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Research Article

The Bolzano-Poincaré Type Theorems

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In 1883–1884, Henri Poincaré announced the result about the structure of the set of zeros of function $f : I^n \to R^n$, or alternatively the existence of solutions of the equation $f(x) = 0$. In the case $n = 1$ the Poincaré Theorem is well known Bolzano Theorem. In 1940 Miranda rediscovered the Poincaré Theorem. Except for few isolated results it is essentially a non-algorithmic theory. The aim of this article is to introduce an algorithmical proof of the Theorem "On the existence of a chain" and for $n = 3$ an algorithmical proof of the Bolzano-Poincaré Theorem and to show the equivalence of Poincaré, Brouwer and "On the existence of a chain" theorems.

1. Introduction

It is well known how influential topology was for the development of many other branches of mathematics and economics. Among many others, let us recall significant place of fixed point theorems of Brouwer and Banach which served as a main tool in solving problems in differential equations, theory of fractals and problems of market equilibrium. Some of these applications raised a question of computability of the fixed points. In [1, 2] Steinhaus presented following conjecture: *Let some segments of the chessboard be mined. Assume that the king cannot go across the chessboard from the left edge to the right one without meeting a mined square. Then the rook can go from upper edge to the lower one moving exclusively on mined segments*.

According to Surówka [3] several proofs of the Steinhaus Chessboard Theorem seem to be incomplete or use induction on the size of the chessboard [4].

The simple proof of the Steinhaus Chessboard Theorem was presented in [5]. In [6] the following generalization of the Steinhaus Chessboard Theorem was published: Theorem [On the existence of a chain] *For an arbitrary decomposition of n-dimensional cube I^n onto k^n cubes and an arbitrary coloring function $F : T(k) \to \{1, \ldots, n\}$ for some natural number $i \in \{1, \ldots, n\}$ there exists an ith colored chain $P_1, \ldots, P_r$ such that $P_1 \cap I_i^+ \neq \emptyset$ and $P_r \cap I_i^- \neq \emptyset$.*

This theorem was the main tool in the proof (see [6]) of the Bolzano-Poincaré theorem (see [7, 8]). In the first part of our paper an algorithm of finding the chain will be presented and will be shown that the theorem "on the existence of a chain", the Bolzano-Poincaré theorem, and the Brouwer fixed point theorem are equivalent (for more informations see [9, 10]).

2. Theorems

Let $I^n := [0,1]^n$ be the n-dimensional cube in $\mathbf{R}^n$.

Its *ith opposite faces* are defined as follows:

$$I_i^- := \{x \in I^n : x(i) = 0\}, \qquad I_i^+ := \{x \in I^n : x(i) = 1\}. \tag{2.1}$$

Let

$$\partial I^n := \bigcup_{i=1}^{n} (I_i^- \cup I_i^+) \tag{2.2}$$

be the *boundary* of the cube I^n.

Let k be an arbitrary natural number.

We call the family

$$T(k) := \left\{ \left[\frac{i_1}{k}, \frac{i_1+1}{k}\right] \times \cdots \times \left[\frac{i_n}{k}, \frac{i_n+1}{k}\right] : i_j \in \{0, \dots, k-1\} \right\} \tag{2.3}$$

the decomposition of I^n into k^n cubes.

The map $F : T(k) \to \{1, \dots, n\}$ is said to be a *coloring function* of the decomposition $T(k)$.

The sequence $P_1, \dots, P_r$ where $P_l \in T(k)$ for $l = 1, \dots, r$ is said to be an *ith colored chain*, if for all $l \in \{1, \dots, r\}$ $F(P_l) = i$ and $P_j \cap P_{j+1} \neq \emptyset$ for $j = 1, \dots, r-1$.

The set $C = \{-1/2k, 1/2k, \dots, 1 + 1/2k\}^n$ is said to be the *n-dimensional combinatorial cube.*

Its *ith opposite faces* are defined as follows:

$$\begin{aligned} C_i^- &= \left\{ z \in C : z(i) = -\frac{1}{2k} \right\}, \\ C_i^+ &= \left\{ z \in C : z(i) = 1 + \frac{1}{2k} \right\}. \end{aligned} \tag{2.4}$$

Let

$$\partial C = \bigcup_{i=1}^{n} C_i^- \cup C_i^+ \tag{2.5}$$

be the *boundary* of the *n-dimensional combinatorial cube.*

Let $e_i = (0, \dots, 0, 1/k, 0, \dots, 0)$, $e_i(i) = 1/k$ be the *ith basic vector.*

An ordered set $S = [z_0, \dots, z_n] \subset C$ is said to be an *n-simplex* if there exists permutation α of set $\{1, \dots, n\}$ such that $z_1 = z_0 + e_{\alpha(1)} \cdots z_n = z_{n-1} + e_{\alpha(n)}$.

Any subset $[z_0, \dots, z_{i-1}, z_{i+1}, \dots, z_n] \subset S$, $i = 0, \dots, n$ is said to be an *(n−1)-face of the n-simplex S*.

Every map $\Phi : C \to \{1, \dots, n\}$ is said to be a *coloring map* of C.

The set $A \subset C$ we call *n′-colored* if $\Phi(A) = \{1, \dots, n'\}$.

Observation 1. Let $S = [z_0, \dots, z_n] \subset C$ be an n-simplex. Then for each $z_i \in S$ if $[z_0, \dots, z_{i-1}, z_{i+1}, \dots, z_n] \not\subset C_p^\epsilon$ for each $p \in \{1, \dots, n\}$, $\epsilon \in \{+, -\}$ then there exists exactly one n-simplex $S[i] \subset C$ such that $S \cap S[i] = [z_0, \dots, z_{i-1}, z_{i+1}, \dots, z_n]$ else there does not exist such $S[i] \subset C$.

Observation 2. Any $(n-1)$-face of an n-simplex $S \subset C$ is an $(n-1)$-face of exactly one or of two n-simplexes from C depending on whether or not it lies on C_p^ϵ for some $p \in \{1, \dots, n\}$, $\epsilon \in \{+, -\}$.

Observation 3. Each n-colored n-simplex has exactly two n-colored $(n-1)$-faces.

Theorem 2.1 (on the existence of a chain). *For an arbitrary decomposition of n- dimensional cube I^n onto k^n cubes and an arbitrary coloring function $F : T(k) \to \{1, \dots, n\}$ for some natural number $i \in \{1, \dots, n\}$ there exists an ith colored chain $P_1, \dots, P_r$ such that $P_1 \cap I_i^+ \neq \emptyset$ and $P_r \cap I_i^- \neq \emptyset$.*

The algorithm (based on the proof from [6]) is as follows.

Step 1. Let us define the coloring map $\Phi : C \to \{1, \dots, n\}$:

$$\Phi(z) = \begin{cases} F(t) & \text{for } z \in C \setminus \partial C \text{ and } z \in t \\ 1 & \text{for } z \in C_1^- \cup C_2^+ \\ j & \text{for } z \in \left(C_j^- \cup C_{j+1}^+\right) \setminus \left(\displaystyle\bigcup_{l=1}^{j-1} (C_l^- \cup C_{l+1}^+)\right),\ j = 2, \dots, n-1 \\ n & \text{for } z \in (C_n^- \cup C_1^+) \setminus \left(\displaystyle\bigcup_{l=1}^{n-1} (C_l^- \cup C_{l+1}^+)\right). \end{cases} \tag{2.6}$$

Step 2. Let us take n-colored n-simplex $S_1 = [z_0^1, \dots, z_{n-1}^1, z_n^1]$ where $z_0^1 = (-1/2k, -1/2k, \dots, -1/2k)$,

$$\begin{gathered} z_1^1 = z_0^1 + e_1, \dots, z_{n-1}^1 = z_{n-2}^1 + e_{n-1}, \\ z_n^1 = \left(\frac{1}{2k}, \frac{1}{2k}, \dots, \frac{1}{2k}\right) = z_{n-1}^1 + e_n. \end{gathered} \tag{2.7}$$

We say that the n-colored $(n-1)$-face $[z_0^1, \dots, z_{n-1}^1]$ is "used".

Let $S = S_1$.

Step 3. Take "unused" n-colored $(n-1)$-face of the n-simplex S.

If this face is contained in C_p^ϵ for some $p \in \{1, \dots, n\}$, $\epsilon \in \{+, -\}$ then go to Step 5. Else this is $(n-1)$-face of exactly one n-simplex S' different to S.

Since that moment this $(n-1)$-face is said to be "used". Go to the Step 4.

Step 4. Let us create the sequence of n-simplexes $S_1, \dots, S_l, S'$.

Let $S = S'$.

Go to Step 3.

Step 5. After finitely many iterations we obtain the sequence $S_1, \ldots, S_m \subset C$ such that $\Phi(S_l \cap S_{l+1}) = \{1, \ldots, n\}$ for $l = 1, \ldots, m-1$. And the n-simplex S_m has the n-colored $(n-1)$-face which is a subset of C_p^{ϵ} for some $p \in \{1, \ldots, n\}$, $\epsilon \in \{+,-\}$. Hence $S_m = [z_0^m, z_1^m, \ldots, z_n^m]$ where $z_0^m = (1-1/2k, 1-1/2k, \ldots, 1-1/2k)$, $z_1^m = z_0^m + e_1$, $z_2^m = z_1^m + e_2, \ldots, z_n^m = z_{n-1}^m + e_n$.

Let us take the smallest index $l^1 \in \{1, 2, \ldots, m\}$ such that $S_{l^1} \cap C_i^+ \neq \emptyset$ for some $i \in \{1, \ldots, n\}$, then let us find the biggest index $l^2 \in \{1, 2, \ldots, l^1\}$ such that $S_{l^2} \cap C_i^- \neq \emptyset$.

Step 6. Then from the chain $S_{l^2+1}, \ldots, S_{l^1}$ choose successively points $z_1, z_2, \ldots, z_r$ in the way that $\Phi(z_j) = i$ for $j = 1, 2, \ldots, r$ and $z_j \neq z_{j+1}$ for $j = 1, 2, \ldots, r-1$, $z_1 \in C \setminus \partial C$ and $z_1 - e_i \in C_i^-$, $z_r \in S_{l^1}$.

Step 7. For the sequence $z_1, \ldots, z_r$ we have the chain $P_1, \ldots, P_r$ where $P_j \in T(k)$ and $z_j \in P_j$ for $j = 1, \ldots, r$.

END

Theorem 2.2 (Bolzano-Poincaré). *Let $f : I^n \to R^n$, $f(x) = (f_1(x), \ldots, f_n(x))$ be a continuous map such that $f_i(I_i^-) \subset (-\infty, 0]$ and $f_i(I_i^+) \subset [0, \infty)$ for $i = 1, \ldots, n$ then there exists $x_0 \in I^n$ such that $f(x_0) = (0, \ldots, 0)$.*

Theorem 2.3 (Brouwer fixed point theorem). *Let $g : I^n \to I^n$, $g(x) = (g_1(x), \ldots, g_n(x))$ be a continuous map then there exists $x_0 \in I^n$ such that $g(x_0) = x_0$.*

Theorem 2.4. *The following theorems are equivalent:*

(1) *Theorem on the existence of a chain*

(2) *Bolzano-Poincaré theorem*

(3) *Brouwer fixed point theorem.*

Proof. "(1)⇒(2)" let us assume that for each $x \in I^n$ $f(x) \neq (0, \ldots, 0)$. Let us define sets:

$U_i = \{x \in I^n : f_i(x) \neq 0\}$ for $i = 1, \ldots, n$, each set U_i is open.

We have $I^n = U_1 \cup \cdots \cup U_n$.

Let us consider the space R^n with the metric $\delta(x, y) = \max\{|x_i - y_i| : i = 1, \ldots, n\}$. From the Lebesgue lemma of covering it follows that there exists $\lambda > 0$ such that for every $k \in \mathbf{N}$ and $1/k < \lambda$ we have for every $t \in T(k)$ there exist $j \in \{1, \ldots, n\}$ such that $t \subset U_j$.

Let us define coloring function $F : T(k) \to \{1, \ldots, n\}$:

$$F(t) := \min\{j : t \subset U_j\}. \tag{2.8}$$

From theorem "on the existing of a chain" there exists ith colored sequence $P_1(k), \ldots, P_{r(k)}(k)$ connecting ith opposite faces of the cube I^n.

The set $W := \bigcup_{l=1}^{r(k)} P_l(k)$ is closed and connected.

The intersections $W \cap I_i^- \neq \emptyset \neq W \cap I_i^+$, Hence there exists $x, y \in W$ such that $f_i(x) < 0$ and $f_i(y) > 0$. Since $f(x)$ is the continuous map, hence $f_i(W)$ is connected in $\mathbf{R}$. Hence the set $f_i(W)$ is an interval containing $[f_i(x), f_i(y)]$. From the Bolzano theorem there exists $c \in W$ such that $f_i(c) = 0$.

Contradiction.

"(2)⇒(3)" let $f(x) = x - g(x)$. The function $f(x)$ fulfills the assumptions of the Bolzano-Poincaré theorem. hence there exist $c \in I^n$ such that $f(c) = 0$.

So $g(c) = c$.

"(3)⇒(1)" let us assume that there exists decomposition of n-dimensional cube I^n onto k^n cubes and a coloring function $F : T(k) \rightarrow \{1, \dots, n\}$ such that for each $i \in \{1, \dots, n\}$ there is no ith colored chain connecting I_i^- and I_i^+.

Let $C_i = \{t \in T(k) : F(t) = i\}$.

Let $\mathcal{L}_i$ be the family of components of $\bigcup C_i \subset I^n$.

$$\begin{aligned} C_i^- &= \{l \in \mathcal{L}_i : l \cap I_i^- \neq \emptyset\}, \\ C_i^+ &= \{l \in \mathcal{L}_i : l \cap I_i^+ \neq \emptyset\}, \\ C_i^0 &= \{l \in \mathcal{L}_i : l \cap (I_i^- \cup I_i^+) = \emptyset\}. \end{aligned} \tag{2.9}$$

The subsets of I^n:

$$\begin{aligned} A_i &= \bigcup C_i^- \cup \bigcup C_i^0 \cup \left\{x \in I^n : x(i) \in \left[0, \frac{1}{2k}\right]\right\}, \\ B_i &= \bigcup C_i^+ \cup \left\{x \in I^n : x(i) \in \left[1 - \frac{1}{2k}, 1\right]\right\} \end{aligned} \tag{2.10}$$

are closed and disjoint.

I^n with the Euclidean metric is a normal space, hence there exists a continuous map $f_i : I^n \rightarrow [-1/2k, 1/2k]$ such that $f_i(A_i) = 1/2k$ and $f_i(B_i) = -1/2k$.

For each $x \in I^n$ let us define the map $g(x) := x + f(x)$ where $f(x) = (f_1(x), \dots, f_n(x))$.

Observe that $g : I^n \rightarrow I^n$ is continuous map. Take an arbitrary $x \in I^n$.

There exists $t \in T(k)$ such that $x \in t$. The cube t is a subset of A_i or B_i for some $i \in \{1, \dots, n\}$. We have $g_i(x) = x(i) + 1/2k$ or $g_i(x) = x(i) - 1/2k$.

Hence the function $g(x)$ has no fixed point. Contradiction. □

3. Poincaré Theorem for $n = 3$

3.1. *The Basic Algorithm*

Let k be an arbitrary natural number.

We have the decomposition of I^3 into k^3 cubes.

Assume w.l.o.g. that $f_i(I_i^-) \subset (-\infty, 0)$ and $f_i(I_i^+) \subset (0, \infty)$ for $i = 1, 2, 3$. Let $d : I^3 \times I^3 \rightarrow R$ be the Euclidean metric.

Observe that there exist $\epsilon^* > 0$ such that for each $x \in I^3$, $d(x, I_i^-) < \epsilon^*$ and for each $y \in I^3$, $d(y, I_i^+) < \epsilon^*$ we have $f_i(x) < 0$, $f_i(y) > 0$, $i = 1, 2, 3$.

3.1.1. Surface

Let k be a natural number, such that $1/k < \epsilon^*$.

The center of each $t \in T(k)$, $t = [i_1/k, (i_1 + 1)/k] \times \cdots \times [i_3/k, (i_3 + 1)/k]$ is defined as follows:

$$t_c = \left(\frac{i_1}{k} + \frac{1}{2k}, \frac{i_2}{k} + \frac{1}{2k}, \frac{i_3}{k} + \frac{1}{2k}\right). \tag{3.1}$$

Let us define coloring map $\phi_1 : T(k) \to \{0,1\}$

$$\phi_1(t) = \begin{cases} 0 & f_1(t_c) \le 0 \\ 1 & f_1(t_c) > 0. \end{cases} \tag{3.2}$$

Algorithm for surface is as follows.

Step 1. Let

$$\begin{aligned} A_0 &= \{t \in T(k) : t \cap I_1^- \neq \emptyset\}, \\ A_1 &= \{t \in T(k) : t \cap A_0 \neq \emptyset,\ \phi_1(t) = 1\}, \\ B &= \{t \cap t' : t \in A_0,\ t' \in A_1\}. \end{aligned} \tag{3.3}$$

Step 2. If $C = \{t \in T(k) \setminus A_0 : \dim[t \cap A_0] = 2, \phi_1(t) = 0\} = \emptyset$ then END.
Otherwise do Step 3.

Step 3. Add elements of the set C to A_0.
Next

$$\begin{aligned} A_1 &= \{t \in T(k) : t \cap A_0 \neq \emptyset,\ \phi_1(t) = 1\}, \\ B &= \{t \cap t' : t \in A_0,\ t' \in A_1,\ t \cap t' \neq \emptyset\} \end{aligned} \tag{3.4}$$

and go to Step 2.

Since $T(k)$ is finite, hence after finitely many steps set C is empty (the procedure ends).

Let us consider the family B. We may assume that B is closed under finite intersections.

The elements $b \in B$, such that $\dim[b] = 2,\ \dim[b] = 1,\ \dim[b] = 0$ are called squares, edges, and vertices.

Observation 4. The $\bigcup B$ separates cube I^3 between I_1^- and I_1^+.

Observation 5. Each edge $b \in B$ if $b \subset \partial I^3$ it is an edge of exactly 1 square, else it is an edge of 2 or 4 squares.

3.1.2. *Modification of B*

Let us divide each element of $\{a \in A_0 : a \cap \bigcup B \neq \emptyset\}$ onto 27 cubes (in the natural way).

Denote the set consisting of all this cubes by T'.

Create coloring map $\phi_1' : T' \to \{0,1\}$ as follows:

$$\phi_1'(t') = \begin{cases} 0 & t' \cap A_1 = \emptyset \\ 1 & t' \cap A_1 \neq \emptyset. \end{cases} \tag{3.5}$$

Now$B' = \{t \cap t' : t, t' \in T',\ \phi_1'(t) = 0,\ \phi_1'(t') = 1, t \cap t' \neq \emptyset\}$.

Observation 6. Any edge of B' is an edge of exactly one or of two squares from B' depending on whether or not it lies on ∂I^3.

Let us define coloring $\phi_2 : \{t \in B' : t \text{ is a square}\} \longrightarrow \{0,1\}$:

$$\phi_2(t) = \begin{cases} 0 & f_2(t_c) \le 0 \\ 1 & f_2(t_c) > 0, \end{cases} \tag{3.6}$$

where t_c is the center of square t.

The edge $t \in B'$ is said to be 2-coloured if there exists squares $s, s' \in B'$ such that $s \cap s' = t$ and $\phi_2(\{s, s'\}) = \{0,1\}$.

Observation 7. The vertex of 2-coloured edge is a subset of exactly one or even number of 2-coloured edges depending on whether or not it lies on ∂I^3.

Observation 8. The components of $\bigcup B' \cap \partial I^3$ are broken lines without self-cutting.

Observation 9. The number of broken lines lying on I_3^- and connecting I_2^- and I_2^+ is odd.

Lemma 3.1. *The number of 2-coloured edges from B', which one of vertices lies on I_3^- is odd.*

Proof. Let us consider components of the set $\bigcup B' \cap I_3^-$.

We have odd number of broken lines connecting I_2^- and I_2^+ and the number of the rest components is arbitrary.

Let us see that $\bigcup B' \subset I^3 \setminus I_1^- \cup I_1^+$.

So, the number of 2-coloured edges from B', which one of vertices lies on I_3^- is odd if it lies on broken line connecting I_2^- and I_2^+ else it is even (using the definition of ϕ_2).

According to Observation 9 this ends the proof. □

3.1.3. *Broken Line Connecting I_3^- and I_3^+*

Step 1. Let $E_0 = \{t \in B' : t \text{ is a 2-coloured edge, } t \cap I_3^- \neq \emptyset\}$,

$$E_1 = \emptyset. \tag{3.7}$$

Step 2. Take $e \in E_0 \setminus E_1$.

Add e to E_1.

The vertex $v \in e \cap I_3^-$ is said to be used.

Go to Step 3.

Step 3. Take unused vertex u of the last added edge to the set E_1.

If $u \in I_3^+$ END.

Otherwise,

If $u \in I_3^-$ go to Step 2.

Else go to Step 4.

Step 4. Take unused vertex u of the last added edge to the set E_1.

Next take 2-coloured edge $e \in B' \setminus E_1$ such that $v \in e$.

Now vertice v is said to be used.
Add e to the set E_1.
Go to Step 3.

First of all the number of 2-coloured edges from B', which one of vertices lies on I_3^- is odd (Lemma 3.1).

The second each vertex of 2-coloured edge is a subset of exactly one or even number of 2-coloured edges depending on whether or not it lies on ∂I^3 (Observation 7).

This arguments allows one to say that procedure is well defined.

Now our broken line connecting I_3^+ and I_3^- is created as follows:
let e_1 be the last added element to E_1.
If $e_i \cap I_3^- = \emptyset$ then e_{i+1} is previous added element to E_1
else Stop.

We obtained the sequence of edges $\{e_1, e_2, \dots, e_m\} \subset B'$. Let us define coloring $\phi_3 : \{t \in B' : t \text{ is an edge of } e_i\} \to \{0,1\}$ where $i \in \{1, \dots, m\}$:

$$\phi_3(t) = \begin{cases} 0 & f_3(t) \leq 0 \\ 1 & f_3(t) > 0. \end{cases} \tag{3.8}$$

It is easy to see that $\phi_3(e_1) = \{1\}$ and $\phi_3(e_m) = \{0\}$.

So starting from e_1 we search with order the first edge $e_k \in \{e_1, e_2, \dots, e_m\}$ such that $\phi_3(e_k) = \{0,1\}$.

3.2. Topological Part

For each $k \in N$, $1/k < \epsilon^*$ we have

(i) $v_k, v_k' \in e_k$ such that $f_3(v_k) \leq 0$ and $f_3(v_k') > 0$,

(ii) $u_k, u_k' \in t_u \cup t_u'$ such that $f_2(u_k) \leq 0$ and $f_2(u_k') > 0$ where t_u, t_u' are squares from B and $e_k \cap t_u \neq \emptyset \neq e_k \cap t_u'$,

(iii) $w_k, w_k' \in t_w \cup t_w'$ such that $f_1(w_k) \leq 0$ and $f_1(w_k') > 0$ where t_w is a cube from A_0, t_w' is a cube from A_1 and $e_k \cap t_w \neq \emptyset \neq e_k \cap t_w'$.

Define the sets $W_k := \operatorname{conv}\{v_k, v_k', u_k, u_k', w_k, w_k'\}$.
For each W_k there exist $c_k^1, c_k^2, c_k^3 \in W_k$ such that

$$f_1\left(c_k^1\right) = f_2\left(c_k^2\right) = f_3\left(c_k^3\right) = 0. \tag{3.9}$$

Without loss of generality we can assume that $\lim_{k\to\infty} c_k^1 = c$.

Moreover, $\lim_{k\to\infty} \operatorname{diam}[W_k] = 0$. So for each $c_k' \in W_k$ the fact $d(c, c_k') \leq d(c, c_k^1) + d(c_k^1, c_k')$ yields

$$\lim_{k\to\infty} c_k^1 = \lim_{k\to\infty} c_k^2 = \lim_{k\to\infty} c_k^3 = c. \tag{3.10}$$

So $f(c) = 0$ ends proof.

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