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Research Article

Duality Property for Positive Weak Dunford-Pettis Operators

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We prove that an operator is weak Dunford-Pettis if its adjoint is one but the converse is false in general, and we give some necessary and sufficient conditions under which each positive weak Dunford-Pettis operator has an adjoint which is weak Dunford-Pettis.

1. Introduction and Notation

Let us recall that an operator T from a Banach space E into another F is called Dunford-Pettis if it carries weakly compact subsets of E onto compact subsets of F. The operator T is said to be weak Dunford-Pettis if $y'_n(T(x_n))$ converges to 0 whenever (x_n) converges weakly to 0 in E and (y'_n) converges weakly to 0 in F.

The class of weak Dunford-Pettis operators was used by Aliprantis and Burkinshaw [1] and Kalton and Saab [2] when they studied the domination property of Dunford-Pettis operators. As this latter class [3], weak Dunford-Pettis operators do not satisfy the duality property. In fact, there exist weak Dunford-Pettis operators whose adjoints are not weak Dunford-Pettis. For example, as the Banach space $l^1(l^2_n)$ has the Schur property, its identity operator $\mathrm{Id}_{l^1(l^2_n)}$ is Dunford-Pettis and then weak Dunford-Pettis, but its adjoint $\mathrm{Id}_{l^\infty(l^2_n)}$, which is the identity operator of the Banach space $l^\infty(l^2_n)$, is not weak Dunford-Pettis (because the Banach space $l^\infty(l^2_n)$ does not have the Dunford-Pettis property (see [4], page 22)). However, each operator is weak Dunford-Pettis if its adjoint is.

On the other hand, if E and F are two Banach spaces such that F is reflexive, then the class of weak Dunford-Pettis operators from E into F coincides with that of Dunford-Pettis operators from E into F, and therefore some results of [5] can be applied here to give some answers to our duality problem.

Morever, if E and F are both reflexive, then the class of weak Dunford-Pettis operators from E into F coincides with that of compact operators from E into F, and hence if $T : E \to F$ is an operator such that T is weak Dunford-Pettis, then its adjoint $T' : F' \to E'$ is weak Dunford-Pettis.

Also, if E and F are two Banach spaces such that E' or F' has the Dunford-Pettis property, then each operator from F' into E' is weak Dunford-Pettis, and hence each weak Dunford-Pettis $T : E \to F$ has an adjoint $T' : F' \to E'$ which is one.

As we have already done for Dunford-Pettis operators [3] and almost Dunford-Pettis operators [6], one of the aims of this paper is to characterize Banach lattices for which each weak Dunford-Pettis operator has an adjoint which is weak Dunford-Pettis.

We refer the reader to [5] for unexplained terminologies on Banach lattice theory and positive operators.

2. Some Preliminaries

Let us recall that an operator T from a Banach lattice E into a Banach space X is said to be AM-compact if it carries each order-bounded subset of E onto a relatively compact set of X. In [7], we used this class of operators to introduce Banach lattices which satisfy the AM-compactness property. In fact, a Banach lattice E is said to have the AM-compactness property if every weakly compact operator defined on E, and taking values in a Banach space X, is AM-compact. For an example, the Banach lattice $L^2[0,1]$ does not have the AM-compactness property, but l^1 has the AM-compactness property.

It follows from [7, Proposition 3.1] that a Banach lattice E has the AM-compactness property if and only if for every weakly null sequence (f_n) of E', we have $|f_n| \to 0$ for $\sigma(E', E)$.

On the other hand, if E is a Banach lattice, then

(1) the lattice operations in the topological dual E' are called sequentially continuous if the sequence $(|f_n|)$ converges to 0 in $\sigma(E', E'')$ whenever the sequence (f_n) converges to 0 in $\sigma(E', E'')$;

(2) the lattice operations in E' are called weak* sequentially continuous if the sequence $(|f_n|)$ converges to 0 in the weak* topology $\sigma(E', E)$ whenever the sequence (f_n) converges to 0 in $\sigma(E', E)$.

A Banach space (resp., Banach lattice) E has the Dunford-Pettis (resp., weak Dunford-Pettis) property if every weakly compact operator T defined on E (and taking values in a Banach space F) is Dunford-Pettis (resp., almost Dunford-Pettis, i.e., the sequence $(\|T(x_n)\|)$ converges to 0 for every weakly null sequence (x_n) consisting of pairwise disjoint elements in E).

We need to recall, from [7], the following sufficient conditions for which a Banach lattice has the AM-compactness property.

Theorem 2.1 (see [7]). *Let E be a Banach lattice. Then E has the AM-compactness property if one of the following assertions is valid:*

(1) *the norm of E is order continuous and E has the Dunford-Pettis property,*

(2) *the topological dual E' is discrete,*

(3) *the lattice operations in E' are weakly sequentially continuous,*

(4) *the lattice operations in E' are weak* sequentially continuous.*

Remarks 2.2. There exists a Banach lattice E such that

(1) the norm of E' is order continuous but E does not have the AM-compactness property nor the weak Dunford-Pettis property. In fact, consider $E = L^2[0,1]$, the norm of $E' = L^2[0,1]$, is order continuous but $L^2[0,1]$ does not have the AM-compactness property nor the weak Dunford-Pettis property;

(2) the norm of E' is not order continuous, but E has the AM-compactness property or the weak Dunford-Pettis property. In fact, consider $E = l^1$, the norm of $E' = l^\infty$, is not order continuous but l^1 has the AM-compactness property and the weak Dunford-Pettis property;

(3) E has the AM-compact property but not the weak Dunford-Pettis property. In fact, consider $E = l^2$, it has the AM-compactness property but not the weak Dunford-Pettis property;

(4) E has the weak Dunford-Pettis property but not the AM-compactness property. In fact, consider $E = l^\infty$, it has the weak Dunford-Pettis property but not the AM-compactness property;

(5) the norms of E and E' are order continuous, but E does not have the Dunford-Pettis property. In fact, consider $E = l^2$, the norms of $E = l^2$ and $E' = l^2$, are order continuous but l^2 does not have the Dunford-Pettis property;

(6) the norms of E and E' are not order continuous, but E has the Dunford-Pettis property. In fact, consider $E = l^1 \oplus l^\infty$, the norms of $E = l^1 \oplus l^\infty$ and $E' = l^\infty \oplus (l^\infty)'$, are not order continuous but $l^1 \oplus l^\infty$ has the Dunford-Pettis property;

(7) the topological dual E' is discrete with an order continuous norm, and E does not have the weak Dunford-Pettis property. In fact, consider $E = l^2$, the topological dual $E' = l^2$, is discrete with an order continuous norm and l^2 does not have the weak Dunford-Pettis property;

(8) the topological dual E' is not discrete and its norm is not order continuous, but it has the weak Dunford-Pettis property. In fact, consider $E = (l^\infty)'$, the topological dual $E' = (l^\infty)''$, is not discrete and its norm is not order continuous but it has the weak Dunford-Pettis property.

A Banach space E is said to have the Schur property if every sequence in E weakly convergent to zero is norm convergent to zero. For an example, the Banach space l^1 has the Schur property.

Note that the Schur property implies the Dunford-Pettis property, and hence the weak Dunford-Pettis property, but the weak Dunford-Pettis property does not imply the Schur property. In fact, the Banach space c_0 has the weak Dunford-Pettis property (because it has the Dunford-Pettis property), but it does not have the Schur property.

The following result gives some sufficient conditions for which the topological dual, of a Banach lattice, has the Schur property.

Theorem 2.3. *Let E be a Banach lattice. Then E' has the Schur property if one of the following assertions is valid:*

(1) *the norm of E' is order continuous, E has the AM-compactness property and the weak Dunford-Pettis property,*

(2) *the norms of E and E' are order continuous and E has the Dunford-Pettis property,*

(3) *the topological dual E' is discrete with an order continuous norm and E has the weak Dunford-Pettis property.*

Proof. (1) Let $(f_n) \subset E'$ be a sequence such that $f_n \to 0$ in $\sigma(E', E'')$. Since E has the AM-compactness property, then $|f_n| \to 0$ in $\sigma(E', E)$ (Proposition 3.1 of [7]).

Now, by Corollary 2.7 of Dodds and Fremlin [8], to show that $\|f_n\| \to 0$, it suffices to prove that $f_n(x_n) \to 0$ for every norm-bounded disjoint sequence $(x_n) \subset E^+$. To this end, let (x_n) be a such sequence of E^+. Since the norm of E' is order continuous, it follows from Corollary 2.9 of Dodds and Fremlin [8] that $x_n \to 0$ in $\sigma(E, E')$. And as E has the weak Dunford-Pettis property, we obtain $f_n(x_n) \to 0$. This proves that E' has the Schur property.

For (2) and (3), it follows from Theorem 2.1 that E has the AM-compactness property. Finally, assertion (1) of the present theorem ends the proof. □

Remarks 2.4. (1) There exists a Banach lattice F which has the AM-compactness property but its topological dual F' does not have the Schur property. In fact, consider $F = l^1$, it has the AM-compactness property but $F' = l^\infty$ does not have the Schur property.

(2) If the topological dual F', of a Banach lattice F, has the Schur property, then F' is discrete, and hence F has the AM-compact property (see Theorem 2.1).

3. Duality Property for Weak Dunford-Pettis Operators

Now, we study the duality property of weak Dunford-Pettis operators. Our first result proves that each operator is weak Dunford-Pettis whenever its adjoint is one.

Theorem 3.1. *Let E and F be two Banach spaces, and let T be an operator from E into F. If the adjoint T' is weak Dunford-Pettis from F' into E', then T is weak Dunford-Pettis.*

Proof. Let (x_n) (resp., (y'_n)) be a sequence of E (resp., of F') such that $x_n \to 0$ in $\sigma(E, E')$ (resp., $y'_n \to 0$ in $\sigma(F', F'')$). We have to prove that $y'_n(T(x_n)) \to 0$. For this, let $\tau : E \to E''$ be the canonical injection of E into its topological bidual E''. Since τ is continuous for the topologies $\sigma(E, E')$ and $\sigma(E'', E''')$, we obtain $\tau(x_n) \to 0$ for $\sigma(E'', E''')$.

Now, as $y'_n \to 0$ in $\sigma(F', F'')$ and the adjoint T' is weak Dunford-Pettis from F' into E', we deduce that $\tau(x)(T'(y'_n)) \to 0$. But we know that

$$\tau(x_n)\left(T'\left(y'_n\right)\right) = T'\left(y'_n\right)(x_n) = y'_n(T(x_n)) \quad \text{for each } n. \tag{3.1}$$

Hence $y'_n(T(x_n)) \to 0$, and this ends the proof. □

Let us recall from [5] that a norm-bounded subset A of a Banach space X is said to be Dunford-Pettis whenever every weakly compact operator from X to an arbitrary Banach space Y carries A to a norm relatively compact set of Y. This is equivalent to saying that A is Dunford-Pettis if and only if every weakly null sequence (f_n) of X' converges uniformly to zero on the set A, that is, $\sup_{x\in \mathbf{A}}|f_n(x)| \to 0$ (see Theorem 5.98 of [5]).

Now, we give some sufficient conditions for which each positive weak Dunford-Pettis operator has an adjoint which is Dunford-Pettis.

Theorem 3.2. *Let E and F be two Banach lattices. Then each positive weak Dunford-Pettis operator $T : E \to F$ has an adjoint $T' : F' \to E'$ which is Dunford-Pettis (and then weak Dunford-Pettis) if one of the following assertions is valid:*

(1) *the norm of E' is order continuous and E has the AM-compactness property,*

(2) *the norm of E' is order continuous and F has the AM-compactness property,*

(3) *the norms of E and E' are order continuous,*

(4) *F' has the Schur property.*

Proof. For (1), (2), and (3), let $T : E \to F$ be a positive weak Dunford-Pettis operator and let $(f_n) \subset F'$ be a sequence such that $f_n \to 0$ in $\sigma(F', F'')$. In the three cases we have $|T'(f_n)| \to 0$ in $\sigma(E', E)$, in fact, consider the following.

(1) As $T'(f_n) \to 0$ in $\sigma(E', E'')$ and E has the AM-compactness property, then $|T'(f_n)| \to 0$ for $\sigma(E', E)$.

(2) Since $f_n \to 0$ in $\sigma(F', F'')$ and F has the AM-compactness property, then $|f_n| \to 0$ in $\sigma(F', F)$. Hence, $T'(|f_n|) \to 0$ in $\sigma(E', E)$. Now, from $|T'(f_n)| \le T'(|f_n|)$ for each n, we conclude that $|T'(f_n)| \to 0$ in $\sigma(E', E)$.

(3) Since the norm of E is order continuous, $[-x, x]$ is weakly compact for each $x \in E^+$. As T is weak Dunford-Pettis, we conclude that $T([-x, x])$ is a Dunford-Pettis set, and then for each $x \in E^+$, $\sup_{y \in T([-x,x])} |f_n(y)| \to 0$. Now, from $\sup_{y \in T([-x,x])} |f_n(y)| = |T'(f_n)|(x)$ for each n, we obtain $|T'(f_n)|(x) \to 0$ for each $x \in E^+$, and hence $|T'(f_n)| \to 0$ in $\sigma(E', E)$.

On the other hand, by Corollary 2.7 of Dodds and Fremlin [8], to prove that $\|T'(f_n)\| \to 0$, it suffices to show that $[T'(f_n)](x_n) \to 0$ for every norm-bounded disjoint sequence $(x_n) \subset E^+$. To this end, let (x_n) be a norm-bounded disjoint sequence of E^+. Since the norm of E' is order continuous, it follows from Corollary 2.9 of Dodds and Fremlin [8] that $x_n \to 0$ in $\sigma(E, E')$. Hence, as T is a weak Dunford-Pettis operator, we obtain $f_n(T(x_n)) \to 0$. And from

$$[T'(f_n)](x_n) = f_n(T(x_n)) \quad \text{for each } n, \tag{3.2}$$

we derive that $[T'(f_n)](x_n) \to 0$, and hence T' is Dunford-Pettis.

(4) In this case, each operator $T : E \to F$ has an adjoint $T' : F' \to E'$ which is Dunford-Pettis. □

Remarks 3.3. There exist Banach lattices E and F and a weakly Dunford-Pettis operator T from E into F such that the adjoint T' is not Dunford-Pettis in the following situations:

(1) if the topological dual E' has an order continuous norm. In fact, if $E = F = l^\infty$, we note that $E' = (l^\infty)'$ has an order continuous norm and its identity operator $\mathrm{Id}_{l^\infty} : l^\infty \to l^\infty$ is weak Dunford-Pettis but its adjoint $\mathrm{Id}_{(l^\infty)'} : (l^\infty)' \to (l^\infty)'$ is not Dunford-Pettis. However, it is weak Dunford-Pettis because $(l^\infty)'$ has the Dunford-Pettis property,

(2) if E has the AM-compactness property (resp., F has the AM-compactness property, E has an order continuous norm). In fact, if $E = F = l^1$, we note that l^1 has the AM-compactness property (resp. its norm is order continuous) and its identity operator $\mathrm{Id}_{l^1} : l^1 \to l^1$ is weak Dunford-Pettis but its adjoint $\mathrm{Id}_{l^\infty} : l^\infty \to l^\infty$ is not Dunford-Pettis. However, it is weak Dunford-Pettis because l^∞ has the Dunford-Pettis property.

As a consequence of Theorems 2.1 and 3.2, we obtain the following.

Corollary 3.4. *Let E and F be two Banach lattices. Then each positive weak Dunford-Pettis operator $T : E \to F$ has an adjoint $T' : F' \to E'$ which is weak Dunford-Pettis if one of the following assertions is valid:*

(1) *the topological dual E' is discrete with an order continuous norm,*

(2) *the norm of E' is order continuous and F' is discrete,*

(3) *the norm of E' is order continuous and the lattice operations in F' are weakly sequentially continuous,*

(4) *the norm of E' is order continuous and the lattice operations in F' are weak* sequentially continuous,*

(5) *the norms of E' and F are order continuous and F has the Dunford-Pettis property,*

(6) *the norms of E and E' are order continuous,*

(7) *E' or F' has the Dunford-Pettis property.*

Proof. For (1), (2), (3), (4), and (5), it follows from Theorem 2.1 that E or F has the AM-compactness property. Since the norm of E' is order continuous, Theorem 3.2 implies that each positive weak Dunford-Pettis operator $T : E \to F$ has an adjoint $T' : F' \to E'$ which is Dunford-Pettis (and then weak Dunford-Pettis).

(6) Follows from (3) of Theorem 3.2.

(7) In this case each operator $T : E \to F$ has an adjoint $T' : F' \to E'$ which is weak Dunford-Pettis. □

For the converse of Theorem 3.2, we have the following.

Theorem 3.5. *Let E and F be two Banach lattices. If each positive weak Dunford-Pettis operator $T : E \to F$ has an adjoint $T' : F' \to E'$ which is Dunford-Pettis, then one of the following assertions is valid:*

(1) *the norm of E' is order continuous,*

(2) *F' has the Schur property.*

Proof. Assume by way of contradiction that the norm of E' is not order continuous and F' does not have the Schur property. We have to construct a positive weak Dunford-Pettis operator $T : E \to F$ such that its adjoint $T' : F' \to E'$ is not Dunford-Pettis.

Since the norm of E' is not order continuous, it follows from the proof of Theorem 1 of Wickstead [9] the existence of a sublattice H of E, which is isomorphic to l^1, and a positive projection $P : E \to l^1$.

On the other hand, since F' does not have the Schur property, there exists a weakly null sequence $(f_n) \subset F'$ such that $\|f_n\| = 1$ for all n. Moreover, there exists a sequence $(y_n) \subset F^+$ with $\|y_n\| \leq 1$ and some $\varepsilon_0 > 0$ such that $|f_n(y_n)| \geq \varepsilon_0$ for all n.

Now, we consider the operator $T = S \circ P : E \to l^1 \to F$, where S is the operator defined by

$$S : l^1 \to F, \qquad (\lambda_n) \longmapsto \sum_n \lambda_n y_n. \tag{3.3}$$

Since l^1 has the Dunford-Pettis property, the operator T is weak Dunford-Pettis. But its adjoint $T' : F' \to E'$ is not Dunford-Pettis. Indeed, the sequence (f_n) is weakly null in F'. And as the operator $P : E \to l^1$ is surjective, there exist $\delta > 0$ such that $\delta \cdot B_{l^1} \subset P(B_E)$, where B_H is the closed unit ball of $H = E$ or l^1. Hence

$$\begin{aligned} \|T'(f_n)\| &= \sup_{x \in B_E} |T'(f_n)(x)| = \sup_{x \in B_E} |f_n(T(x))| = \sup_{x \in B_E} |f_n \circ S(p(x))| \\ &\geq \delta \cdot \sup_{(\lambda_i)_i \in B_{l^1}} |f_n \circ S((\lambda_i))| \geq \delta \cdot |f_n \circ S((e_n))| \geq \delta \cdot |f_n(y_n)| > \delta . \varepsilon_0, \end{aligned} \tag{3.4}$$

where $(e_i)_{i=1}^{\infty}$ is the canonical bases of l^1.

Then $\|T'(f_n)\| > \delta \cdot \varepsilon_0$ for all n, and we conclude that T' is not Dunford-Pettis. This presents a contradiction. □

Remarks 3.6. Let E and F be two Banach lattices such that F' does not have the Schur property. If each positive weak Dunford-Pettis operator T from E into F has an adjoint T' from F' into E' which is Dunford-Pettis, then

(1) F does not necessarily have the AM-compactness property. In fact, if we take $E = c_0$ and $F = l^\infty$, we observe that each operator T from c_0 into l^∞ has an adjoint T' from $(l^\infty)'$ into l^1 which is Dunford-Pettis (because l^1 has the Schur property), but $F = l^\infty$ does not have the AM-compactness property,

(2) the norm of E is not necessarily order continuous. In fact, if we take $E = c$ and $F = l^\infty$, we note that each operator T from c into l^∞ has an adjoint T' from $(l^\infty)'$ into c' which is Dunford-Pettis (because c' has the Schur property), but the norm of $E = c$ is not order continuous,

(3) E does not necessarily have the AM-compactness property. In fact, if we take $E = l^\infty$ and $F = (l^\infty)'$, we note that each positive weak Dunford-Pettis operator T from l^∞ into $(l^\infty)'$ has an adjoint T' from $(l^\infty)''$ into $(l^\infty)'$ which is Dunford-Pettis (see assertion 2 of Theorem 3.2), but $E = l^\infty$ does not have the AM-compactness property.

Whenever $E = F$, we obtain the following characterization.

Theorem 3.7. *Let E be a Dedekind σ-complete Banach lattice. Then the following assertions are equivalent:*

(1) *each positive weak Dunford-Pettis operator T from E into E has an adjoint which is Dunford-Pettis,*

(2) *the norms of E and E' are order continuous.*

Proof. (1)⇒(2). By Theorem 3.5, the norm of E' is order continuous. We have just to prove that the norm of E is order continuous. Assume that the norm of E is not order continuous, and since E is Dedekind σ-complete, then E contains a closed sublattice isomorphic to l^∞ and there is a positive projection $P : E \to l^\infty$. Let $i : l^\infty \to E$ be the canonical injection of l^∞ into E. Consider the operator defined by

$$T = i \circ P : E \longrightarrow l^\infty \longrightarrow E. \tag{3.5}$$

Since l^∞ has the Dunford-Pettis property, the positive operator T is weak Dunford-Pettis. But its adjoint $T' : E' \to E'$ is not Dunford-Pettis. If not, the adjoint of the composed operator

$$P \circ T \circ i : l^\infty \longrightarrow E \longrightarrow E \longrightarrow l^\infty \tag{3.6}$$

would be Dunford-Pettis. But $(P \circ T \circ i)' = (\mathrm{Id}_{l^\infty})' = \mathrm{Id}_{(l^\infty)'}$ is not Dunford-Pettis (because $(l^\infty)'$ does not have the Schur property). This presents a contradiction, and hence E has an order continuous norm.

(2)⇒(1). It follows from (3) of Theorem 3.2. □

4. Complements on the Duality of Almost Dunford-Pettis Operators

In [6], we studied the duality for almost Dunford-Pettis operators. In this section we use the AM-compactness property to give some new results.

Let us recall that an operator T from a Banach lattice E into a Banach space F is said to be almost Dunford-Pettis if the sequence $(\|T(x_n)\|)$ converges to 0 for every weakly null sequence (x_n) consisting of pairwise disjoint elements in E.

Note that the adjoint of a positive almost Dunford-Pettis operator is not necessarily Dunford-Pettis. In fact, the identity operator of the Banach space l^1 is almost Dunford-Pettis but its adjoint, which is the identity of the Banach space l^∞, is not Dunford-Pettis.

The following result gives some sufficient conditions for which each positive almost Dunford-Pettis operator has an adjoint which is Dunford-Pettis.

Theorem 4.1. *Let E and F be two Banach lattices. Then each positive almost Dunford-Pettis operator $T : E \to F$ has an adjoint $T' : F' \to E'$ which is Dunford-Pettis if one of the following assertions is valid:*

(1) *the norm of E' is order continuous and E has the AM-compactness property,*

(2) *the norm of E' is order continuous and F has the AM-compactness property,*

(3) *F' has the Schur property.*

Proof. Note that for (1) and (2), the proof is the same as (1) and (2) of Theorem 3.2. In fact, let $T : E \to F$ be a positive almost Dunford-Pettis operator, and let $(f_n) \subset F$ be a sequence such that $f_n \to 0$ in $\sigma(F', F'')$. By the uniform boundedness Theorem, there exists some $\alpha > 0$ such that $\|f_n\| \leq \alpha$ for all n. In the two cases we have $|T'(f_n)| \to 0$ in $\sigma(E', E)$. In fact, consider the following.

(1) As $T'(f_n) \to 0$ in $\sigma(E', E'')$ and E has the AM-compactness property, then $|T'(f_n)| \to 0$ in $\sigma(E', E)$.

(2) As $f_n \to 0$ in $\sigma(F', F'')$, and since F has the AM-compactness property, then $|f_n| \to 0$ in $\sigma(F', F)$. Hence, $T'(|f_n|) \to 0$ in $\sigma(E', E)$ and from $|T'(f_n)| \le T'(|f_n|)$ for each n, we conclude that $|T'(f_n)| \to 0$ in $\sigma(E', E)$.

Now to prove that $\|T'(f_n)\|_{E'} \to 0$, it suffices to show that $[T'(f_n)](x_n) \to 0$ in every norm-bounded disjoint sequence $(x_n) \subset E^+$ (Corollary 2.7 of Dodds and Fremlin [8]). To this end, let (x_n) be a norm-bounded disjoint sequence of E^+.

Since the norm of E' is order continuous, it follows from Corollary 2.9 of Dodds and Fremlin [8] that $x_n \to 0$ in $\sigma(E, E')$. Hence, as T is almost Dunford-Pettis operator, we obtain $\|T(x_n)\|_F \to 0$. Now, from

$$\left|[T'(f_n)](x_n)\right| = \left|f_n(T(x_n))\right| \le \alpha \cdot \|T(x_n)\|_F \quad \text{for each } n, \tag{4.1}$$

we see that $[T'(f_n)](x_n) \to 0$, and hence T' is Dunford-Pettis.

(3) In this case each operator $T : E \to F$ has an adjoint $T' : F' \to E'$ which is Dunford-Pettis. □

Remarks 4.2. Let E and F be two Banach lattices, and let T be an operator from E into F. Then the adjoint T' is not necessarily Dunford-Pettis whenever T is almost Dunford-Pettis in the following situations.

(1) If the topological dual E' has an order continuous norm. In fact, since the norm of l^∞ is not order continuous and the Banach lattice $(l^\infty)'$ is not discrete, it follows from Theorem 1 of Wickstead [9] the existence of two positive operators $S_1, S_2 : l^\infty \to l^\infty$ such that $0 \le S_1 \le S_2$, S_2 is compact, and S_1 is not compact. Now, as $(l^\infty)'$ has an order continuous norm, Theorem 5.31 of Aliprantis and Burkinshaw [5] implies that S_1 is weakly compact. So, by Theorem 5.44 of Aliprantis and Burkinshaw [5], there exist a reflexive Banach lattice G, lattice homomorphism $Q : l^\infty \to G$, and a positive operator $R : G \to l^\infty$ such that $S_1 = R \circ Q$. We note that Q is not compact (because S_1 is not one).

On the other hand, if we take $E = l^\infty$, $F = G$, and $T = Q$, then $T : l^\infty \to G$ is a weakly compact operator (because G is reflexive), and hence T is Dunford-Pettis (l^∞ has the Dunford-Pettis property) and then T is almost Dunford-Pettis. But its adjoint $T' : G' \to (l^\infty)'$ is not Dunford-Pettis (if not, since G' is reflexive, T' would be compact and so T is compact, which is a contradiction). However, the norm of $E' = (l^\infty)'$ is order continuous.

(2) If E has the AM-compactness property. In fact, if we take $E = F = l^1$, we note that $E = l^1$ has the AM-compactness property and its identity operator $\mathrm{Id}_{l^1} : l^1 \to l^1$ is almost Dunford-Pettis but the adjoint $\mathrm{Id}_{l^\infty} : l^\infty \to l^\infty$ is not Dunford-Pettis.

(3) If F has the AM-compactness property. In fact, if we take $E = F = l^1$, we observe that $F = l^1$ has the AM-compactness property and its identity operator $\mathrm{Id}_{l^1} : l^1 \to l^1$ is almost Dunford-Pettis, but the adjoint $\mathrm{Id}_{l^\infty} : l^\infty \to l^\infty$ is not Dunford-Pettis.

For the converse of Theorem 4.1, we obtain the following.

Theorem 4.3. *Let E and F be two Banach lattices. If each positive almost Dunford-Pettis operator $T : E \to F$ has an adjoint $T' : F' \to E'$ which is Dunford-Pettis, then one of the following assertions is valid:*

(1) *the norm of E' is order continuou,*

(2) *F' has the Schur property.*

Proof. The proof is the same as that of Theorem 3.5 if we observe that the operator T in the proof of Theorem 3.5 is almost Dunford-Pettis (because T admits a factorization through the Banach lattice l^1, which has the Schur property). □

Remarks 4.4. Let E and F be two Banach lattices such that F' does not have the Schur property. If each positive almost Dunford-Pettis operator T from E into F has an adjoint T' from F' into E' which is Dunford-Pettis, then

(1) E does not necessarily have the AM-compactness property. In fact, if we take $E = l^\infty$ and $F = (l^\infty)'$, we note that each positive almost Dunford-Pettis operator T from l^∞ into $(l^\infty)'$ has an adjoint T' from $(l^\infty)''$ into $(l^\infty)'$ which is Dunford-Pettis (see assertion 2 of Theorem 4.1), but $E = l^\infty$ does not have the AM-compactness property,

(2) F does not necessarily have the AM-compactness property. In fact, if we take $E = c_0$ and $F = l^\infty$, we observe that each operator T from c_0 into l^∞ has an adjoint T' from $(l^\infty)'$ into l^1 which is Dunford-Pettis (because l^1 has the Schur property), but $F = l^\infty$ does not have the AM-compactness property.

Finally, we note that there exists a positive weak Dunford-Pettis (resp., Dunford-Pettis) operator $T : E \to F$ whose adjoint $T' : F' \to E'$ is not almost Dunford-Pettis. In fact, the identity operator of the Banach lattice l^1 is weak Dunford-Pettis (resp., Dunford-Pettis) operator but its adjoint, which is the identity of the Banach lattice l^∞, is not almost Dunford-Pettis.

Now, we give a characterization on the duality between weak Dunford-Pettis operators and almost Dunford-Pettis operators.

Theorem 4.5. *Let E and F be two Banach lattices. Then the following assertions are equivalent:*

(1) *each positive weak Dunford-Pettis (resp., Dunford-Pettis, almost Dunford-Pettis) operator $T : E \to F$ has an adjoint $T' : F' \to E'$ which is almost Dunford-Pettis,*

(2) *one of the following assertions is valid:*

 (a) *the norm of E' is order continuous,*

 (b) *F' has the positive Schur property.*

Proof. (1)⇒(2). Assume by way of contradiction that the norm of E' is not order continuous and F' does not have the positive Schur property. We have to construct a positive weak Dunford-Pettis (resp., Dunford-Pettis, almost Dunford-Pettis) operator $T : E \to F$ such that its adjoint $T' : F' \to E'$ is not almost Dunford-Pettis.

Since the norm of E' is not order continuous, it follows from the proof of Theorem 1 of Wickstead [9] the existence of a sublattice H of E, which is isomorphic to l^1, and a positive projection $P : E \to l^1$.

On the other hand, since F' does not have the positive Schur property, it follows from Theorem 3.1 of [10] the existence of a disjoint weakly null sequence $(f_n) \subset (F')^+$ such that (f_n) does not converge to zero for the norm. Moreover, there exists a sequence $(y_n) \subset F^+$ with $\|y_n\| \le 1$, and some $\varepsilon > 0$, a subsequence (g_n) of (f_n) such that $g_n(y_n) \ge \varepsilon$ for all n.

Now, we consider the composed operator

$$T = S \circ P : E \longrightarrow l^1 \longrightarrow F, \tag{4.2}$$

where S is defined by

$$S : l^1 \to F, \qquad (\lambda_n) \longmapsto \sum_n \lambda_n y_n. \tag{4.3}$$

Since l^1 has the Schur property, the operator T is weak Dunford-Pettis (resp. Dunford-Pettis, almost Dunford-Pettis), but its adjoint $T' : F' \to E'$ is not almost Dunford-Pettis. Indeed, (g_n) is a disjoint weakly null sequence in F'. And since the operator $P : E \to l^1$ is surjective, there exist $\delta > 0$ such that $\delta \cdot B_{l^1} \subset P(B_E)$ where B_H is the closed unit ball of $H = E, l^1$. Hence

$$\begin{aligned} \|T'(g_n)\| &= \sup_{x\in B_E}\left|T'(g_n)(x)\right| = \sup_{x\in B_E}\left|g_n(T(x))\right| = \sup_{x\in B_E}\left|g_n \circ S(p(x))\right| \\ &\geq \delta \cdot \sup_{(\lambda_i)_i\in B_{l^1}} \left|g_n \circ S((\lambda_i))\right| \geq \delta \cdot \left|g_n \circ S((e_n))\right| \geq \delta \cdot \left|g_n(y_n)\right| > \delta.\varepsilon_0, \end{aligned} \tag{4.4}$$

where $(e_i)_{i=1}^{\infty}$ is the canonical bases of l^1.

Then $\|T'(g_n)\| > \delta\cdot\varepsilon_0$ for every n, and we conclude that T' is not almost Dunford-Pettis. This presents a contradiction.

(2), (a)⇒(1). Let (f_n) be a disjoint sequence of F' such that $f_n \to 0$ in $\sigma(F', F'')$. We have to prove that $(T'(f_n))$ converges to 0 for the norm of E'. By using Corollary 2.7 of Dodds-Fremlin [8], it suffices to prove that $|T'(f_n)| \to 0$ in $\sigma(E', E)$ and $[T'(f_n)](x_n) \to 0$ for every norm-bounded disjoint sequence $(x_n) \subset E^+$. In fact, as (f_n) is a weakly null sequence with pairwise disjoint terms, it follows from Remark 1 of Wnuk [11] that $|f_n| \to 0$ in $\sigma(F', F'')$, and then $T'(|f_n|) \to 0$ for $\sigma(E', E'')$. Now, since $|T'(f_n)| \leq T'(|f_n|)$ for each n, then $|T'(f_n)| \to 0$ in $\sigma(E', E'')$, and hence $|T'(f_n)| \to 0$ in $\sigma(E', E)$.

On the other hand, since the norm of E' is order continuous, it follows from Corollary 2.9 of Dodds and Fremlin [8] that $x_n \to 0$ in $\sigma(E, E')$. Hence, as T is a weak Dunford-Pettis (resp., Dunford-Pettis, almost Dunford-Pettis) operator, we obtain $[T'(f_n)](x_n) = f_n(T(x_n)) \to 0$, and this proves that T' is almost Dunford-Pettis.

(2), (b)⇒(1). Obvious. □

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