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Research Article

A Moment Problem for Discrete Nonpositive Measures on a Finite Interval

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We will estimate the upper and the lower bounds of the integral $\int_0^1 \Omega(t)d\mu(t)$, where μ runs over all discrete measures, positive on some cones of generalized convex functions, and satisfying certain moment conditions with respect to a given Chebyshev system. Then we apply these estimations to find the error of optimal shape-preserving interpolation.

1. Introduction

Let $\{u_0, \ldots, u_k\}$ be a Chebyshev system on $[0,1]$. A function f, defined on $[0,1]$, is said to be *convex relative to the system* $\{u_0, \ldots, u_k\}$ (we will write $f \in C(u_0, \ldots, u_k)$) if

$$\begin{vmatrix} u_0(t_0) & u_0(t_1) & \cdots & u_0(t_{k+1}) \\ \cdots & \cdots & \cdots & \cdots \\ u_k(t_0) & u_k(t_1) & \cdots & u_k(t_{k+1}) \\ f(t_0) & f(t_1) & \cdots & f(t_{k+1}) \end{vmatrix} \geq 0 \tag{1.1}$$

for all choices of $0 < t_0 < t_1 < \cdots < t_{k+1} < 1$.

In particular, if $u_0(x) \equiv 1$, then $C(u_0)$ is a cone of all increasing functions on $(0,1)$. If $u_0(x) \equiv 1$, $u_1(x) \equiv x$, then $C(u_0, u_1)$ is a cone of all convex functions on $(0,1)$. The review of some results of the theory of generalized convex functions can be found in the book in [1].

Let $k \geq 0$, $\sigma = (\sigma_0, \ldots, \sigma_k) \in \mathbb{R}^{k+1}$ with $\sigma_i \in \{-1, 0, 1\}$, $\sigma_0, \sigma_k \neq 0$. As usual, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{R}^n$ denotes the vector space of all real n-tuples (columns).

Denote by W_{l+1} the set of all continuous functions defined on $[0,1]$ and convex relative to the system $\{u_0, \ldots, u_l\}$, that is,

$$W_{l+1} := \{f \in \mathbb{C}[0,1] : f \in C(u_0, \ldots, u_l)\}, \quad l = 0, \ldots, k-1. \tag{1.2}$$

Denote $W_0 := \{f \in \mathbb{C}[0,1] : f \geq 0\}$. Following ideas of [2] we consider the cone

$$W_{0,k}(\sigma) = \bigcap_{l=0}^{k} \sigma_l W_l. \tag{1.3}$$

For example, if $k = 2$, $\sigma = (1, 0, 1)$, $u_0(x) \equiv 1$, $u_1(x) \equiv x$, then $W_{0,2}(\sigma)$ is the cone of all positive and convex continuous functions defined on $[0,1]$.

Let $0 \leq x_1 < x_2 < \cdots < x_n \leq 1$, and denote $Ig = (g(x_1), \ldots, g(x_n))^T \in \mathbb{R}^n$, $g \in \mathbb{C}[0,1]$. Let

$$V_{0,k}(\sigma) := \{If \in \mathbb{R}^n : f \in W_{0,k}(\sigma)\}. \tag{1.4}$$

Denote by

$$V_{0,k}^*(\sigma) := \left\{\mu \in \mathbb{R}^n : (If)^T \mu \geq 0 \ \forall If \in V_{0,k}(\sigma)\right\} \tag{1.5}$$

the dual cone.

Let $\{f_0, \ldots, f_p\}$ be a Chebyshev system on $[0,1]$. Let us consider the moment space with respect to the system $\{f_0, \ldots, f_p\}$ defined by

$$M_{p+1,k}(\sigma) := \left\{c = (c_0, \ldots, c_p) \in \mathbb{R}^{p+1} : (If_i)^T \mu = c_i, \ i = 0, \ldots, p\right\}, \tag{1.6}$$

where μ runs over $V_{0,k}^*(\sigma)$.

Given $c^0 = (c_0^0, c_1^0, \ldots, c_p^0) \in M_{p+1,k}(\sigma)$, denote

$$K_{0,k}\left(c^0\right) = \left\{\mu \in V_{0,k}^*(\sigma) : (If_i)^T \mu = c_i^0, \ i = 0, 1, \ldots, p\right\}. \tag{1.7}$$

In this paper we find the lower and upper bound of the value $(If)^T \mu$, where $\mu \in K_{0,k}(c^0)$. This problem is similar to the classical moment problem (see, e.g., [1, Chapter 2] and [3, Chapter 4]), but the measure we are interested in is discrete and positive on some cones of generalized convex functions.

The main result of this paper can be stated as follows.

Theorem 1.1. *Let c^0 be an internal point of $M_{p+1,k}(\sigma)$, and let $f \in \mathbb{C}[0,1]$ be such that P_+ and P_- are nonempty sets, then*

$$\sup_{\mu \in K_{0,k}(c^0)} (If)^T \mu = \inf_{g \in P_-} g\left(c^0\right), \tag{1.8}$$

$$\inf_{\mu \in K_{0,k}(c^0)} (If)^T \mu = \sup_{g \in P_+} g\left(c^0\right), \tag{1.9}$$

where

$$\begin{aligned} P_+ &= \left\{g \in \operatorname{Span}\left\{f_0, \ldots, f_p\right\} : I(g-f) \in V_{0,k}(\sigma)\right\}, \\ P_- &= \left\{g \in \operatorname{Span}\left\{f_0, \ldots, f_p\right\} : I(f-g) \in V_{0,k}(\sigma)\right\}. \end{aligned} \tag{1.10}$$

Note that the motivation of consideration of the problems

$$\sup_{\mu \in K_{0,k}(c^0)} (If)^T \mu, \qquad \inf_{\mu \in K_{0,k}(c^0)} (If)^T \mu \tag{1.11}$$

has arisen from the theory of shape-preserving approximation. As we will show in Section 3, the estimation of the error of optimal recovery by means of shape-preserving algorithms can be reduced to the problems of type (1.11).

2. Duality Theorems and the Proof of Theorem 1.1

First we consider a conic programming problem, and we prove weak and strong duality theorems relative to this problem.

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $m < n$, $m, n \in \mathbb{N}$.

Consider the problem

$$\min_{x \in M} c^T x, \ \text{ where } M := \left\{x \in \mathbb{R}^n : Ax = b, x \in V_{0,k}^*(\sigma)\right\}. \tag{2.1}$$

It follows from [4], that the dual problem can be written in the following way:

$$\max_{y \in M^*} b^T y, \quad \text{where } M^* := \left\{y \in \mathbb{R}^m : A^T y + s = c, s \in V_{0,k}(\sigma)\right\}. \tag{2.2}$$

Lemma 2.1. *The set $Q := \{Ax : x \in V_{0,k}^*(\sigma)\}$ is a nonempty, convex, closed set.*

Proof. It is clear that Q is a convex set. Moreover, since the origin of $\mathbb{R}^n$ belongs to Q, the set Q is nonempty. To show that Q is closed, suppose that q^k is a sequence in Q, such that $q^k \to q$. Our goal is to show that $q \in Q$.

Consider the optimization problem

$$\min_{x \in V_{0,k}^*(\sigma)} \|q - Ax\|_\infty, \tag{2.3}$$

where $\|\cdot\|_\infty$ defined by $\|a\|_\infty = \max_i |a_i|$, $a = (a_i)_{i=1}^n \in \mathbb{R}^n$.

It can be rewritten as follows:

$$\min\ t, \tag{2.4}$$

where minimum is taken over all $x \in V_{0,k}^*(\sigma)$ such that

$$t \geq q_i - \left(Ax^T\right)_{i'} \quad i = 1, \ldots, n, \tag{2.5}$$

$$-t \geq -q_i + \left(Ax^T\right)_{i'} \quad i = 1, \ldots, n. \tag{2.6}$$

Note that $x \in V_{0,k}^*(\sigma)$ if and only if $x^T v^* \geq 0$, where v^* runs over all extreme rays of the cone $V_{0,k}(\sigma)$. Thus, the set of all $x \in V_{0,k}^*(\sigma)$ satisfying inequalities (2.5), (2.6) is a nonlinear polyhedron. It is obvious that there is an optimal solution (x^*, t^*) such that $t^* \geq 0$. Assume that $t^* > 0$. Since $q^k \to q$, there is an index k' such that $\|q - q^{k'}\|_\infty = t' < t^*$, where $q^{k'} \in Q$. Let $x' \in V_{0,k}^*(\sigma)$ be such that $q^{k'} = Ax'$. It implies that (x', t') is a feasible solution of the system (2.5), (2.6). It contradicts that (x^*, t^*) is optimal. Thus, we have $t^* = 0$ which implies $q = Ax^*$, and therefore $q \in Q$. □

Lemma 2.2. *Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$. Only one of the following sets is not empty:*

$$\left\{x \in \mathbb{R}^n : Ax = b, x \in V_{0,k}^*(\sigma)\right\}, \tag{2.7}$$

$$\left\{y \in \mathbb{R}^m : -A^T y \in V_{0,k}(\sigma), b^T y > 0\right\}. \tag{2.8}$$

Proof. Assume the opposite, that is, there exist $x_* \in \mathbb{R}^n$ and $y_* \in \mathbb{R}^m$ which belong to the sets (2.7) and (2.8), respectively. It follows from $-A^T y_* \in V_{0,k}(\sigma)$ and $x_* \in V_{0,k}^*(\sigma)$ that $0 \geq (A^T y_*)^T x_* = (y_*^T A)x_* = y_*^T(Ax_*) = y_*^T b = b^T y_*$. This contradicts to $b^T y_* > 0$. Hence, we conclude that at most one of (2.7) or (2.8) is not empty.

Now, it remains to show that if (2.7) is empty, then (2.8) is not. Consider the nonempty closed and convex set $Q = \{Ax, x \in V_{0,k}^*(\sigma)\}$. Since (2.7) is empty, we have $b \notin Q$. It follows from the separating hyperplane theorem that there exists $y \in \mathbb{R}^m$ such that $y^T(Ax) < y^T b$ for all $x \in V_{0,k}^*(\sigma)$. As $0 \in Q$, $b^T y > 0$. Since $b^T y > 0$, the definition of set (2.7) implies $-A^T y \in V_{0,k}(\sigma)$. □

Lemma 2.3. *Suppose the feasible sets M and M^* of problems* (2.1) *and* (2.2) *are both not empty. Let $x_* \in \mathbb{R}^n$ be the optimal solution of* (2.1) *and $y_* \in \mathbb{R}^m$ the optimal solution of* (2.2). *Then $b^T y_* \leq c^T x_*$.*

Proof. The proposition follows from

$$b^T y_* = (Ax_*)^T y_* = \left(x_*^T A^T\right) y_* \leq x_*^T \left(A^T y_* + s\right) = x_*^T c, \tag{2.9}$$

where $s \in V_{0,k}(\sigma)$, $x_* \in V_{0,k}^*(\sigma)$, and $A^T y_* + s = c$ by definition. □

Theorem 2.4 (strong duality theorem). *If the problem* (2.1) *has an optimal solution* $x_* \in \mathbb{R}^n$, *then the problem* (2.2) *also has an optimal solution* $y_* \in \mathbb{R}^m$ *and*

$$b^T y_* = c^T x_*. \tag{2.10}$$

Proof. Assume that the feasible set M of the problem (2.1) is not empty, and denote by $x_* \in \mathbb{R}^n$ the optimal solution of the problem (2.1). Let us show that the set of all (x,t), $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, satisfying

$$Ax - bt = 0, \quad c^T x - \left(c^T x_*\right)t = -1 < 0, \quad x \in V_{0,k}^*(\sigma),\ t \geq 0, \tag{2.11}$$

is empty.

Assume that (x', t') satisfies the system (2.11) and $t' > 0$. Then (x'/t') is a solution of (2.1) and $c^T(x'/t') < c^T x_*$, which contradicts to the optimality of x_*. On the other hand, if $t' = 0$, then $x_* + x'$ is a solution of (2.1), and $c^T(x_* + x') = c^T x_* - 1 < c^T x_*$, which contradicts to the optimally of x_*.

Now, it follows from [4]that there is $y_* \in \mathbb{R}^m$ such that $c - A^T y_* \geq 0$ and $-c^T x_* + b^T y_* \geq 0$. It implies that y_* is a solution of (2.2). Moreover, it follows from Lemma 2.2 that $b^T y_* \leq c^T x_*$. □

Now we are ready to prove Theorem 1.1. Note that the set $M_{p+1,k}(\sigma)$ defined in Section 1 is a closed convex cone. Let $0 \leq t_0 < t_1 < \cdots < t_p \leq 1$ be arbitrary points in $[0,1]$. Since $\{f_0, \ldots, f_p\}$ is a Chebyshev system, we may conclude that points

$$c_i = \left(f_0(t_i), \ldots, f_p(t_i)\right) \in M_{p+1,k}(\sigma), \quad i = 0, \ldots, p, \tag{2.12}$$

are linearly independent. Thus, the cone $M_{p+1,k}(\sigma)$ is not contained in any p-dimensional subspaces.

Proof of Theorem 1.1. We will prove (1.9). Consider the conic programming problem

$$\min_{\mu \in K_{0,k}(c^0)} (If)^T \mu. \tag{2.13}$$

Denote

$$M_{0,k}^*(\sigma) := \left\{ y \in \mathbb{R}^{p+1} : I\left(f - \sum_{i=0}^{p} y_i f_i \right) \in V_{0,k}(\sigma) \right\}. \tag{2.14}$$

The dual problem of the problem (2.13) is the problem

$$\max_{y \in M_{0,k}^*(\sigma)} c^T y. \tag{2.15}$$

It follows from Lemma 2.3 that

$$\inf_{\mu\in K_{0,k}(c^0)} (If)^T\mu = \max_{y\in M^*_{0,k}(\sigma)} y^T c^0. \tag{2.16}$$

Equality (1.9) follows from the equality

$$\max_{y\in M^*_{0,k}(\sigma)} y^T c^0 = \sup_{g\in P_+} g\left(c^0\right). \tag{2.17}$$

Equality (1.8) can be proved similarly. □

3. The Error of Optimal Interpolation by Means of Shape Preserving Algorithms

Let $0 \le x_1 < x_2 < \cdots < x_n \le 1$, $If = (f(x_1),\ldots,f(x_n))^T \in \mathbb{R}^n$, $f \in \mathbb{C}[0,1]$. Let Φ denote the class of all linear algorithms $A : \mathbb{R}^n \to \mathbb{R}$ based on information I. The error of the problem of optimal linear interpolation of $f \in \mathbb{C}[0,1]$ at point $\zeta \in [0,1]$ on the basis of information If, $f \in W$, is defined by

$$e_\zeta(f,I) := \inf_{A\in\Phi}\left|f(\zeta) - A(If)\right|. \tag{3.1}$$

Note that for every $A \in \Phi$ there exists $\mu \in \mathbb{R}^n$ such that $A(If) = (If)^T\mu$ for all $f \in \mathbb{C}[0,1]$. Then

$$e_\zeta(f,I) = \inf_{\mu\in\mathbb{R}^n}\left|f(\zeta) - (If)^T\mu\right|. \tag{3.2}$$

Optimal recovery problems arise in many applications of the approximation theory and have received much attention. In-depth study can be found in papers [5, 6], and in book in [7].

In various applications it is necessary to approximate a function preserving properties such as monotonicity, convexity, and concavity. In the theory of shape-preserving approximation by means of polynomials and splines the last 25 years have seen extensive research. The most significant results were summarized in [8, 9].

If a function f has some shape properties, then it usually means that the element f belongs to a cone in $\mathbb{C}[0,1]$.

One of the tasks of the theory of shape-preserving approximation is to estimate value (3.1), where infimum is taken over all linear algorithms, which are satisfied additional (shape-preserving) properties.

Let K be a cone in $\mathbb{C}\ [0,1]$. Let $\Phi(K)$ denote the class of all linear algorithms $A : \mathbb{R}^n \to \mathbb{R}$, based on information I and such that $A(v) \ge 0$ for all $v \in V$, $V := \{If : f \in K\} \subset \mathbb{R}^n$.

Define by

$$e_\zeta(f,I,K) := \inf_{A\in\Phi(K)}\left|f(\zeta) - A(If)\right| \tag{3.3}$$

the error of optimal linear interpolation of $f \in \mathbb{C}[0,1]$ at point $\zeta \in [0,1]$ on the basis of information If, $f \in W$, *with respect to the cone* K.

Denote by $V^* := \{u \in \mathbb{R}^n : u^T v \geq 0 \text{ for all } v \in V\}$ the cone dual to V. Note that for every $A \in \Phi(K)$ there exists $\mu \in V^*$ such that $A(If) = (If)^T \mu$ for all $f \in \mathbb{C}[0,1]$. Then

$$e_\zeta(f, I, K) = \inf_{\mu \in V^*} \left| f(\zeta) - (If)^T \mu \right|. \tag{3.4}$$

The next proposition demonstrates how we can use Theorem 1.1 to obtain the error of optimal linear interpolation.

We will consider the case $k = 2$, $\sigma = (1, 0, 1)$, $u_0(x) \equiv 1$, $u_1(x) \equiv x$. Then $W_{0,2}(\sigma)$ is the cone of all positive and convex functions on $[0,1]$, $V_{0,2}(\sigma) = \{If : f \in W_{0,2}(\sigma)\}$ and, $V^*_{0,2}(\sigma) = \{\mu \in \mathbb{R}^n : (If)^T \mu \geq 0 \text{ for all } If \in V_{0,2}(\sigma)\}$.

In the next proposition we consider the problem of interpolation by means of shape-preserving algorithms A, which have some properties of shape-preserving projections (i.e., $A(f) = f$ for every f from a certain finite-dimensional subspace). Note that a deep study of linear shape preserving projections was undertaken in papers [10–12].

Corollary 3.1. *Let* $f \in \mathbb{C}[0,1]$ *be a strictly convex function on* $[0,1]$, $\zeta \in [0,1]$, *and let* $1 \leq k \leq n-1$ *be such that* $x_k < \zeta < x_{k+1}$. *Denote*

$$D := \left\{ \mu \in V^*_{0,2}(\sigma) : (Iu_i)^T \mu = u_i(\zeta),\ i = 0, 1 \right\}. \tag{3.5}$$

Then

$$\inf_{\mu \in D} \left| f(\zeta) - (If)^T \mu \right| = (x_{k+1} - \zeta)(\zeta - x_k)[x_k, \zeta, x_{k+1}]f, \tag{3.6}$$

where $[x_k, \zeta, x_{k+1}]f$ *denotes the divided difference of* f *at* $x_k < \zeta < x_{k+1}$.

Proof. Consider the problem

$$\min_{\mu \in D} \left(f(\zeta) - (If)^T \mu \right). \tag{3.7}$$

It follows from Theorem 1.1 that

$$\min_{\mu \in D} \left(f(\zeta) - (If)^T \mu \right) = \max\left(f(\zeta) + y_0 u_0(\zeta) + y_1 u_1(\zeta) \right), \tag{3.8}$$

where maximum is taken over all $y_0, y_1 \in \mathbb{R}$ such that $I(f + y_0 u_0 + y_1 u_1) \in V_{0,2}(\sigma)$.

It follows from $x_k < \zeta < x_{k+1}$ and strictl convexity of f that

$$\max\left(f(\zeta) + y_0 u_0(\zeta) + y_1 u_1(\zeta) \right) = (x_{k+1} - \zeta)(\zeta - x_k)[x_k, \zeta, x_{k+1}]f. \tag{3.9}$$

Consider the problem

$$\min_{\mu\in D}\left(-f(\zeta)+(If)^T\mu\right). \tag{3.10}$$

It follows from Theorem 1.1 that

$$\min_{\mu\in D}\left(-f(\zeta)+(If)^T\mu\right)=\max\left(-f(\zeta)+y_0u_0(\zeta)+y_1u_1(\zeta)\right), \tag{3.11}$$

where maximum is taken over all $y_0, y_1\in\mathbb{R}$ such that $I(-f+y_0u_0+y_1u_1)\in V_{0,2}(\sigma)$.

It follows from $x_k<\zeta<x_{k+1}$ and strictl convexity of f that

$$\max\left(f(\zeta)+y_0u_0(\zeta)+y_1u_1(\zeta)\right)=(x_n-\zeta)(\zeta-x_1)[x_1,\zeta,x_n]f. \tag{3.12}$$

Now (3.6) follows from (3.8), (3.9), (3.11), and (3.12). □

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