcal{B})$ form a GDD$(14, 6h+3; 3, 1)$.

Subcase 4 ($k \geq 3$). Let $A = \{a_1, a_2, a_3, a_4, a_5\}$. Suppose that $A \subseteq X_k$ and $X' = X_k - A$. Since X' is a $(6k-3)$-set, it follows that $\text{STS}(X') \neq \emptyset$ and $\text{KTS}(X') \neq \emptyset$. Choose $\mathcal{B}_1 \in \text{STS}(X')$ and let $\mathcal{K} \in \text{KTS}(X')$ with $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{3k-2}$ as its parallel classes. Let $\mathcal{B}_2 = \bigcup_{i=1}^{5} (a_i * \mathcal{P}_i) \cup \bigcup_{i=6}^{3k-1} \mathcal{P}_i$.

Since $X_k \cup Y_h$ is a set of size $6(k+h)+5$, we choose a PBD$(6(k+h)+5)$, $(X_k \cup Y_h, \mathcal{B}_3)$, as in Theorem 2.2 in which $A \in \mathcal{B}_3$. Let $\mathcal{B}_3' = \mathcal{B}_3 - \{A\}$. Since A is a 5-set and Y_h is a $(6h+3)$-set, it follows, by Theorem 2.5(c) and (d), that there exist $\mathcal{B}_4 \in \mathrm{BIBD}(5,3,3)$ and $\mathcal{B}_5 \in \mathrm{BIBD}(Y_h,3,2)$. Thus, we can see that $(X_k, Y_h; \mathcal{B})$ forms a GDD$(6k+2, 6h+3; 3, 1)$, where

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3' \cup \mathcal{B}_4 \cup \mathcal{B}_5. \tag{3.8}$$

Case 2. We now suppose that X and Y be sets of size $6k+5$ and $6h+6$, respectively. We suppose further that $a \in X$ and $X' = X - \{a\}$. By Lemma 3.3(b), we have GDD$(X', Y; 2, 1) \neq \emptyset$. Choose $\mathcal{B}_1 \in \mathrm{GDD}(X', Y; 2, 1)$ and $\mathcal{B}_2 \in \mathrm{STS}(Y \cup \{a\})$. By Theorem 2.6(e) that GDD$(\{a\}, X'; 1, 3) \neq \emptyset$. Choose $\mathcal{B}_3 \in \mathrm{GDD}(\{a\}, X'; 1, 3)$. Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$, and it is easy to see that $\mathcal{B} \in \mathrm{GDD}(X, Y; 3, 1)$.

Thus, $(6k+5, 6h+6) \in S(3)$. □

Lemma 3.5. *Let h and k be nonnegative integers. Then*

(a) $(6k+6, 6h+6) \in S(4)$ *and* $(6k+6, 6h+6) \in S(6)$*,*

(b) $(6k+6, 6h+4), (6k+4, 6h+6) \in S(4)$ *and* $(6k+6, 6h+4), (6k+4, 6h+6) \in S(6)$*,*

(c) $(6k+2, 6h+2)$ *with* h, k *not both zero,* $(6k+2, 6h+4), (6k+4, 6h+2) \in S(4)$*, and*

(d) $(6k+6, 6h+2), (6k+2, 6h+6) \in S(6)$*.*

Proof. The proofs of (a) and (b) follow from the results of Lemma 3.3(a), and (b), respectively, and Theorem 2.5(c).

(c) We have the following cases.

Case 1 (6k + 2, 6h + 2). Let X_k be a $(6k+2)$-set and Y_h be a $(6h+2)$-set. It is clear that $\mathrm{GDD}(X_0, Y_0; 4, 1) = \emptyset$. We now construct a GDD$(2, 8; 4, 1)$, $(X_0, Y_1; \mathcal{B})$, with $X_0 = \{x, y\}$, $Y_1 = \{a_1, a_2, \ldots, a_8\}$, $A = \{a_1, a_2, a_3\}$, $Y_1' = Y_1 - A$, and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$, where $\mathcal{B}_1 \in \mathrm{BIBD}(A, 3, 4)$, $\mathcal{B}_2 \in \mathrm{STS}(X \cup Y_1')$, $\mathcal{B}_3 = \bigcup_{i=1}^{3}(a_i + E(K(Y_1')))$, and $\mathcal{B}_4 = \{\{a_i, x, y\} : i = 1, 2, 3\}$. We now construct a GDD$(6k+2, 6h+2; 4, 1)$, $(X_k, Y_h; \mathcal{B})$, in general case, where $k \geq 0$ and $h \geq 1$. We first let $A = \{a_1, a_2, a_3\} \subseteq Y_h$, $Y_h' = Y_h - A$, and we will use a result on the existence of GDD$(1, 6h-1; 1, 4)$ which has been shown in Theorem 2.6(f), namely, GDD$(\{a\}, Y_h'; 1, 4) \neq \emptyset$. Therefore, we can choose $\mathcal{B}_i \in \mathrm{GDD}(\{a_i\}, Y_h'; 1, 4)$, $\mathcal{B}_4 \in \mathrm{BIBD}(A, 3, 4)$, $\mathcal{B}_5 \in \mathrm{STS}(X_k \cup Y_h')$, and $\mathcal{B}_6 = \bigcup_{i=1}^{3} F_i$, where $F_i \in \mathrm{STS}(X_k \cup \{a_i\})$. We can see that $\mathcal{B} \in \mathrm{GDD}(X_k, Y_h; 4, 1)$, where $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \cup \mathcal{B}_5 \cup \mathcal{B}_6$.

Case 2 (6k + 2, 6h + 4). Let X_k be a $(6k+2)$-set and Y_h be a $(6h+4)$-set. It is easy to see that GDD$(X_0, Y_0; 4, 1) \neq \emptyset$ by constructing $(X_0, Y_0; \mathcal{B})$ as follows. Let $X_0 = \{a, b\}$, $Y_0 = \{1, 2, 3, 4\}$, and $\mathcal{B} = \bigcup_{i=1}^{4}\{\{i, a, b\}\} \cup \mathcal{P}_2 \cup \mathcal{P}_2$, where $\mathcal{P}_2 = \{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}$. We now turn to more general cases. Suppose that $a \in Y_h$ and $Y_h' = Y_h - \{a\}$. Since Y_h' is a $(6h+3)$-set, it follows that KTS$(Y_h') \neq \emptyset$. Choose $\mathcal{B}_1 \in \mathrm{KTS}(Y_h')$ with parallel classes $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{3h+1}$. Let $\mathcal{B}_2 = (a * \mathcal{P}_1) \cup (a * \mathcal{P}_2) \cup (\bigcup_{i=3}^{3h+1} \mathcal{P}_i)$. We have shown in Lemma 3.4(d) that GDD$(X_k, Y_h'; 3, 1) \neq \emptyset$. Choose $\mathcal{B}_3 \in \mathrm{GDD}(X_k, Y_h'; 3, 1)$ and $\mathcal{B}_4 \in \mathrm{STS}(X_k \cup \{a\})$. We can see that $\mathcal{B} \in \mathrm{GDD}(X_k, Y_h; 4, 1)$, where $\mathcal{B} = \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$.

(d) Let X_k be a $(6k+6)$-set and Y_h be a $(6h+2)$-set. Let $X_0 = \{a_1, a_2, \ldots, a_6\}$ and $Y_0 = \{a, b\}$. Let $\mathcal{B}_1 = \{\{a_i, a, b\} : i = 1, 2, \ldots, 6\}$, $\mathcal{B}_2 \in \mathrm{BIBD}(X_0, 3, 6)$. Then $\mathcal{B}_1 \cup \mathcal{B}_2 \in \mathrm{GDD}(X_0, Y_0; 6, 1)$.

Next we will show that $\mathrm{GDD}(X_k, Y_0; 6, 1) \neq \emptyset$ by letting $X_k = \{a_1, a_2, \ldots, a_{6k+6}\}$, $Y_0 = \{a, b\}$, $A = \{a_1, a_2, \ldots, a_5\}$ and $X'_k = X_k - A$. Let $\mathcal{B}_1 = \{\{a_i, a, b\} : i = 1, 2, \ldots, 5\}$, $\mathcal{B}_2 \in \mathrm{STS}(\{a, b\} \cup X'_k)$, and $\mathcal{B}_3 \in \mathrm{BIBD}(A, 3, 6)$. Theorem 2.6(b) shows an existence of a $\mathrm{GDD}(1, 6h+1; 1, 6)$. Let $\mathcal{B}_4 = \bigcup_{i=1}^{5} \mathcal{B}'_i$, where $\mathcal{B}'_i \in \mathrm{GDD}(\{a_i\}, X'_k; 1, 6)$. It is easy to check that $\mathcal{B} \in \mathrm{GDD}(X_k, Y_0; 6, 1)$, where $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$.

Finally, let $h \geq 1$, $a \in Y$ and $Y'_h = Y_h - \{a\}$. We now choose $\mathcal{B}_1 \in \mathrm{BIBD}(X_k, 3, 4)$, $\mathcal{B}_2 \in \mathrm{BIBD}(Y'_h, 3, 4)$, $\mathcal{B}_3 \in \mathrm{STS}(X_k \cup Y'_h)$, and $\mathcal{B}_4 \in \mathrm{STS}(X_k \cup \{a\})$. By Theorem 2.6(b) that $\mathrm{GDD}(\{a\}, Y'_h; 1, 6) \neq \emptyset$. Choose $\mathcal{B}_5 \in \mathrm{GDD}(\{a\}, Y'_h; 1, 6)$. Thus, we can check that $\mathcal{B} \in \mathrm{GDD}(X_k, Y_h; 6, 1)$, where $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \cup \mathcal{B}_5$. Thus $(6k+6, 6h+2) \in S(6)$ for all positive integers h, k. □

Now we have an existence of a $\mathrm{GDD}(m, n; r, 1)$ for $r = 1, 2, \ldots, 6$ whenever m and n are not equal to 2, so we can readily extend to any $6t + r$ by the following lemma.

Lemma 3.6. *Let m and n be positive integers with $m \neq 2$ and $n \neq 2$. If there exists a* $\mathrm{GDD}(m, n; r, 1)$ *with $r \geq 1$, then a* $\mathrm{GDD}(m, n; 6t + r, 1)$, $t \geq 0$, *exists*

Proof. Let X be an m-set and Y be an n-set. By assumption we have $\mathrm{GDD}(X, Y; r, 1) \neq \emptyset$. Choose $\mathcal{B}_1 \in \mathrm{GDD}(X, Y; r, 1)$. Since m and n are not equal to 2, by Theorem 2.5(a) there exist $\mathcal{B}_2 \in \mathrm{BIBD}(X, 3, 6t)$ and $\mathcal{B}_3 \in \mathrm{BIBD}(Y, 3, 6t)$. It is easy to see that $(X, Y; \mathcal{B})$ forms a $\mathrm{GDD}(m, n; 6t + r, 1)$, where $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. Thus $(m, n) \in S(6t + r)$ with $r \geq 1$. □

Finally, we have the main result as in the following.

Theorem 3.7. *Let m and n be positive integers with $m \neq 2$ and $n \neq 2$. There exists a* $\mathrm{GDD}(m, n; \lambda, 1)$, $\lambda \geq 1$ *if and only if*

(1) $3 \mid \lambda[m(m-1) + n(n-1)] + 2mn$ *and*

(2) $2 \mid \lambda(m-1) + n$ *and* $2 \mid \lambda(n-1) + m$.

Proof. The proof follows from Lemmas 3.1–3.6. □

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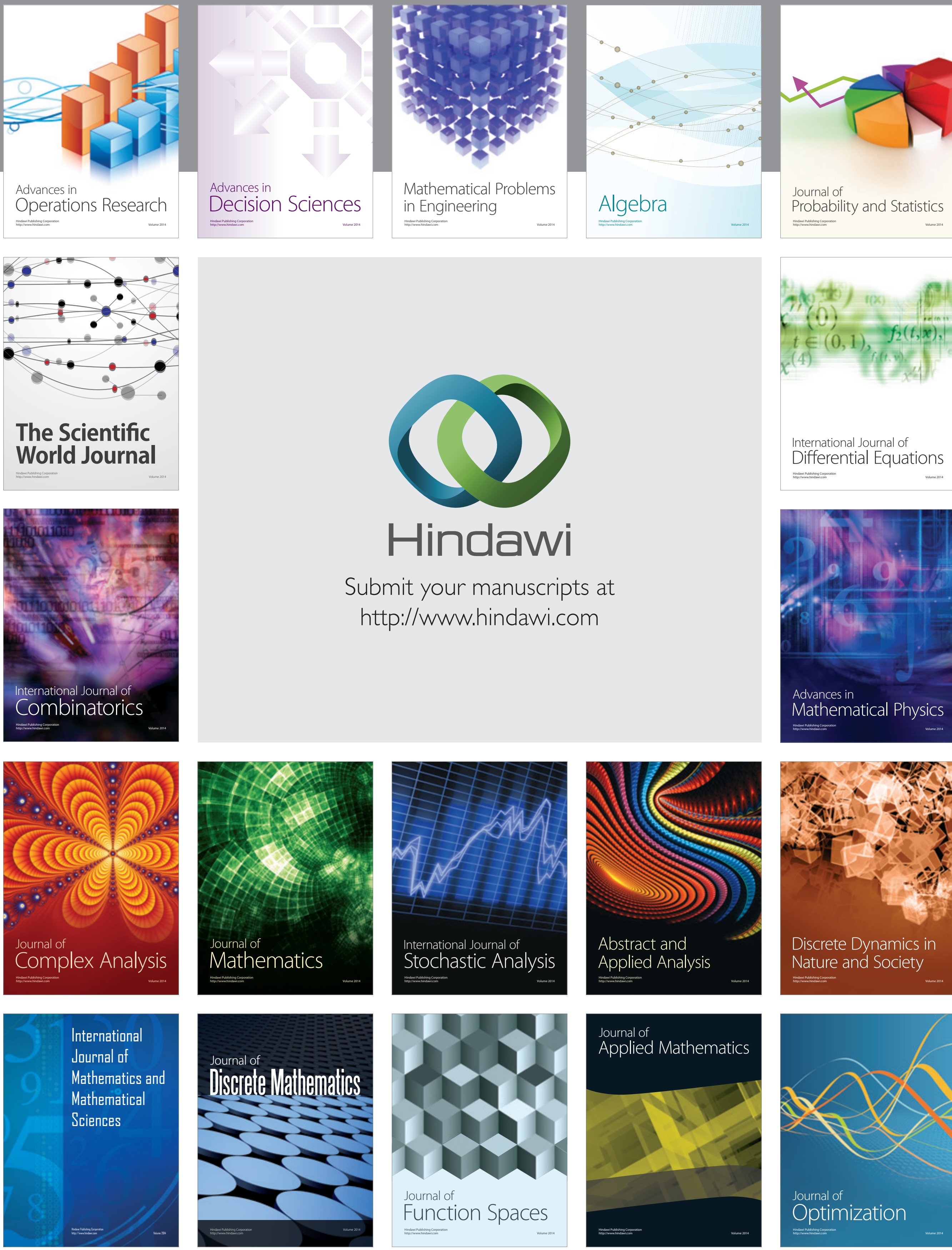
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