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Research Article

Integrability of the Bakirov System: A Zero-Curvature Representation

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For the Bakirov system, which is known to possess only one higher-order local generalized symmetry, we explicitly find a zero-curvature representation containing an essential parameter.

1. Introduction

Bakirov [1] discovered that the following evidently integrable triangular system of a linear PDE with a source determined by another linear PDE,

$$u_t = u_{xxxx} + v^2, \qquad v_t = \frac{1}{5} v_{xxxx}, \tag{1.1}$$

possesses only one higher-order (x, t)-independent local generalized symmetry of order not exceeding 53, namely, a sixth-order one. Beukers et al. [2] extended the result of Bakirov to (x, t)-independent local generalized symmetries of unlimited order. Bilge [3] found a formal recursion operator for the Bakirov system (1.1) and showed that the structure of the operator's nonlocal terms prevents the generation of local higher symmetries from the known sixth-order symmetry. Sergyeyev [4] showed that the existence of such a formal recursion operator is essentially a consequence of the triangular form of the Bakirov system. Finally, Sergyeyev [5] proved that the Bakirov system possesses only one higher-order local generalized symmetry, namely, the sixth-order one found by Bakirov, even if (x, t)-dependent symmetries are taken into account. Due to these results, the Bakirov system looks quite different from other known integrable systems which possess infinite algebras of higher symmetries.

In the present paper, we explicitly find a linear spectral problem associated with the Bakirov system (1.1), in the form of a zero-curvature representation containing an essential parameter. Section 2 gives necessary preliminaries. In Section 3, we find for the system (1.1) a 4×4 zero-curvature representation containing a parameter, and we prove in Section 4 that this parameter cannot be removed by gauge transformations. Section 5 gives concluding remarks. We believe that the obtained Lax pair of the Bakirov system will be useful for future studies on the relation between Lax pairs and higher symmetries of integrable PDEs.

2. Preliminaries

A zero-curvature representation (ZCR) of a system of PDEs (see, e.g., [6] and references therein) is the compatibility condition

$$D_t X = D_x T - [X, T] \tag{2.1}$$

of the overdetermined linear problem

$$\Psi_x = X\Psi, \qquad \Psi_t = T\Psi, \tag{2.2}$$

where D_t and D_x stand for the total derivatives, X and T are $n \times n$ matrix functions of independent and dependent variables and finite-order derivatives of dependent variables, the square brackets denote the matrix commutator, Ψ is a column of n functions of independent variables, and the ZCR (2.1) is satisfied by any solution of the represented system of PDEs. Two ZCRs are equivalent if they are related by a gauge transformation

$$\begin{gathered} X' = GXG^{-1} + (D_x G)G^{-1}, \\ T' = GTG^{-1} + (D_t G)G^{-1}, \\ \Psi' = G\Psi, \quad \det G \neq 0 \end{gathered} \tag{2.3}$$

of the linear problem (2.2), where G is a $n \times n$ matrix function of independent and dependent variables and finite-order derivatives of dependent variables.

3. Zero-Curvature Representation

Our aim is to find a ZCR (2.1) of the Bakirov system (1.1). Assuming for simplicity that $X = X(u, v)$ and $T = T(u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx})$ and using (1.1), we rewrite (2.1) in the equivalent form

$$\left(u_{xxxx} + v^2\right)\frac{\partial X}{\partial u} + \frac{1}{5}v_{xxxx}\frac{\partial X}{\partial v} - D_x T + [X, T] = 0. \tag{3.1}$$

Since (3.1) cannot be a system of ODEs restricting solutions of (1.1), it must be an identity with respect to u and v, and therefore u, v, and all derivatives of u and v should be treated as formally independent quantities in (3.1). This allows us to solve (3.1) and obtain the following expressions for the matrices X and T:

$$\begin{aligned} X &= Pu + Qv + R,\\ T &= Pu_{xxx} + \frac{1}{5}Qv_{xxx} + [R,P]u_{xx} + \frac{1}{5}[R,Q]v_{xx} + [R,[R,P]]u_x \\ &\quad + \frac{1}{5}[R,[R,Q]]v_x + [R,[R,[R,P]]]u + \frac{1}{5}[R,[R,[R,Q]]]v + S, \end{aligned} \tag{3.2}$$

where P, Q, R, and S are any $n \times n$ constant matrices satisfying the following commutator relations:

$$\begin{gathered} P = -\frac{1}{5}[Q,[R,[R,[R,Q]]]], \qquad [P,Q] = 0, \\ [P,[Q,R]] = 0, \qquad [P,[R,P]] = 0, \qquad [Q,[R,Q]] = 0, \\ [[R,P],[R,Q]] = 0, \qquad [P,[R,[R,[R,P]]]] = 0, \\ [[R,P],[R,[R,Q]]] = 0, \qquad [P,S] + [R,[R,[R,[R,P]]]] = 0, \\ [Q,S] + \frac{1}{5}[R,[R,[R,[R,Q]]]] = 0, \qquad [R,S] = 0. \end{gathered} \tag{3.3}$$

We have to find a solution of (3.3) which should be nontrivial in the following sense: X contains both u and v, that is, (2.1) gives expressions for both equations of (1.1); $[X,T] \neq 0$, because commutative ZCRs are simply matrices of conservation laws (for this reason, and without loss of generality, the matrices P, Q, R, and S are set to be traceless); X contains a parameter (essential or spectral) which cannot be removed (gauged out) by gauge transformations (2.3). We solve (3.3), using the *Mathematica* computer algebra system [7], successively taking Q in all possible Jordan forms, suppressing the excessive arbitrariness of solutions by transformations (2.3) with constant G and increasing the matrix dimension n if necessary. The cases of 2×2 and 3×3 matrices contain no nontrivial solutions of (3.3), while the 4×4 case gives us the following:

$$P = \begin{pmatrix} 0 & 0 & \frac{8}{5}z\left(-3+6z-11z^2\right)\alpha^3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.4}$$

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.5}$$

$$R = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & z\alpha & 0 & \alpha \\ 0 & 0 & (-1+2z)\alpha & 0 \\ 0 & (-3+6z-11z^2)\alpha & 0 & -3z\alpha \end{pmatrix}, \tag{3.6}$$

$$S = \begin{pmatrix} S_{11} & 0 & 0 & 0 \\ 0 & S_{22} & 0 & S_{24} \\ 0 & 0 & S_{33} & 0 \\ 0 & S_{42} & 0 & S_{44} \end{pmatrix} \tag{3.7}$$

with

$$\begin{gathered} S_{11} = \frac{8}{5}\left(2 - 12z + 21z^2 - 18z^3 + 3z^4\right)\alpha^4, \\ S_{22} = \frac{8}{5}\left(3 - 10z + 15z^2 - 4z^4\right)\alpha^4, \\ S_{24} = \frac{8}{5}\left(-1 + 3z + z^2 - 3z^3\right)\alpha^4, \\ S_{33} = \frac{8}{5}\left(-8 + 28z - 39z^2 + 22z^3 - 7z^4\right)\alpha^4, \\ S_{42} = \frac{8}{5}\left(3 - 15z + 26z^2 - 18z^3 - 29z^4 + 33z^5\right)\alpha^4, \\ S_{44} = \frac{8}{5}\left(3 - 6z + 3z^2 - 4z^3 + 8z^4\right)\alpha^4, \end{gathered} \tag{3.8}$$

where α is an arbitrary parameter and z is any of the four roots

$$\begin{gathered} z_{1,2} = \frac{1}{20}\left(9 + i\sqrt{39} \pm \sqrt{-138 - 2i\sqrt{39}}\right), \\ z_{3,4} = \frac{1}{20}\left(9 - i\sqrt{39} \pm \sqrt{-138 + 2i\sqrt{39}}\right) \end{gathered} \tag{3.9}$$

of the algebraic equation

$$3 - 12z + 21z^2 - 18z^3 + 10z^4 = 0. \tag{3.10}$$

Consequently, a nontrivial ZCR (2.1) of the Bakirov system (1.1) is determined by the following 4×4 matrices X and T:

$$X = \begin{pmatrix} \alpha & v & \frac{8}{5}z(-3+6z-11z^2)\alpha^3 u & 0 \\ 0 & z\alpha & v & \alpha \\ 0 & 0 & (-1+2z)\alpha & 0 \\ 0 & (-3+6z-11z^2)\alpha & 0 & -3z\alpha \end{pmatrix}, \tag{3.11}$$

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ 0 & T_{22} & T_{23} & T_{24} \\ 0 & 0 & T_{33} & 0 \\ 0 & T_{42} & T_{43} & T_{44} \end{pmatrix} \tag{3.12}$$

with

$$\begin{aligned}
T_{11} &= \frac{8}{5}\left(2-12z+21z^2-18z^3+3z^4\right)\alpha^4, \\
T_{12} &= \frac{1}{5}\left(-4\left(2-3z+6z^2+3z^3\right)\alpha^3 v - 2\left(1-2z+5z^2\right)\alpha^2 v_x + (1-z)\alpha v_{xx} + v_{xxx}\right), \\
T_{13} &= \frac{8}{5}z\left(-3+6z-11z^2\right)\alpha^3\left(8(1-z)^3\alpha^3 u + 4(1-z)^2\alpha^2 u_x + 2(1-z)\alpha u_{xx} + u_{xxx}\right), \\
T_{14} &= \frac{1}{5}\alpha\left(4z(-3+z)\alpha^2 v - 2(1+z)\alpha v_x - v_{xx}\right), \\
T_{22} &= \frac{8}{5}\left(3-10z+15z^2-4z^4\right)\alpha^4, \\
T_{23} &= \frac{1}{5}\left(4\left(-2+9z-18z^2+19z^3\right)\alpha^3 v - 2\left(1-2z+5z^2\right)\alpha^2 v_x + (1-z)\alpha v_{xx} + v_{xxx}\right), \\
T_{24} &= \frac{8}{5}\left(-1+3z+z^2-3z^3\right)\alpha^4, \\
T_{33} &= \frac{8}{5}\left(-8+28z-39z^2+22z^3-7z^4\right)\alpha^4, \\
T_{42} &= \frac{8}{5}\left(3-15z+26z^2-18z^3-29z^4+33z^5\right)\alpha^4, \\
T_{43} &= \frac{1}{5}\left(-3+6z-11z^2\right)\alpha\left(4z(-3+5z)\alpha^2 v + (2-6z)\alpha v_x + v_{xx}\right), \\
T_{44} &= \frac{8}{5}\left(3-6z+3z^2-4z^3+8z^4\right)\alpha^4.
\end{aligned} \tag{3.13}$$

Let us remember that, in (3.11) and (3.13), z is any of the four roots (3.9) of (3.10) and α is an arbitrary parameter.

4. Essential Parameter

Now, we have to prove that α is an essential parameter, that is, that α cannot be removed from the obtained ZCR by a gauge transformation (2.3). We do this, using the method of gauge-invariant description of ZCRs of evolution equations [6, 8] (see also the independent work [9], based on the very general and abstract study of ZCRs [10]). Since the matrix X (3.11) does not contain derivatives of u and v, the two characteristic matrices of the obtained ZCR are simply $C_u = \partial X/\partial u = P$ and $C_v = \partial X/\partial v = Q$. We take one of them, $C_u = P$ (3.4), introduce the operator ∇_x, defined as $\nabla_x M = D_x M - [X, M]$ for any 4×4 matrix function M, compute $\nabla_x C_u$, and find that

$$\nabla_x C_u + 2(1-z)\alpha C_u = 0. \tag{4.1}$$

In the terminology of [6, 8], relation (4.1) is one of the two closure equations of the cyclic basis. The scalar coefficient $2(1-z)\alpha$ in (4.1) is an invariant with respect to the gauge transformations (2.3), because the matrices C_u and $\nabla_x C_u$ are transformed as $C'_u = GC_uG^{-1}$ and $\nabla'_x C'_u = G(\nabla_x C_u)G^{-1}$ (see [8, 9]). The explicit dependence of the invariant $2(1-z)\alpha$ on the parameter α shows that this parameter cannot be "gauged out" from the matrix X (3.11).

5. Conclusion

We believe that the ZCR of the Bakirov system, found in this paper, can be used in future studies of the relation between Lax pairs, recursion operators, generalized symmetries, and conservation laws. The following problems arise from the obtained result. Is it possible to derive a recursion operator for the Bakirov system from the obtained ZCR, for example, through the cyclic basis technique [6]? If yes, is that recursion operator different from the formal recursion operator found in [3]? And why does not the obtained ZCR generate an infinite sequence of nontrivial local conservation laws for the Bakirov system, for example, through the standard techniques described in [11]?

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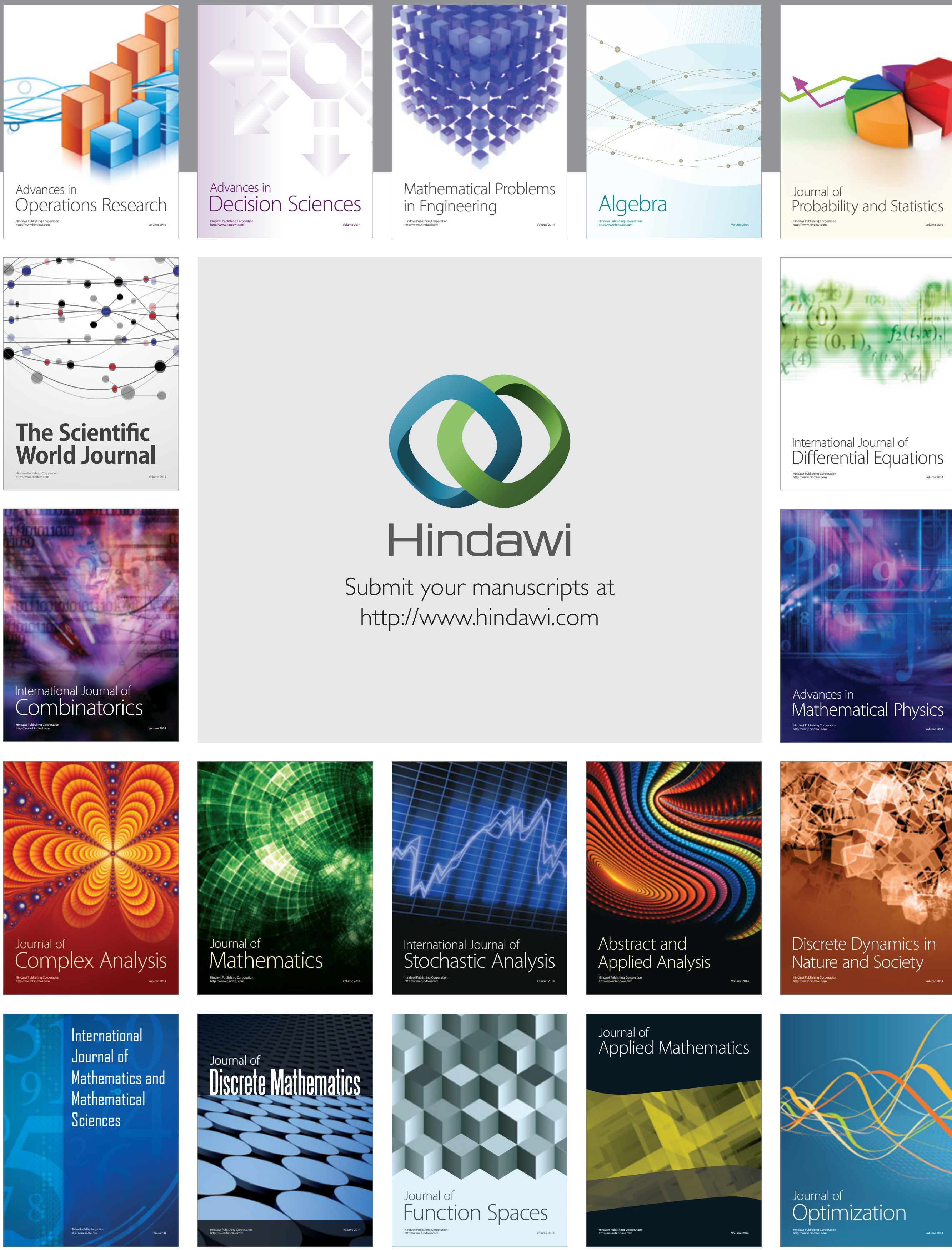
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