Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 185346, 25 pages
doi:10.1155/2012/185346

*Research Article*

# Two Sufficient Conditions for Hamilton and Dominating Cycles

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Received 24 March 2012; Accepted 14 August 2012

Academic Editor: Marianna Shubov



We prove that if $G$ is a 2-connect graph of size $q$ (the number of edges) and minimum degree $\delta$ with $\delta \geq \sqrt{2q/3+\epsilon/12}-1/2$, where $\epsilon = 11$ when $\delta = 2$ and $\epsilon = 31$ when $\delta \geq 3$, then each longest cycle in $G$ is a dominating cycle. The exact analog of this theorem for Hamilton cycles follows easily from two known results according to Dirac and Nash-Williams: each graph with $\delta \geq \sqrt{q+5/4}-1/2$ is hamiltonian. Both results are sharp in all respects.

## 1. Introduction

Only finite undirected graphs without loops or multiple edges are considered. We reserve $n$, $q$, $\delta$, and $\kappa$ to denote the number of vertices (order), the number of edges (size), the minimum degree, and the connectivity of a graph, respectively. A graph $G$ is hamiltonian if $G$ contains a hamiltonian cycle, that is, a cycle of length $n$. Further, a cycle $C$ in $G$ is called a dominating cycle if the vertices in $G\backslash C$ are mutually nonadjacent. A good reference for any undefined terms is [1].

The following two well-known theorems provide two classic sufficient conditions for Hamilton and dominating cycles by linking the minimum degree $\delta$ and order $n$.

**Theorem A** (see [2]). *Every graph with $\delta \geq (1/2)n$ is hamiltonian.*

**Theorem B** (see [3]). *If $G$ is a 2-connect graph with $\delta \geq (1/3)(n+2)$, then each longest cycle in $G$ is a dominating cycle.*

The exact analog of Theorem A that links the minimum degree $\delta$ and size $q$ easily follows from Theorem A and a particular result according to Nash-Williams [4] (see Theorem 1.1 below).

**Theorem 1.1.** *Every graph is hamiltonian if*

$$\delta \geq \sqrt{q + \frac{5}{4}} - \frac{1}{2}. \tag{1.1}$$

The hypothesis in Theorem 1.1 is equivalent to $q \leq \delta^2 + \delta - 1$ and cannot be relaxed to $q \leq \delta^2 + \delta$ due to the graph $K_1 + 2K_\delta$ consisting of two copies of $K_{\delta+1}$ and having exactly one vertex in common. Hence, Theorem 1.1 is best possible.

The main goal of this paper is to prove the exact analog of Theorem B for dominating cycles based on another similar relation between $\delta$ and $q$.

**Theorem 1.2.** *Let G be a 2-connect graph with*

$$\delta \geq \sqrt{\frac{2q}{3} + \frac{\epsilon}{12}} - \frac{1}{2}, \tag{1.2}$$

*where $\epsilon = 11$ when $\delta = 2$ and $\epsilon = 31$ when $\delta \geq 3$. Then each longest cycle in G is a dominating cycle.*

To show that Theorem 1.2 is sharp, suppose first that $\delta = 2$, implying that the hypothesis in Theorem 1.2 is equivalent to $q \leq 8$. The graph $K_1 + 2K_2$ shows that the connectivity condition $\kappa \geq 2$ in Theorem 1.2 cannot be relaxed by replacing it with $\kappa \geq 1$. The graph with vertex set $\{v_1, v_2, \ldots, v_8\}$ and edge set

$$\{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1, v_1v_7, v_7v_8, v_8v_4\} \tag{1.3}$$

shows that the size bound $q \leq 8$ cannot be relaxed by replacing it with $q \leq 9$. Finally, the graph $K_2 + 3K_1$ shows that the conclusion "each longest cycle in $G$ is a dominating cycle" cannot be strengthened by replacing it with "$G$ is hamiltonian." Analogously, we can use $K_1 + 2K_\delta$, $K_2 + 3K_{\delta-1}$, and $K_\delta + (\delta+1)K_1$, respectively, to show that Theorem 1.2 is sharp when $\delta \geq 3$. So, Theorem 1.2 is best possible in all respects.

To prove Theorems 1.1 and 1.2, we need two known results, the first of which is belongs Nash-Williams [4].

**Theorem C** (see [4]). *If $\delta = (n-1)/2$, then either G is hamiltonian or $G = K_1 + 2K_\delta$, or $G = \overline{K}_{\delta+1} + G_\delta$, where $G_\delta$ denote an arbitrary graph on $\delta$ vertices.*

The next theorem provides a lower bound for the length of a longest cycle in 2-connected graphs according to Dirac [2].

**Theorem D** (see [2]). *Every 2-connected graph either has a hamiltonian cycle or has a cycle of length at least $2\delta$.*

## 2. Notations and Preliminaries

The set of vertices of a graph $G$ is denoted by $V(G)$ and the set of edges by $E(G)$. For $S$, a subset of $V(G)$, we denote by $G\backslash S$ the maximum subgraph of $G$ with vertex set $V(G)\backslash S$.

We write $G[S]$ for the subgraph of $G$ induced by $S$. For a subgraph $H$ of $G$, we use $G\backslash H$ short for $G\backslash V(H)$. The neighborhood of a vertex $x \in V(G)$ will be denoted by $N(x)$. Set $d(x) = |N(x)|$. Furthermore, for a subgraph $H$ of $G$ and $x \in V(G)$, we define $N_H(x) = N(x) \cap V(H)$ and $d_H(x) = |N_H(x)|$.

A simple cycle (or just a cycle) $C$ of length $t$ is a sequence $v_1v_2 \cdots v_tv_1$ of distinct vertices $v_1, \ldots, v_t$ with $v_iv_{i+1} \in E(G)$ for each $i \in \{1, \ldots, t\}$, where $v_{t+1} = v_1$. When $t = 2$, the cycle $C = v_1v_2v_1$ on two vertices $v_1, v_2$ coincides with the edge $v_1v_2$, and when $t = 1$, the cycle $C = v_1$ coincides with the vertex $v_1$. So, all vertices and edges in a graph can be considered as cycles of lengths 1 and 2, respectively.

Paths and cycles in a graph $G$ are considered as subgraphs of $G$. If $Q$ is a path or a cycle, then the length of $Q$, denoted by $|Q|$, is $|E(Q)|$. We write $Q$ with a given orientation by $\vec{Q}$. For $x, y \in V(Q)$, we denote by $x\vec{Q}y$ the subpath of $Q$ in the chosen direction from $x$ to $y$. For $x \in V(Q)$, we denote the $h$th successor and the $h$th predecessor of $x$ on $\vec{Q}$ by $x^{+h}$ and $x^{-h}$, respectively. We abbreviate $x^{+1}$ and $x^{-1}$ by $x^+$ and $x^-$, respectively.

### *Special Definitions*

Let $G$ be a graph, $C$ a longest cycle in $G$, and $P = x\vec{P}y$ a longest path in $G\backslash C$ of length $\overline{p} \geq 0$. Let $\xi_1, \xi_2, \ldots, \xi_s$ be the elements of $N_C(x) \cup N_C(y)$ occurring on $C$ in a consecutive order. Set

$$I_i = \xi_i\vec{C}\xi_{i+1}, \quad I_i^* = \xi_i^+\vec{C}\xi_{i+1}^- \quad (i = 1, 2, \ldots, s), \tag{2.1}$$

where $\xi_{s+1} = \xi_1$.

(∗1) We call $I_1, I_2, \ldots, I_s$ elementary segments on $C$ created by $N_C(x) \cup N_C(y)$.

(∗2) We call a path $L = z\vec{L}w$ an intermediate path between two distinct elementary segments $I_a$ and $I_b$ if

$$z \in V(I_a^*), \quad w \in V(I_b^*), \quad V(L) \cap V(C \cup P) = \{z, w\}. \tag{2.2}$$

(∗3) The set of all intermediate paths between elementary segments $I_{i_1}, I_{i_2}, \ldots, I_{i_t}$ will be denoted by $\Upsilon(I_{i_1}, I_{i_2}, \ldots, I_{i_t})$.

**Lemma 2.1.** *Let $G$ be a graph, $C$ a longest cycle in $G$, and $P = x\vec{P}y$ a longest path in $G\backslash C$ of length $\overline{p} \geq 1$. If $|N_C(x)| \geq 2$, $|N_C(y)| \geq 2$ and $N_C(x) \neq N_C(y)$, then*

$$|C| \geq \begin{cases} 3\delta + \max\{\sigma_1, \sigma_2\} - 1 \geq 3\delta, & \text{if } \overline{p} = 1, \\ \max\{2\overline{p} + 8, 4\delta - 2\overline{p}\}, & \text{if } \overline{p} \geq 2, \end{cases} \tag{2.3}$$

*where $\sigma_1 = |N_C(x) \backslash N_C(y)|$ and $\sigma_2 = |N_C(y) \backslash N_C(x)|$.*

**Lemma 2.2.** *Let $G$ be a graph, $C$ a longest cycle in $G$, and $P = x\vec{P}y$ a longest path in $G\backslash C$ of length $\overline{p} \geq 0$. If $N_C(x) = N_C(y)$, $|N_C(x)| \geq 2$ and $I_a$, $I_b$ are elementary segments induced by $N_C(x) \cup N_C(y)$, then*

*(a1) if $L$ is an intermediate path between $I_a$ and $I_b$, then*

$$|I_a| + |I_b| \geq 2\overline{p} + 2|L| + 4, \tag{2.4}$$

*(a2) if $\Upsilon(I_a, I_b) \subseteq E(G)$ and $|\Upsilon(I_a, I_b)| = i$ for some $i \in \{1, 2, 3\}$, then*

$$|I_a| + |I_b| \geq 2\overline{p} + i + 5. \tag{2.5}$$

**Lemma 2.3.** *Let $G$ be a graph, $S$ a cut set in $G$, and $H$ a connected component of $G\backslash S$ of order $h$. Then*

$$q_H \geq \frac{h(2\delta - h + 1)}{2}, \tag{2.6}$$

*where $q_H = |\{xy \in E(G) : \{x, y\} \cap V(H) \neq \emptyset\}|$.*

**Lemma 2.4.** *Let $G$ be a 2-connect graph. If $\delta \geq (n-2)/3$, then either*

$$q \geq \begin{cases} 9 & \text{when } \delta = 2, \\ \dfrac{3(\delta-1)(\delta+2)}{2} & \text{when } \delta \geq 3, \end{cases} \tag{2.7}$$

*or each longest cycle in $G$ is a dominating cycle.*

## 3. Proofs

*Proof of Lemma 2.1.* Put

$$A_1 = N_C(x)\backslash N_C(y), \qquad A_2 = N_C(y)\backslash N_C(x), \qquad M = N_C(x) \cap N_C(y). \tag{3.1}$$

By the hypothesis, $N_C(x) \neq N_C(y)$, implying that

$$\max\{|A_1|, |A_2|\} \geq 1. \tag{3.2}$$

Let $\xi_1, \xi_2, \ldots, \xi_s$ be the elements of $N_C(x) \cup N_C(y)$ occuring on $C$ in a consecutive order. Put $I_i = \xi_i\vec{C}\xi_{i+1} (i = 1, 2, \ldots, s)$, where $\xi_{s+1} = \xi_1$. Clearly, $s = |A_1| + |A_2| + |M|$. Since $C$ is extreme, $|I_i| \geq 2 (i = 1, 2, \ldots, s)$. Next, if $\{\xi_i, \xi_{i+1}\} \cap M \neq \emptyset$ for some $i \in \{1, 2, \ldots, s\}$, then $|I_i| \geq \overline{p} + 2$. Further, if either $\xi_i \in A_1$, $\xi_{i+1} \in A_2$ or $\xi_i \in A_2$, $\xi_{i+1} \in A_1$, then again $|I_i| \geq \overline{p} + 2$.

*Case* 1. ($\overline{p} = 1$).

*Case* 1.1 ($|A_i| \geq 1 (i = 1,2)$). It follows that among $I_1, I_2, \ldots, I_s$ there are $|M| + 2$ segments of length at least $\overline{p} + 2$. Observing also that each of the remaining $s - (|M| + 2)$ segments has a length at least 2, we have

$$\begin{aligned} |C| &\geq (\overline{p} + 2)(|M| + 2) + 2(s - |M| - 2) \\ &= 3(|M| + 2) + 2(|A_1| + |A_2| - 2) \\ &= 2|A_1| + 2|A_2| + 3|M| + 2. \end{aligned} \tag{3.3}$$

Since $|A_1| = d(x) - |M| - 1$ and $|A_2| = d(y) - |M| - 1$,

$$|C| \geq 2d(x) + 2d(y) - |M| - 2 \geq 3\delta + d(x) - |M| - 2. \tag{3.4}$$

Recalling that $d(x) = |M| + |A_1| + 1$, we get

$$|C| \geq 3\delta + |A_1| - 1 = 3\delta + \sigma_1 - 1. \tag{3.5}$$

Analogously, $|C| \geq 3\delta + \sigma_2 - 1$. So,

$$|C| \geq 3\delta + \max\{\sigma_1, \sigma_2\} - 1 \geq 3\delta. \tag{3.6}$$

*Case* 1.2 (either $|A_1| \geq 1, |A_2| = 0$ or $|A_1| = 0, |A_2| \geq 1$). Assume without loss of generality that $|A_1| \geq 1$ and $|A_2| = 0$, that is, $|N_C(y)| = |M| \geq 2$ and $s = |A_1| + |M|$. Hence, among $I_1, I_2, \ldots, I_s$ there are $|M| + 1$ segments of length at least $\overline{p} + 2 = 3$. Taking into account that each of the remaining $s - (|M| + 1)$ segments has a length at least 2 and $|M| + 1 = d(y)$, we get

$$\begin{aligned} |C| &\geq 3(|M| + 1) + 2(s - |M| - 1) = 3d(y) + 2(|A_1| - 1) \\ &\geq 3\delta + |A_1| - 1 = 3\delta + \max\{\sigma_1, \sigma_2\} - 1 \geq 3\delta. \end{aligned} \tag{3.7}$$

*Case* 2 ($\overline{p} \geq 2$). We first prove that $|C| \geq 2\overline{p} + 8$. Since $|N_C(x)| \geq 2$ and $|N_C(y)| \geq 2$, there are at least two segments among $I_1, I_2, \ldots, I_s$ of length at least $\overline{p} + 2$. If $|M| = 0$, then clearly $s \geq 4$ and

$$|C| \geq 2(\overline{p} + 2) + 2(s - 2) \geq 2\overline{p} + 8. \tag{3.8}$$

Otherwise, since $\max\{|A_1|, |A_2|\} \geq 1$, there are at least three elementary segments of length at least $\overline{p} + 2$, that is,

$$|C| \geq 3(\overline{p} + 2) \geq 2\overline{p} + 8. \tag{3.9}$$

So, in any case, $|C| \geq 2\overline{p} + 8$.

To prove that $|C| \geq 4\delta - 2\overline{p}$, we distinguish two main cases.

*Case* 2.1 ($|A_i| \geq 1(i = 1,2)$). It follows that among $I_1, I_2, \ldots, I_s$ there are $|M| + 2$ segments of length at least $\overline{p} + 2$. Further, since each of the remaining $s - (|M| + 2)$ segments has a length at least 2, we get

$$\begin{aligned}|C| &\geq (\overline{p} + 2)(|M| + 2) + 2(s - |M| - 2) \\ &= (\overline{p} - 2)|M| + (2\overline{p} + 4|M| + 4) + 2(|A_1| + |A_2| - 2) \\ &\geq 2|A_1| + 2|A_2| + 4|M| + 2\overline{p}.\end{aligned} \tag{3.10}$$

Observing also that

$$|A_1| + |M| + \overline{p} \geq d(x), \qquad |A_2| + |M| + \overline{p} \geq d(y), \tag{3.11}$$

we have

$$2|A_1| + 2|A_2| + 4|M| + 2\overline{p} \geq 2d(x) + 2d(y) - 2\overline{p} \geq 4\delta - 2\overline{p}, \tag{3.12}$$

implying that $|C| \geq 4\delta - 2\overline{p}$.

*Case* 2.2 (either $|A_1| \geq 1$, $|A_2| = 0$ or $|A_1| = 0$, $|A_2| \geq 1$). Assume without loss of generality that $|A_1| \geq 1$ and $|A_2| = 0$, that is, $|N_C(y)| = |M| \geq 2$ and $s = |A_1| + |M|$. It follows that among $I_1, I_2, \ldots, I_s$ there are $|M| + 1$ segments of length at least $\overline{p} + 2$. Observing also that $|M| + \overline{p} \geq d(y) \geq \delta$, that is, $2\overline{p} + 4|M| \geq 4\delta - 2\overline{p}$, we get

$$\begin{aligned}|C| &\geq (\overline{p} + 2)(|M| + 1) \geq (\overline{p} - 2)(|M| - 1) + 2\overline{p} + 4|M| \\ &\geq 2\overline{p} + 4|M| \geq 4\delta - 2\overline{p}.\end{aligned} \tag{3.13}$$

□

*Proof of Lemma 2.2.* Let $\xi_1, \xi_2, \ldots, \xi_s$ be the elements of $N_C(x)$ occuring on $C$ in a consecutive order. Put $I_i = \xi_i \overrightarrow{C} \xi_{i+1} (i = 1,2,\ldots,s)$, where $\xi_{s+1} = \xi_1$. To prove $(a1)$, let $L = z\overrightarrow{L}w$ be an intermediate path between elementary segments $I_a$ and $I_b$ with $z \in V(I_a^*)$ and $w \in V(I_b^*)$. Put

$$\begin{gathered}\left|\xi_a \overrightarrow{C} z\right| = d_1, \qquad \left|z \overrightarrow{C} \xi_{a+1}\right| = d_2, \qquad \left|\xi_b \overrightarrow{C} w\right| = d_3, \qquad \left|w \overrightarrow{C} \xi_{b+1}\right| = d_4, \\ C' = \xi_a x \overrightarrow{P} y \xi_b \overleftarrow{C} z \overrightarrow{L} w \overrightarrow{C} \xi_a.\end{gathered} \tag{3.14}$$

Clearly,

$$\left|C'\right| = |C| - d_1 - d_3 + |L| + |P| + 2. \tag{3.15}$$

Since $C$ is extreme, we have $|C| \geq |C'|$, implying that $d_1 + d_3 \geq \overline{p} + |L| + 2$. By a symmetric argument, $d_2 + d_4 \geq \overline{p} + |L| + 2$. Hence

$$|I_a| + |I_b| = \sum_{i=1}^{4} d_i \geq 2\overline{p} + 2|L| + 4. \tag{3.16}$$

The proof of $(a1)$ is complete. To prove $(a2)$, let $\Upsilon(I_a, I_b) \subseteq E(G)$ and $|\Upsilon(I_a, I_b)| = i$ for some $i \in \{1, 2, 3\}$.

*Case* 1 ($i = 1$). It follows that $\Upsilon(I_a, I_b)$ consists of a unique intermediate edge $L = zw$. By $(a1)$,

$$|I_a| + |I_b| \geq 2\overline{p} + 2|L| + 4 = 2\overline{p} + 6. \tag{3.17}$$

*Case* 2 ($i = 2$). It follows that $\Upsilon(I_a, I_b)$ consists of two edges $e_1,\ e_2$. Put $e_1 = z_1 w_1$ and $e_2 = z_2 w_2$, where $\{z_1, z_2\} \subseteq V(I_a^*)$ and $\{w_1, w_2\} \subseteq V(I_b^*)$.

*Case* 2.1 ($z_1 \neq z_2$ and $w_1 \neq w_2$). Assume without loss of generality that $z_1$ and $z_2$ occur in this order on $I_a$.

*Case* 2.1.1. $w_2$ and $w_1$ occur in this order on $I_b$.

Put

$$\begin{gathered}
\left|\xi_a \vec{C} z_1\right| = d_1, \qquad \left|z_1 \vec{C} z_2\right| = d_2, \qquad \left|z_2 \vec{C} \xi_{a+1}\right| = d_3, \\
\left|\xi_b \vec{C} w_2\right| = d_4, \qquad \left|w_2 \vec{C} w_1\right| = d_5, \qquad \left|w_1 \vec{C} \xi_{b+1}\right| = d_6, \\
C' = \xi_a \vec{C} z_1 w_1 \overleftarrow{C} w_2 z_2 \vec{C} \xi_b x \vec{P} y \xi_{b+1} \vec{C} \xi_a.
\end{gathered} \tag{3.18}$$

Clearly,

$$\begin{aligned}
\left|C'\right| &= |C| - d_2 - d_4 - d_6 + |\{e_1\}| + |\{e_2\}| + |P| + 2 \\
&= |C| - d_2 - d_4 - d_6 + \overline{p} + 4.
\end{aligned} \tag{3.19}$$

Since $C$ is extreme, $|C| \geq |C'|$, implying that $d_2 + d_4 + d_6 \geq \overline{p} + 4$. By a symmetric argument, $d_1 + d_3 + d_5 \geq \overline{p} + 4$. Hence

$$|I_a| + |I_b| = \sum_{i=1}^{6} d_i \geq 2\overline{p} + 8. \tag{3.20}$$

*Case* 2.1.2. $w_1$ and $w_2$ occur in this order on $I_b$.

Putting

$$C' = \xi_a \vec{C} z_1 w_1 \vec{C} w_2 z_2 \vec{C} \xi_b x \vec{P} y \xi_{b+1} \vec{C} \xi_a, \tag{3.21}$$

we can argue as in Case 2.1.1.

*Case* 2.2 (either $z_1 = z_2$, $w_1 \neq w_2$ or $z_1 \neq z_2$, $w_1 = w_2$). Assume without loss of generality that $z_1 \neq z_2$, $w_1 = w_2$ and $z_1, z_2$ occur in this order on $I_a$. Put

$$\begin{gathered}
\left|\xi_a \vec{C} z_1\right| = d_1, \qquad \left|z_1 \vec{C} z_2\right| = d_2, \qquad \left|z_2 \vec{C} \xi_{a+1}\right| = d_3, \\
\left|\xi_b \vec{C} w_1\right| = d_4, \qquad \left|w_1 \vec{C} \xi_{b+1}\right| = d_5, \\
C' = \xi_a x \vec{P} y \xi_b \overleftarrow{C} z_1 w_1 \vec{C} \xi_a, \\
C'' = \xi_a \vec{C} z_2 w_1 \overleftarrow{C} \xi_{a+1} x \vec{P} y \xi_{b+1} \vec{C} \xi_a.
\end{gathered} \tag{3.22}$$

Clearly,

$$\begin{gathered}
\left|C'\right| = |C| - d_1 - d_4 + |\{e_1\}| + |P| + 2 = |C| - d_1 - d_4 + \overline{p} + 3, \\
\left|C''\right| = |C| - d_3 - d_5 + |\{e_2\}| + |P| + 2 = |C| - d_3 - d_5 + \overline{p} + 3.
\end{gathered} \tag{3.23}$$

Since $C$ is extreme, $|C| \geq |C'|$ and $|C| \geq |C''|$, implying that

$$d_1 + d_4 \geq \overline{p} + 3, \qquad d_3 + d_5 \geq \overline{p} + 3. \tag{3.24}$$

Hence,

$$|I_a| + |I_b| = \sum_{i=1}^{5} d_i \geq d_1 + d_3 + d_4 + d_5 + 1 \geq 2\overline{p} + 7. \tag{3.25}$$

*Case* 3 ($i = 3$). It follows that $\Upsilon(I_a, I_b)$ consists of three edges $e_1,\ e_2,\ e_3$. Let $e_i = z_i w_i$ ($i = 1,2,3$), where $\{z_1, z_2, z_3\} \subseteq V(I_a^*)$ and $\{w_1, w_2, w_3\} \subseteq V(I_b^*)$. If there are two independent edges among $e_1,\ e_2,\ e_3$, then we can argue as in Case 2.1. Otherwise, we can assume without loss of generality that $w_1 = w_2 = w_3$ and $z_1,\ z_2,\ z_3$ occur in this order on $I_a$. Put

$$\begin{gathered}
\left|\xi_a \vec{C} z_1\right| = d_1, \qquad \left|z_1 \vec{C} z_2\right| = d_2, \qquad \left|z_2 \vec{C} z_3\right| = d_3, \\
\left|z_3 \vec{C} \xi_{a+1}\right| = d_4, \qquad \left|\xi_b \vec{C} w_1\right| = d_5, \qquad \left|w_1 \vec{C} \xi_{b+1}\right| = d_6, \\
C' = \xi_a x \vec{P} y \xi_b \overleftarrow{C} z_1 w_1 \vec{C} \xi_a, \\
C'' = \xi_a \vec{C} z_3 w_1 \overleftarrow{C} \xi_{a+1} x \vec{P} y \xi_{b+1} \vec{C} \xi_a.
\end{gathered} \tag{3.26}$$

Clearly,

$$\begin{gathered}
\left|C'\right| = |C| - d_1 - d_5 + |\{e_1\}| + \overline{p} + 2, \\
\left|C''\right| = |C| - d_4 - d_6 + |\{e_3\}| + \overline{p} + 2.
\end{gathered} \tag{3.27}$$

Since $C$ is extreme, we have $|C| \geq |C'|$ and $|C| \geq |C''|$, implying that

$$d_1 + d_5 \geq \overline{p} + 3, \qquad d_4 + d_6 \geq \overline{p} + 3. \tag{3.28}$$

Hence,

$$|I_a| + |I_b| = \sum_{i=1}^{6} d_i \geq d_1 + d_4 + d_5 + d_6 + 2 \geq 2\overline{p} + 8. \tag{3.29}$$

□

*Proof of Lemma 2.3.* Put

$$V(H) = \{v_1, \ldots, v_h\}, \qquad |N(v_i) \cap S| = \beta_i \quad (i = 1, \ldots, h). \tag{3.30}$$

Observing that $h \geq d(v_i) - \beta_i + 1 \geq \delta - \beta_i + 1$ for each $i \in \{1, 2, \ldots, h\}$, we have $\beta_i \geq \delta - h + 1$ $(i = 1, 2, \ldots, h)$. Therefore,

$$\begin{aligned} q_H &= q(H) + \sum_{i=1}^{h} \beta_i = \frac{1}{2} \sum_{i=1}^{h} d_H(v_i) + \sum_{i=1}^{h} \beta_i, \\ &= \frac{1}{2} \sum_{i=1}^{h} (d_H(v_i) + \beta_i) + \frac{1}{2} \sum_{i=1}^{h} \beta_i = \frac{1}{2} \sum_{i=1}^{h} d(v_i) + \frac{1}{2} \sum_{i=1}^{h} (\delta - h + 1), \\ &\geq \frac{1}{2} h\delta + \frac{1}{2} h(\delta - h + 1) = \frac{h(2\delta - h + 1)}{2}. \end{aligned} \tag{3.31}$$

□

*Proof of Lemma 2.4.* Let $C$ be a longest cycle in $G$ and $P = x_1 \vec{P} x_2$ a longest path in $G \backslash C$ of length $\overline{p}$. If $|V(P)| \leq 1$, then $C$ is a dominating cycle and we are done. Let $|V(P)| \geq 2$, that is, $\overline{p} \geq 1$. By the hypothesis, $|C| + \overline{p} + 1 \leq n \leq 3\delta + 2$. Further, by Theorem D, $|C| \geq 2\delta$. From these inequalities, we get

$$n \leq 3\delta + 2, \qquad |C| \leq 3\delta - \overline{p} + 1, \quad 1 \leq \overline{p} \leq \delta + 1. \tag{3.32}$$

Let $\xi_1, \xi_2, \ldots, \xi_s$ be the elements of $N_C(x_1) \cup N_C(x_2)$ occuring on $C$ in a consecutive order. Put

$$I_i = \xi_i \vec{C} \xi_{i+1}, \qquad I_i^* = \xi_i^+ \vec{C} \xi_{i+1}^- \quad (i = 1, 2, \ldots, s), \tag{3.33}$$

where $\xi_{s+1} = \xi_1$.

*Case* 1 ($\delta = 2$). Let $Q$ be a longest path in $G$ with $Q = \xi\vec{Q}\eta$ and $V(Q) \cap V(C) = \{\xi, \eta\}$. Since $C$ is extreme, we have $|\xi\vec{C}\eta| \geq |Q|$ and $|\eta\vec{C}\xi| \geq |Q|$, implying that

$$|C| = \left|\xi\vec{C}\eta\right| + \left|\eta\vec{C}\xi\right| \geq 2|Q|. \tag{3.34}$$

Since $\kappa \geq 2$ and $\overline{p} \geq 1$, we have $|Q| \geq 3$. By (3.34), $|C| \geq 2|Q| \geq 6$, implying that $q \geq |C| + |Q| \geq 9$.

*Case* 2. ($\delta \geq 3$).

*Case* 2.1 ($\overline{p} = 1$). By (3.32),

$$|C| \leq 3\delta. \tag{3.35}$$

*Case* 2.1.1 ($N_C(x_1) \neq N_C(x_2)$). It follows that $\max\{\sigma_1, \sigma_2\} \geq 1$, where

$$\sigma_1 = |N_C(x_1) \backslash N_C(x_2)|, \qquad \sigma_2 = |N_C(x_2) \backslash N_C(x_1)|. \tag{3.36}$$

By Lemma 2.1, $|C| \geq 3\delta$. Recalling (3.35), we get $|C| = 3\delta$. If $\max\{\sigma_1, \sigma_2\} \geq 2$, then by Lemma 2.1, $|C| \geq 3\delta + 1$, contradicting (3.35). Let $\max\{\sigma_1, \sigma_2\} = 1$. Clearly, $s \geq \delta$ and $|I_i| \geq 3$ ($i = 1, 2, \ldots, s$). Further, if $s \geq \delta + 1$, then $|C| \geq 3s \geq 3\delta + 3$, again contradicting (3.35). Let $s = \delta$, implying that $|I_i| = 3$ ($i = 1, 2, \ldots, s$). By Lemma 2.2, $\Upsilon(I_1, I_2, \ldots, I_s) = \emptyset$. Let $H_1, H_2, \ldots, H_{s+1}$ be the connected components of $G \backslash \{\xi_1, \xi_2, \ldots, \xi_s\}$ with $V(H_i) = V(I_i^*)$ ($i = 1, 2, \ldots, s$) and $V(H_{s+1}) = \{x_1, x_2\}$. For each $i \in \{1, 2, \ldots, s+1\}$, put

$$h_i = |V(H_i)|, \qquad q_i = \left|\{xy \in E(G) : \{x, y\} \cap V(H_i) \neq \emptyset\}\right|. \tag{3.37}$$

Clearly, $h_i = 2$ ($i = 1, 2, \ldots, s+1$). By Lemma 2.3,

$$q_i \geq \frac{h_i(2\delta - h_i + 1)}{2} = 2\delta - 1 \quad (i = 1, 2, \ldots, s+1), \tag{3.38}$$

implying that

$$q \geq \sum_{i=1}^{s+1} q_i \geq (s+1)(2\delta - 1) = (\delta + 1)(2\delta - 1) > \frac{3(\delta - 1)(\delta + 2)}{2}. \tag{3.39}$$

*Case* 2.1.2 ($N_C(x_1) = N_C(x_2)$). Clearly, $s \geq \delta - 1$. If $s \geq \delta$, then we can argue as in Case 2.1.1. Let $s = \delta - 1$. Further, if $|I_i| + |I_j| \geq 10$ for some distinct $i, j \in \{1, 2, \ldots, s\}$, then $|C| \geq 10 + 3(s-2) = 3\delta + 1$, contradicting (3.35). Hence

$$|I_i| + \left|I_j\right| \leq 9 \quad \text{for each distinct } i, j \in \{1, 2, \ldots, s\}. \tag{3.40}$$

*Claim* 1. $\Upsilon(I_1, I_2, \ldots, I_s) \subseteq E(G)$ and

(1) if $\max_i |I_i| \le 4$ then $|\Upsilon(I_1, I_2, \ldots, I_s)| \le 3$,

(2) if $\max_i |I_i| = 5$ then $|\Upsilon(I_1, I_2, \ldots, I_s)| \le \delta - 1$,

(3) if $\max_i |I_i| = 6$ then $|\Upsilon(I_1, I_2, \ldots, I_s)| \le 2(\delta - 2)$.

*Proof.* If $\Upsilon(I_1, I_2, \ldots, I_s) = \emptyset$ then we are done. Otherwise, $\Upsilon(I_a, I_b) \neq \emptyset$, for some distinct $a, b \in \{1, 2, \ldots, s\}$. By definition, there is an intermediate path $L$ between $I_a$ and $I_b$. If $|L| \ge 2$, then by Lemma 2.2,

$$|I_a| + |I_b| \ge 2\overline{p} + 2|L| + 4 \ge 10, \tag{3.41}$$

contradicting (3.40). Otherwise, $|L| = 1$ and therefore, $\Upsilon(I_1, I_2, \ldots, I_s) \subseteq E(G)$. By Lemma 2.2, $|I_a| + |I_b| \ge 2\overline{p} + 6 = 8$. Combining this with (3.40), we have

$$8 \le |I_a| + |I_b| \le 9. \tag{3.42}$$

Furthermore, if $|\Upsilon(I_a, I_b)| \ge 3$, then by Lemma 2.2, $|I_a| + |I_b| \ge 2\overline{p} + 8 = 10$, contradicting (3.42). So,

$$1 \le \left|\Upsilon(I_i, I_j)\right| \le 2 \quad \text{for each distinct } i, j \in \{1, 2, \ldots, s\}. \tag{3.43}$$

Put $r = |\{i : |I_i| \ge 4\}|$. If $r \ge 4$, then $|C| \ge 16 + 3(s - 4) = 3\delta + 1$, contradicting (3.35). Further, if $r = 0$, then by Lemma 2.2, $\Upsilon(I_1, I_2, \ldots, I_s) = \emptyset$. Let $1 \le r \le 3$.

*Case a*1 ($r = 3$). It follows that $|I_i| \ge 4$ ($i = a, b, c$) for some distinct $a, b, c \in \{1, 2, \ldots, s\}$ and $|I_i| = 3$ for each $i \in \{1, 2, \ldots, s\} \setminus \{a, b, c\}$. Recalling that $s = \delta - 1$ and $|C| = 3\delta$, we have $|I_a| = |I_b| = |I_c| = 4$, that is, $\max_i |I_i| = 4$. By Lemma 2.2, $|\Upsilon(I_i, I_j)| \le 1$ for each distinct $i, j \in \{a, b, c\}$. Moreover, we have $|\Upsilon(I_i, I_j)| = 0$ if either $i \notin \{a, b, c\}$ or $j \notin \{a, b, c\}$. So,

$$|\Upsilon(I_1, I_2, \ldots, I_s)| = |\Upsilon(I_a, I_b, I_c)| \le 3. \tag{3.44}$$

*Case a*2 ($r = 2$). It follows that $|I_a| \ge 4$ and $|I_b| \ge 4$ for some distinct $a, b \in \{1, 2, \ldots, s\}$ and $|I_i| = 3$ for each $i \in \{1, 2, \ldots, s\} \setminus \{a, b\}$. By (3.42), we can assume without loss of generality that either $|I_a| = |I_b| = 4$ or $|I_a| = 5$, $|I_b| = 4$.

*Case a*2.1 ($|I_a| = |I_b| = 4$). It follows that $\max_i |I_i| = 4$. By Lemma 2.2, $|\Upsilon(I_a, I_b)| \le 1$ and $\Upsilon(I_i, I_j) = \emptyset$ if $\{i, j\} \neq \{a, b\}$, implying that $|\Upsilon(I_1, I_2, \ldots, I_s)| = |\Upsilon(I_a, I_b)| \le 1$.

*Case a*2.2 ($|I_a| = 5$, $|I_b| = 4$). It follows that $\max_i |I_i| = 5$. By Lemma 2.2, we have $|\Upsilon(I_a, I_b)| \le 2$ and $|\Upsilon(I_a, I_i)| \le 1$ for each $i \in \{1, 2, \ldots, s\} \setminus \{a, b\}$. Furthermore, $\Upsilon(I_i, I_j) = \emptyset$ if $a \notin \{i, j\}$. Thus, $|\Upsilon(I_1, I_2, \ldots, I_s)| \le \delta - 1$.

*Case* $a3$ $(r = 1)$. It follows that $|I_a| \geq 4$ for some $a \in \{1, 2, \ldots, s\}$ and $|I_i| = 3$ for each $i \in \{1, 2, \ldots, s\} \setminus \{a\}$. By (3.42), $4 \leq |I_a| \leq 6$.

*Case* $a3.1$ $(|I_a| = 4)$. It follows that $\max_i |I_i| = 4$. By Lemma 2.2, $\Upsilon(I_a, I_i) = \emptyset$ for each $i \in \{1, 2, \ldots, s\} \setminus \{a\}$, implying that $|\Upsilon(I_1, I_2, \ldots, I_s)| = 0$.

*Case* $a3.2$ $(|I_a| = 5)$. It follows that $\max_i |I_i| = 5$. By Lemma 2.2, $|\Upsilon(I_a, I_i)| \leq 1$ for each $i \in \{1, 2, \ldots, s\} \setminus \{a\}$ and $\Upsilon(I_i, I_j) = \emptyset$ if $a \notin \{i, j\}$, that is, $|\Upsilon(I_1, I_2, \ldots, I_s)| \leq \delta - 2$.

*Case* $a3.3$ $(|I_a| = 6)$. It follows that $\max_i |I_i| = 6$. By Lemma 2.2, $|\Upsilon(I_a, I_i)| \leq 2$ for each $i \in \{1, 2, \ldots, s\} \setminus \{a\}$ and $\Upsilon(I_i, I_j) = \emptyset$ if $a \notin \{i, j\}$, that is, $|\Upsilon(I_1, I_2, \ldots, I_s)| \leq 2(\delta - 2)$. Claim 1 is proved.

Let $e \in \Upsilon(I_1, I_2, \ldots, I_s)$ and let $e = zw$, where $z \in V(I_a^*)$ and $w \in V(I_b^*)$ for some distinct $a, b \in \{1, 2, \ldots, s\}$. Put $G' = G \setminus e$. Form a graph $G''$ in the following way. If $d(z) \geq \delta$ and $d(w) \geq \delta$ in $G'$ then we take $G'' = G'$. Next, suppose that $d(z) = \delta - 1$ and $d(w) \geq \delta$ in $G'$. Put

$$U_1 = (\{\xi_1, \xi_2, \ldots, \xi_s\} \cup V(I_a^*)) \setminus \{z\}, \qquad U_2 = \left(\{\xi_1, \xi_2, \ldots, \xi_s\} \cup V\left(I_b^*\right)\right) \setminus \{w\}. \tag{3.45}$$

If $U_1 \subseteq N(z)$, then clearly $d(z) \geq |U_1| = \delta$ in $G'$, contradicting the hypothesis. Otherwise, $zv \notin E(G')$ for some $v \in U_1$ and we take $G'' = G' + \{zv\}$. Finally, if $d(z) = d(w) = \delta - 1$, then as above, $zv \notin E(G')$ and $wu \notin E(G')$ for some $v \in U_1$, $u \in U_2$ and we take $G'' = G' + \{zv, wu\}$. Clearly, $\delta(G'') = \delta(G)$ and $q = q(G) \geq q(G'') - 1$. This procedure may be repeated for all edges of $\Upsilon(I_1, I_2, \ldots, I_s)$. The resulting graph $G^*$ satisfies the following conditions:

$$\delta(G^*) = \delta(G), \qquad q(G) \geq q(G^*) - |\Upsilon(I_1, I_2, \ldots, I_s)|. \tag{3.46}$$

In fact,

$$G^* = (G \setminus \Upsilon(I_1, I_2, \ldots, I_s)) + E^*, \tag{3.47}$$

where $E^*$ consists of at most $2|\Upsilon(I_1, I_2, \ldots, I_s)|$ appropriate new edges such that $G^* \setminus \{\xi_1, \xi_2, \ldots, \xi_s\}$ is disconnected. Let $H_1, H_2, \ldots, H_t$ be the connected components of $G^* \setminus \{\xi_1, \xi_2, \ldots, \xi_s\}$ with $V(I_i^*) \subseteq V(H_i)$ $(i = 1, 2, \ldots, s)$ and $V(H_{s+1}) = \{x_1, x_2\}$. For each $i \in \{1, 2, \ldots, s+1\}$, put

$$h_i = |V(H_i)|, \qquad q_i = \left|\{xy \in E(G^*) : \{x, y\} \cap V(H_i) \neq \emptyset\}\right|. \tag{3.48}$$

Clearly, $h_i \geq 2$ $(i = 1, 2, \ldots, s+1)$. If $h_i \geq 6$ for some $i \in \{1, 2, \ldots, s\}$, then

$$n \geq \sum_{i=1}^{s+1} h_i + |\{\xi_1, \xi_2, \ldots, \xi_s\}| \geq 6 + 3s = 3\delta + 3, \tag{3.49}$$

contradicting (3.32). Otherwise, $2 \le h_i \le 5 \le 2\delta - 1$ $(i = 1, 2, \ldots, s+1)$. It follows that $(h_i - 2)(2\delta - h_i - 1) \ge 0$ which is equivalent to

$$\frac{h_i(2\delta - h_i + 1)}{2} \ge 2\delta - 1 \quad (i = 1, 2, \ldots, s+1). \tag{3.50}$$

*Case* 2.1.2.1 ($\max_i|I_i| \le 4$). By (3.50) and Lemma 2.3, $q_i(G^*) \ge 2\delta - 1$ $(i = 1, 2, \ldots, s+1)$. Hence

$$q(G^*) \ge \sum_{i=1}^{s+1} q_i(G^*) \ge (s+1)(2\delta - 1) = \delta(2\delta - 1). \tag{3.51}$$

Using (3.46) and Claim 1, we have

$$q \ge q(G^*) - 3 \ge \delta(2\delta - 1) - 3 \ge \frac{3(\delta - 1)(\delta + 2)}{2}. \tag{3.52}$$

*Case* 2.1.2.2 ($\max_i|I_i| = 5$). Assume without loss of generality that $\max_i|I_i| = |I_1| = 5$, that is, $4 \le h_1 \le 5$. By (3.50) and Lemma 2.3, $q_i(G^*) \ge 2\delta - 1$ $(i = 2, \ldots, s+1)$ and

$$q_1(G^*) \ge \frac{h_1(2\delta - h_1 + 1)}{2} \ge 2(2\delta - 3). \tag{3.53}$$

Hence

$$q(G^*) \ge s(2\delta - 1) + 2(2\delta - 3) = 2\delta^2 + \delta - 5. \tag{3.54}$$

By (3.46) and Claim 1,

$$q \ge q(G^*) - (\delta - 1) \ge 2\delta^2 - 4 > \frac{3(\delta - 1)(\delta + 2)}{2}. \tag{3.55}$$

*Case* 2.1.2.3 ($\max_i|I_i| = 6$). Assume without loss of generality that $\max_i|I_i| = |I_1| = 6$, that is, $h_1 = 5$. By (3.50) and Lemma 2.3, $q_i(G^*) \ge 2\delta - 1$ $(i = 2, \ldots, s+1)$ and

$$q_1(G^*) \ge \frac{h_1(2\delta - h_1 + 1)}{2} = 5(\delta - 2). \tag{3.56}$$

Hence

$$q(G^*) \ge s(2\delta - 1) + 5(\delta - 2) = 2\delta^2 + 2\delta - 9. \tag{3.57}$$

By (3.46) and Claim 1,

$$q \ge q(G^*) - 2(\delta - 2) \ge 2\delta^2 - 5 > \frac{3(\delta - 1)(\delta + 2)}{2}. \tag{3.58}$$

*Case* 2.2 ($\overline{p} \geq 2$). According to (3.32), we can distinguish five main cases, namely, $2 \leq \overline{p} \leq \delta - 3$, $\overline{p} = \delta - 2$, $\overline{p} = \delta - 1$, $\overline{p} = \delta$, and $\overline{p} = \delta + 1$.

*Case* 2.2.1 ($2 \leq \overline{p} \leq \delta - 3$). It follows that $|N_C(x_i)| \geq \delta - \overline{p} \geq 3$ $(i = 1, 2)$ and

$$\delta \geq 5, \quad \delta - \overline{p} \geq 3. \tag{3.59}$$

If $N_C(x_1) \neq N_C(x_2)$, then by (3.59) and Lemma 2.1, $|C| \geq 4\delta - 2\overline{p} \geq 3\delta - \overline{p} + 3$, contradicting (3.32). Let $N_C(x_1) = N_C(x_2)$. Clearly, $s \geq |N_C(x_1)| - (|V(P)| - 1) \geq \delta - \overline{p}$ and $|I_i| \geq \overline{p} + 2$ $(i = 1, 2, \ldots, s)$. If $s \geq \delta - \overline{p} + 1$, then

$$\begin{aligned} |C| &\geq s(\overline{p} + 2) \geq (\delta - \overline{p} + 1)(\overline{p} + 2), \\ &= (\delta - \overline{p} - 1)(\overline{p} - 1) + 3\delta - \overline{p} + 1 \geq 3\delta - \overline{p} + 3, \end{aligned} \tag{3.60}$$

again contradicting (3.32). Let $s = \delta - \overline{p}$. It means that $x_1 x_2 \in E(G)$, that is, $G[V(P)]$ is hamiltonian. By symmetric arguments, $N_C(y) = N_C(x_1)$ for each $y \in V(P)$. Assume that $\Upsilon(I_1, I_2, \ldots, I_s) \neq \emptyset$, that is, $\Upsilon(I_a, I_b) \neq \emptyset$ for some elementary segments $I_a$ and $I_b$. By the definition, there is an intermediate path $L$ between $I_a$ and $I_b$. If $|L| \geq 2$, then by Lemma 2.2

$$|I_a| + |I_b| \geq 2\overline{p} + 2|L| + 4 \geq 2\overline{p} + 8. \tag{3.61}$$

Hence

$$\begin{aligned} |C| &= |I_a| + |I_b| + \sum_{i \in \{1, \ldots, s\} \setminus \{a, b\}} |I_i| \geq 2\overline{p} + 8 + (s - 2)(\overline{p} + 2), \\ &= (\delta - \overline{p} - 2)(\overline{p} - 1) + 3\delta - \overline{p} + 2 \geq 3\delta - \overline{p} + 3, \end{aligned} \tag{3.62}$$

contradicting (3.32). Thus, $|L| = 1$, that is, $\Upsilon(I_1, I_2, \ldots, I_s) \subseteq E(G)$. By Lemma 2.2,

$$|I_a| + |I_b| \geq 2\overline{p} + 2|L| + 4 = 2\overline{p} + 6, \tag{3.63}$$

which yields

$$\begin{aligned} |C| &= |I_a| + |I_b| + \sum_{i \in \{1, \ldots, s\} \setminus \{a, b\}} |I_i| \geq 2\overline{p} + 6 + (s - 2)(\overline{p} + 2) \\ &= (s - 2)(\overline{p} - 2) + 4\delta - 2\overline{p} - 2 \geq 3\delta - \overline{p} - 2 + (\delta - \overline{p}). \end{aligned} \tag{3.64}$$

If $\delta - \overline{p} \geq 4$, then $|C| \geq 3\delta - \overline{p} + 2$, contradicting (3.32). Let $\delta - \overline{p} \leq 3$. Recalling (3.59), we have $\delta - \overline{p} = 3$, that is, $\overline{p} = \delta - 3$ and $s = \delta - \overline{p} = 3$. Hence, $|C| \geq s(\overline{p} + 2) = 3(\delta - 1)$. On the other hand, by (3.32) and the fact that $\overline{p} \geq 2$, we have $|C| \leq 3\delta - \overline{p} + 1 \leq 3\delta - 1$. Thus

$$3\delta - 3 \leq |C| \leq 3\delta - 1. \tag{3.65}$$

Put $G' = G\backslash\Upsilon(I_1, I_2, I_3)$. As in Case 2.1.2, form a graph $G^*$ by adding at most $2|\Upsilon(I_1, I_2, I_3)|$ new edges in $G'$ such that $\delta(G^*) = \delta(G)$ and $G^*\backslash\{\xi_1, \xi_2, \xi_3\}$ are disconnected. We denote $G^* = G$ immediately if $\Upsilon(I_1, I_2, I_3) = \emptyset$. Hence

$$q(G) \geq q(G^*) - |\Upsilon(I_1, I_2, I_3)|. \tag{3.66}$$

Let $H_1, H_2, \ldots, H_t$ be the connected components of $G^* \setminus \{\xi_1, \xi_2, \xi_3\}$ with $V(I_i^*) \subseteq V(H_i)(i = 1, 2, 3)$ and $V(P) \subseteq V(H_4)$. Since $x_1x_2 \in E(G)$ (i.e., $G[V(P)]$ is hamiltonian) and $P$ is extreme, we have $V(H_4) = V(P)$. Using notation (3.48) for $G^*$, we have $h_i \geq |I_i| - 1 \geq \overline{p} + 1 = \delta - 2$ $(i = 1, 2, 3)$ and $h_4 = \delta - 2$. If $h_i \geq \delta + 1$ for some $i \in \{1, 2, 3\}$, then

$$n \geq h_1 + h_2 + h_3 + h_4 + s \geq 4\delta - 2. \tag{3.67}$$

By (3.59), $\delta \geq 5$, implying that $4\delta - 2 \geq 3\delta + 3$ and $n \geq 3\delta + 3$, contradicting (3.32). Let $\delta - 2 \leq h_i \leq \delta$ $(i = 1, 2, 3, 4)$. It follows that

$$\frac{h_i(2\delta - h_i + 1)}{2} \geq \frac{(\delta - 2)(\delta + 3)}{2} \quad (i = 1, 2, 3, 4). \tag{3.68}$$

By Lemma 2.3, $q_i(G^*) \geq (\delta - 2)(\delta + 3)/2$ $(i = 1, 2, 3, 4)$, implying that

$$q(G^*) \geq \sum_{i=1}^{4} q_i(G^*) \geq 2(\delta - 1)(\delta + 3). \tag{3.69}$$

If $|\Upsilon(I_1, I_2, I_3)| \geq 4$, then $|\Upsilon(I_a, I_b)| \geq 2$ for some distinct $a, b \in \{1, 2, 3\}$. By Lemma 2.2

$$|I_a| + |I_b| \geq 2\overline{p} + 7 = 2\delta + 1 \tag{3.70}$$

and hence $|C| \geq 3\delta$, contradicting (3.65). So, $|\Upsilon(I_1, I_2, I_3)| \leq 3$. By (3.66) and (3.69),

$$q \geq q(G^*) - 3 \geq 2(\delta - 1)(\delta + 3) - 3 \geq \frac{3(\delta - 1)(\delta + 2)}{2}. \tag{3.71}$$

*Case* 2.2.2 ($\overline{p} = \delta - 2$). It follows that $|N_C(x_i)| \geq \delta - \overline{p} = 2$ $(i = 1, 2)$. By (3.32),

$$|C| \leq 3\delta + 1 - \overline{p} = 2\delta + 3. \tag{3.72}$$

If $N_C(x_1) \neq N_C(x_2)$, then by Lemma 2.1, $|C| \geq 4\delta - 2\overline{p} = 2\delta + 4$, contradicting (3.72). Let $N_C(x_1) = N_C(x_2)$. Further, if $s \geq 3$, then

$$|C| \geq s(\overline{p} + 2) \geq 3\delta = 2\delta + (\overline{p} + 2) \geq 2\delta + 4, \tag{3.73}$$

again contradicting (3.72). Let $s = 2$. It follows that $x_1x_2 \in E(G)$, that is, $G[V(P)]$ is hamiltonian. By symmetric arguments, $N_C(y) = N_C(x_1) = \{\xi_1, \xi_2\}$ for each $y \in V(P)$. Clearly, $|I_i| \geq \overline{p} + 2 = \delta$ $(i = 1, 2)$.

*Case* 2.2.2.1 ($\Upsilon(I_1, I_2) = \emptyset$). It follows that $G \setminus \{\xi_1, \xi_2\}$ is disconnected. Let $H_1, H_2, \ldots, H_t$ be the connected components of $G \setminus \{\xi_1, \xi_2\}$ with $V(I_i^*) \subseteq V(H_i)$ $(i = 1, 2)$ and $V(P) \subseteq V(H_3)$. Since $P$ is extreme and $G[V(P)]$ is hamiltonian, we have $V(H_3) = V(P)$. By notation (3.48), $h_i \geq |I_i| - 1 \geq \delta - 1$ $(i = 1, 2)$ and $h_3 = \delta - 1$. If $h_i \geq \delta + 3$ for some $i \in \{1, 2\}$, then

$$n \geq h_1 + h_2 + h_3 + |\{\xi_1, \xi_2\}| \geq 3\delta + 3, \tag{3.74}$$

contradicting (3.32). So, $\delta - 1 \leq h_i \leq \delta + 2$ $(i = 1, 2, 3)$. By Lemma 2.3,

$$q_i \geq \frac{h_i(2\delta - h_i + 1)}{2} \geq \frac{(\delta - 1)(\delta + 2)}{2} \quad (i = 1, 2, 3). \tag{3.75}$$

Hence,

$$q \geq \sum_{i=1}^{3} q_i \geq \frac{3(\delta - 1)(\delta + 2)}{2}. \tag{3.76}$$

*Case* 2.2.2.2 ($\Upsilon(I_1, I_2) \neq \emptyset$). By definition, there is an intermediate path $L$ between $I_1$ and $I_2$. If $|L| \geq 2$, then by Lemma 2.2

$$|C| = |I_1| + |I_2| \geq 2\overline{p} + 2|L| + 4 \geq 2\delta + 4, \tag{3.77}$$

contradicting (3.72). Otherwise, $\Upsilon(I_1, I_2) \subseteq E(G)$. Further, if $|\Upsilon(I_1, I_2)| \geq 3$, then by Lemma 2.2

$$|C| = |I_1| + |I_2| \geq 2\overline{p} + 8 = 2\delta + 4, \tag{3.78}$$

again contradicting (3.72). Thus $|\Upsilon(I_1, I_2)| \leq 2$.

*Case* 2.2.2.2.1 ($|\Upsilon(I_1, I_2)| = 1$). Put $G' = G \setminus \Upsilon(I_1, I_2)$. As in Case 2.1.2, form a graph $G^*$ by adding at most two new edges in $G'$ such that $\delta(G^*) = \delta(G)$, $G^* \setminus \{\xi_1, \xi_2\}$ is disconnected and $q(G) \geq q(G^*) - 1$. Let $H_1, H_2, \ldots, H_t$ be the connected components of $G^* \setminus \{\xi_1, \xi_2\}$ with $V(I_i^*) \subseteq V(H_i)$ $(i = 1, 2)$ and $V(P) = V(H_3)$. Using notation (3.48) for $G^*$, as in Case 2.2.2.1, we have $\delta - 1 \leq h_i \leq \delta + 2$ $(i = 1, 2, 3)$. By Lemma 2.2, $|I_1| + |I_2| \geq 2\overline{p} + 6 = 2\delta + 2$. It means that $\max_i |I_i| \geq \delta + 1$, that is, $\max_i h_i \geq \delta$. Assume without loss of generality that $h_1 \geq \delta$. By Lemma 2.3,

$$\begin{aligned} q_1(G^*) &\geq \frac{h_1(2\delta - h_1 + 1)}{2} \geq \frac{\delta(\delta + 1)}{2}, \\ q_i(G^*) &\geq \frac{h_i(2\delta - h_i + 1)}{2} \geq \frac{(\delta - 1)(\delta + 2)}{2} \quad (i = 2, 3), \end{aligned} \tag{3.79}$$

implying that

$$q(G^*) \geq \frac{\delta(\delta+1)}{2} + (\delta-1)(\delta+2). \tag{3.80}$$

Hence

$$q \geq q(G^*) - 1 \geq \frac{\delta(\delta+1)}{2} + (\delta-1)(\delta+2) - 1 \geq \frac{3(\delta-1)(\delta+2)}{2}. \tag{3.81}$$

*Case* 2.2.2.2.2 ($|\Upsilon(I_1, I_2)| = 2$). By Lemma 2.2,

$$|C| = |I_1| + |I_2| \geq 2\overline{p} + 7 = 2\delta + 3. \tag{3.82}$$

Recalling (3.72), we get $|C| = 2\delta + 3$ and $V(G) = V(P \cup C)$. Put $G' = G \backslash \Upsilon(I_1, I_2)$. As in Case 2.1.2, form a graph $G^*$ by adding at most four new edges in $G'$ such that $\delta(G^*) = \delta(G)$, $G^* \backslash \{\xi_1, \xi_2\}$ is disconnected and $q(G) \geq q(G^*) - 2$. Let $H_1, H_2$, and $H_3$ be the connected components of $G^* \backslash \{\xi_1, \xi_2\}$ with $V(H_i) = V(I_i^*)(i = 1,2)$ and $V(H_3) = V(P)$. Using notation (3.48) for $G^*$, we have as in Case 2.2.2.1, $\delta - 1 \leq h_i \leq \delta + 2$ $(i = 1,2,3)$. Since $|I_i| \geq \delta$ $(i = 1,2)$ and $|C| = |I_1| + |I_2| = 2\delta + 3$, we can assume without loss of generality that either $|I_1| = \delta + 2$, $|I_2| = \delta + 1$ or $|I_1| = \delta + 3$, $|I_2| = \delta$.

*Case* 2.2.2.2.2.1 ($|I_1| = \delta + 2$, $|I_2| = \delta + 1$). It follows that $h_1 = \delta + 1$, $h_2 = \delta$ and $h_3 = \delta - 1$. By Lemma 2.3,

$$\begin{aligned} q_i(G^*) &\geq \frac{h_i(2\delta - h_i + 1)}{2} = \frac{\delta(\delta+1)}{2} \quad (i = 1,2), \\ q_3(G^*) &\geq \frac{h_3(2\delta - h_3 + 1)}{2} = \frac{(\delta-1)(\delta+2)}{2}. \end{aligned} \tag{3.83}$$

Hence

$$q \geq \sum_{i=1}^{3} q_i(G^*) - 2 \geq \delta(\delta+1) - 2 + \frac{(\delta-1)(\delta+2)}{2} = \frac{3(\delta-1)(\delta+2)}{2}. \tag{3.84}$$

*Case* 2.2.2.2.2.2 ($|I_1| = \delta + 3$, $|I_2| = \delta$). Let $\Upsilon(I_1, I_2) = \{e_1, e_2\}$, where

$$e_1 = y_1 z_1, \qquad e_2 = y_2 z_2, \qquad \{y_1, y_2\} \subseteq V(I_1^*), \ \{z_1, z_2\} \subseteq V(I_2^*). \tag{3.85}$$

If $y_1 \neq y_2$ and $z_1 \neq z_2$, then as in proof of Lemma 2.2 (Case 2.1),

$$|C| = |I_1| + |I_2| \geq 2\overline{p} + 8 = 2(\delta - 2) + 8 = 2\delta + 4, \tag{3.86}$$

contradicting (3.72). Let either $y_1 \neq y_2$ and $z_1 = z_2$ or $y_1 = y_2$ and $z_1 \neq z_2$.

*Case* 2.2.2.2.2.2.2.1 ($y_1 \neq y_2$ and $z_1 = z_2$). Assume without loss of generality that $y_1, y_2$ occur on $I_1$ in this order. If $y_2 = y_1^+$, then

$$|C| \geq \left|\xi_1 \vec{C} y_1 z_1 y_2 \vec{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1\right| = 2\delta + 4, \tag{3.87}$$

contradicting (3.72). Let $y_2 \neq y_1^+$, that is, $|y_1 \vec{C} y_2| \geq 2$. Put

$$\begin{aligned} C' &= \xi_1 \vec{C} y_2 z_1 \overleftarrow{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1, \\ C'' &= \xi_1 \overleftarrow{C} z_1 y_1 \vec{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1. \end{aligned} \tag{3.88}$$

Clearly,

$$\begin{aligned} |C| \geq \left|C'\right| &= \left|\xi_1 \vec{C} y_1\right| + \left|y_1 \vec{C} y_2\right| + |\{e_2\}| + \left|\xi_2 \vec{C} z_1\right| + (\overline{p} + 2), \\ |C| \geq \left|C''\right| &= \left|\xi_1 \overleftarrow{C} z_1\right| + |\{e_1\}| + \left|y_1 \vec{C} y_2\right| + \left|y_2 \vec{C} \xi_2\right| + (\overline{p} + 2). \end{aligned} \tag{3.89}$$

By summing and observing that

$$\left|\xi_1 \vec{C} y_1\right| + \left|y_1 \vec{C} y_2\right| + \left|y_2 \vec{C} \xi_2\right| + \left|\xi_2 \vec{C} z_1\right| + \left|z_1 \vec{C} \xi_1\right| = |C|, \tag{3.90}$$

we get

$$2|C| \geq |C| + \left|y_1 \vec{C} y_2\right| + 2(\overline{p} + 2) + 2 \geq |C| + 2\delta + 4. \tag{3.91}$$

Hence $|C| \geq 2\delta + 4$, again contradicting (3.72).

*Case* 2.2.2.2.2.2.2.2 ($y_1 = y_2$ and $z_1 \neq z_2$). Assume without loss of generality that $z_2, z_1$ occur on $I_2$ in this order.

*Case* 2.2.2.2.2.2.2.2.1 ($\delta \geq 6$). If $|\xi_1 \vec{C} y_1| \geq \delta - 1$ and $|y_1 \vec{C} \xi_2| \geq \delta - 1$, then $|I_1| \geq 2\delta - 2 \geq \delta + 4$, contradicting the hypothesis. Thus, we can assume without loss of generality that $|\xi_1 \vec{C} y_1| \leq \delta - 2$. If $y_1^- = \xi_1$, then

$$|C| \geq \left|\xi_1 \overleftarrow{C} z_2 y_1 \vec{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1\right| \geq 2\delta + 5, \tag{3.92}$$

contradicting (3.72). Let $y_1^- \neq \xi_1$, that is, $y_1^- \in V(I_1^*)$. Since $\Upsilon(I_1, I_2) = \{y_1z_1, y_1z_2\}$, we have $N(y_1^-) \subset V(I_1)$. If $N(y_1^-) \cap V(y_1^+\vec{C}\xi_2^-) = \emptyset$, then $|N(y_1^-)| \leq \delta - 1$, a contradiction. Otherwise, $y_1^-w \in E(G)$ for some $w \in V(y_1^+\vec{C}\xi_2^-)$. Put

$$\begin{aligned} R &= \xi_1\vec{C}y_1^-w\overleftarrow{C}y_1, \\ C' &= \xi_1\vec{R}y_1z_1\overleftarrow{C}\xi_2x_2\overleftarrow{P}x_1\xi_1, \\ C'' &= \xi_1\overleftarrow{C}z_2y_1\vec{C}\xi_2x_2\overleftarrow{P}x_1\xi_1. \end{aligned} \tag{3.93}$$

Clearly,

$$\begin{aligned} |C| &\geq |C'| = |R| + |\{y_1z_1\}| + |z_1\overleftarrow{C}\xi_2| + (\overline{p}+2), \\ |C| &\geq |C''| = |\xi_1\overleftarrow{C}z_1| + |z_1\overleftarrow{C}z_2| + |\{y_1z_2\}| + |y_1\vec{C}\xi_2| + (\overline{p}+2). \end{aligned} \tag{3.94}$$

By summing and observing that $|R| \geq |\xi_1\vec{C}y_1| + 1$, we get

$$2|C| \geq \left(|\xi_1\vec{C}y_1| + |y_1\vec{C}\xi_2| + |\xi_2\vec{C}z_1| + |z_1\vec{C}\xi_1|\right) + 2(\overline{p}+2) + 4 = |C| + 2\delta + 4. \tag{3.95}$$

Hence $|C| \geq 2\delta + 4$, contradicting (3.72).

*Case* 2.2.2.2.2.2.2.2 ($\delta = 5$). It follows that

$$|I_1| = \delta + 3 = 8, \qquad |I_2| = \delta = 5, \qquad |C| = 2\delta + 3 = 13. \tag{3.96}$$

If either $|\xi_1\vec{C}y_1| \leq \delta - 2 = 3$ or $|y_1\vec{C}\xi_2| \leq \delta - 2 = 3$, then we can argue as in Case 2.2.2.2.2.2.2.1. Otherwise, $|\xi_1\vec{C}y_1| = |y_1\vec{C}\xi_2| = 4$. If $|z_1\overleftarrow{C}\xi_2| \geq 4$, then

$$|\xi_1\vec{C}y_1z_1\overleftarrow{C}\xi_2x_2\overleftarrow{P}x_1\xi_1| \geq 14 > |C|, \tag{3.97}$$

a contradiction. Let $|z_1\overleftarrow{C}\xi_2| \leq 3$. Similarly, $|\xi_1\overleftarrow{C}z_2| \leq 3$, implying that $I_2 = \xi_2\xi_2^+z_2z_1z_1^+\xi_1$. If $z_1^+z_2 \in E(G)$, then

$$|\xi_1\vec{C}y_1z_1z_1^+z_2\overleftarrow{C}\xi_2x_2\overleftarrow{P}x_1\xi_1| = 14 > |C|, \tag{3.98}$$

a contradiction. So, $N(z_1^+) \subseteq \{\xi_1, \xi_2, z_1, \xi_2^+\}$, again a contradiction, since $|N(z_1^+)| \geq \delta = 5$.

*Case* 2.2.2.2.2.2.2.3 ($\delta = 4$). It follows that

$$|I_1| = \delta + 3 = 7, \qquad |I_2| = \delta = 4, \qquad |C| = 2\delta + 3 = 11. \tag{3.99}$$

Since $|I_1| = 7$, we have either $|\xi_1\vec{C}y_1| \geq 4$ or $|y_1\vec{C}\xi_2| \geq 4$, say $|\xi_1\vec{C}y_1| \geq 4$. Put

$$C' = \xi_1\vec{C}y_1z_1\overleftarrow{C}\xi_2x_2\overleftarrow{P}x_1\xi_1. \tag{3.100}$$

If $|\xi_1\vec{C}y_1| \geq 5$, then $|C'| \geq 12 > |C|$, a contradiction. This means that $|\xi_1\vec{C}y_1| = 4$. If $|z_1\vec{C}\xi_1| = 1$ then $|C'| \geq 12 > |C|$, a contradiction. Let $|z_1\vec{C}\xi_1| \geq 2$. Since $|I_2| = 4$, we have $|z_1\vec{C}\xi_1| = 2$, that is, $I_2 = \xi_2z_2z_1z_1^+\xi_1$. Further, if $z_1^+z_2 \in E(G)$, then

$$\left|\xi_1\vec{C}y_1z_1z_1^+z_2\xi_2x_2\overleftarrow{P}x_1\xi_1\right| = 12 > |C|, \tag{3.101}$$

a contradiction. Let $z_1^+z_2 \notin E(G)$. Since $|\Upsilon(I_1, I_2)| = 2$, we have $N(z_1^+) \subseteq \{\xi_1, \xi_2, z_1\}$, contradicting the fact that $|N(z_1^+)| \geq \delta = 4$.

*Case* 2.2.3 ($\overline{p} = \delta - 1$). By (3.32),

$$|C| \leq 3\delta + 1 - \overline{p} = 2\delta + 2. \tag{3.102}$$

It follows that $|N_C(x_i)| \geq \delta - \overline{p} = 1$ $(i = 1, 2)$.

*Case* 2.2.3.1 ($|N_C(x_i)| \geq 2$ $(i = 1, 2)$). If $N_C(x_1) \neq N_C(x_2)$, then by Lemma 2.1, $|C| \geq 2\overline{p} + 8 = 2\delta + 6$, contradicting (3.102). Let $N_C(x_1) = N_C(x_2)$. If $s \geq 3$, then

$$|C| \geq s(\overline{p} + 2) \geq 3(\delta + 1) > 2\delta + 2, \tag{3.103}$$

contradicting (3.102). Let $s = 2$. It follows that $|C| \geq s(\overline{p} + 2) \geq 2(\delta + 1)$. Recalling (3.102), we get

$$|C| = 2\delta + 2, \qquad |I_1| = |I_2| = \delta + 1, \qquad V(G) = V(C \cup P). \tag{3.104}$$

Assume that $yz \in E(G)$ for some $y \in V(P)$ and $z \in V(C)\backslash\{\xi_1, \xi_2\}$. Besides, we can assume without loss of generality that $z \in V(I_1^*)$. Since $C$ is extreme, we have

$$\left|\xi_1\vec{C}z\right| \geq \left|x_1\vec{P}y\right| + 2, \qquad \left|z\vec{C}\xi_2\right| \geq \left|y\vec{P}x_2\right| + 2, \tag{3.105}$$

implying that

$$|I_1| = \left|\xi_1\vec{C}z\right| + \left|z\vec{C}\xi_2\right| \geq \left|x_1\vec{P}y\right| + \left|y\vec{P}x_2\right| + 4 = \overline{p} + 4 = \delta + 3, \tag{3.106}$$

a contradiction. So, $N_C(y) \subseteq \{\xi_1, \xi_2\}$ for each $y \in V(P)$. On the other hand, by Lemma 2.2, $\Upsilon(I_1, I_2) = \emptyset$ and hence $G \setminus \{\xi_1, \xi_2\}$ is disconnected. Let $H_1, H_2$, and $H_3$ be the connected components of $G\backslash\{\xi_1,\xi_2\}$ with $V(H_i) = V(I_i^*)$ $(i = 1,2)$ and $V(H_3) = V(P)$. Using notation (3.48), we have $h_i = \delta$ $(i = 1,2,3)$. By Lemma 2.3,

$$q_i \geq \frac{h_i(2\delta - h_i + 1)}{2} = \frac{\delta(\delta+1)}{2} \quad (i = 1,2,3), \tag{3.107}$$

implying that

$$q \geq \sum_{i=1}^{3} q_i \geq \frac{3(\delta^2+\delta)}{2} > \frac{3(\delta-1)(\delta+2)}{2}. \tag{3.108}$$

*Case* 2.2.3.2 (either $|N_C(x_1)| = 1$ or $|N_C(x_2)| = 1$). Assume without loss of generality that $|N_C(x_1)| = 1$. Put $N_C(x_1) = \{y_1\}$.

*Case* 2.2.3.2.1 ($N_C(x_2) \neq N_C(x_1)$). It follows that $x_2y_2 \in E(G)$ for some $y_2 \in V(C)\backslash\{y_1\}$ and we can argue as in Case 2.2.3.1.

*Case* 2.2.3.2.2 ($N_C(x_2) = N_C(x_1) = \{y_1\}$). It follows that

$$N(x_i) = (V(P)\backslash\{x_i\}) \cup \{y_1\} \quad (i = 1,2). \tag{3.109}$$

Since $\kappa \geq 2$, there is an edge $zw$ such that $z \in V(P)$ and $w \in V(C)\backslash\{y_1\}$. Since $N_C(x_1) = N_C(x_2) = \{y_1\}$, we have $z \notin \{x_1,x_2\}$. By (3.109), $x_2z^- \in E(G)$. Then replacing $P$ with $x_1\vec{P}z^-x_2\overleftarrow{P}z$, we can argue as in Case 2.2.3.1.

*Case* 2.2.4 ($\overline{p} = \delta$). By (3.32), $|C| \leq 3\delta + 1 - \overline{p} = 2\delta + 1$. Let $Q = \xi\vec{Q}\eta$ be a longest path in $G$ with $V(Q) \cap V(C) = \{\xi,\eta\}$. If $|Q| \geq \delta + 1$, then by (3.34), $|C| \geq 2|Q| \geq 2\delta + 2$, a contradiction. Let

$$|Q| \leq \delta. \tag{3.110}$$

*Case* 2.2.4.1 ($x_1x_2 \notin E(G)$). It follows that $|N_C(x_i)| \geq 1$ $(i = 1,2)$. If $|N_C(x_i)| \geq 2$ for some $i \in \{1,2\}$, then clearly $|Q| \geq \overline{p} + 2 = \delta + 2$, contradicting (3.110). Let $|N_C(x_1)| = |N_C(x_2)| = 1$. Further, if $N_C(x_1) \neq N_C(x_2)$, then again $|Q| \geq \delta + 2$, contradicting (3.110). Let $N_C(x_1) = N_C(x_2) = \{z_1\}$ for some $z_1 \in V(C)$. Since $\kappa \geq 2$, there is a path $L = yz_2$ connecting $P$ and $C$ such that $y \in V(P)$ and $z_2 \in V(C)\backslash\{z_1\}$. Clearly, $y \notin \{x_1,x_2\}$. If $x_2y^- \in E(G)$, then

$$|Q| \geq \left|z_1x_1\vec{P}y^-x_2\overleftarrow{P}yz_2\right| = \delta + 2, \tag{3.111}$$

contradicting (3.110). Let $x_2y^- \notin E(G)$. Further, if $y^- \neq x_1$, then recalling that $x_2x_1 \notin E(G)$ we have $|N_C(x_2)| \geq 2$, a contradiction. Otherwise, $y^- = x_1$ and $|Q| \geq |z_1x_2\overleftarrow{P}yz_2| = \delta + 1$, contradicting (3.110).

*Case* 2.2.4.2 ($x_1x_2 \in E(G)$). Put $C' = x_1\vec{P}x_2x_1$. Since $\kappa \geq 2$, there are two disjoint paths $L_1, L_2$ connecting $C'$ and $C$. Further, since $P$ is extreme, we have $|L_1| = |L_2| = 1$. Let $L_1 = y_1z_1$ and $L_2 = y_2z_2$, where $y_1, y_2 \in V(C')$ and $z_1, z_2 \in V(C)$. Since $C'$ is a hamiltonian cycle in $G[V(P)]$, we can assume that $P$ is chosen such that $x_1 = y_1$. If $|x_1\vec{P}y_2| \leq 2$, then $|Q| \geq |z_1x_1x_2\overleftarrow{P}y_2z_2| \geq \delta + 1$, contradicting (3.110). Let $|x_1\vec{P}y_2| \geq 3$. If $x_2v \in E(G)$ for some $v \in \{y_2^{-1}, y_2^{-2}\}$, then

$$|Q| \geq \left|z_1x_1\vec{P}vx_2\overleftarrow{P}y_2z_2\right| \geq \delta + 1, \tag{3.112}$$

contradicting (3.110). Otherwise, $|N_C(x_2)| \geq 2$, implying that $x_2z_3 \in E(G)$ for some $z_3 \in V(C)\backslash\{z_1\}$. Then

$$|Q| \geq \left|z_1x_1\vec{P}x_2z_3\right| \geq \delta + 2, \tag{3.113}$$

again contradicting (3.110).

*Case* 2.2.5 ($\overline{p} = \delta + 1$). By (3.32), $|C| \leq 3\delta + 1 - \overline{p} = 2\delta$. On the other hand, by Theorem D, $|C| \geq 2\delta$, implying that $|C| = 2\delta$ and $V(G) = V(C \cup P)$. Let $Q = \xi\vec{Q}\eta$ be a longest path in $G$ with $V(Q) \cap V(C) = \{\xi, \eta\}$. If $|Q| \geq \delta + 1$, then by (3.35), $|C| \geq 2|Q| \geq 2\delta + 2$, a contradiction. Let

$$|Q| \leq \delta. \tag{3.114}$$

*Case* 2.2.5.1 ($x_1x_2 \in E(G)$). Put $C' = x_1\vec{P}x_2x_1$. Since $\kappa \geq 2$, there are two disjoint edges $z_1w_1$ and $z_2w_2$ connecting $C'$ and $C$ such that $z_1, z_2 \in V(C')$ and $w_1, w_2 \in V(C)$. Since $C'$ is a hamiltonian cycle in $G[V(P)]$, we can assume without loss of generality that $P$ is chosen such that $z_1 = x_1$. If $|x_1\vec{P}z_2| \leq 3$, then $|Q| \geq |w_1x_1x_2\overleftarrow{P}z_2w_2| \geq \delta + 1$, contradicting (3.114). Let $|x_1\vec{P}z_2| \geq 4$. Further, if $x_2v \in E(G)$ for some $v \in \{z_2^{-1}, z_2^{-2}, z_2^{-3}\}$, then

$$|Q| \geq \left|w_1x_1\vec{P}vx_2\overleftarrow{P}z_2w_2\right| \geq \delta + 1, \tag{3.115}$$

contradicting (3.114). Now let $x_2v \notin E(G)$ for each $v \in \{z_2^{-1}, z_2^{-2}, z_2^{-3}\}$. It follows that $|N_C(x_2)| \geq 2$, that is, $x_2w_3 \in E(G)$ for some $w_3 \in V(C)\backslash\{w_1\}$. But then $|Q| \geq |w_1x_1\vec{P}x_2w_3| = \delta + 3$, contradicting (3.114).

*Case* 2.2.5.2 ($x_1x_2 \notin E(G)$). If $d_P(x_1) + d_P(x_2) \geq |V(P)| = \overline{p} + 1 = \delta + 2$ then by Theorem C, $G[V(P)]$ is hamiltonian and we can argue as in Case 2.2.5.1. Otherwise, $d_P(x_1)+d_P(x_2) \leq \delta+1$, implying that

$$d_C(x_1) + d_C(x_2) \geq \delta - 1 \geq 2. \tag{3.116}$$

Assume without loss of generality that $d_C(x_1) \geq d_C(x_2)$.

*Case* 2.2.5.2.1 ($d_C(x_2) = 0$). It follows that $N(x_2) = V(P)\backslash\{x_1, x_2\}$. By (3.116), $d_C(x_1) \geq 2$. Put $C' = x_1^+\vec{P}x_2x_1^+$. Since $\kappa \geq 2$, there is a path $L = z\vec{L}w$ connecting $C'$ and $C$ such that $z \in V(C')\backslash\{x_1^+\}$ and $w \in V(C)$. If $x_1 \in V(L)$, that is, $x_1z \in E(G)$, then $x_1\vec{P}z^-x_2\overleftarrow{P}zx_1$ is a hamiltonian cycle in $G[V(P)]$ and we can argue as in Case 2.2.5.1. Let $x_1 \notin V(L)$. Since $V(G) = V(C \cup P)$, we have $L = zw$. Further, since $d_C(x_1) \geq 2$, we have $x_1w_1 \in E(G)$ for some $w_1 \in V(C)\backslash\{w\}$. Hence,

$$|Q| \geq \left|w_1x_1\vec{P}z^-x_2\overleftarrow{P}zw\right| = \delta + 3, \tag{3.117}$$

contradicting (3.114).

*Case* 2.2.5.2.2 ($d_C(x_2) = 1$). Let $N_C(x_2) = \{w_1\}$. By (3.116), $d_C(x_1) \geq 1$. If either $d_C(x_1) \geq 2$ or $N_C(x_1) \neq N_C(x_2)$, then $x_1w \in E(G)$ for some $w \in V(C)\backslash\{w_1\}$ and therefore

$$|Q| \geq \left|wx_1\vec{P}x_2w_1\right| = \delta + 3, \tag{3.118}$$

contradicting (3.114). Otherwise, $N_C(x_1) = N_C(x_2) = \{w_1\}$. Since $\kappa \geq 2$, there is an edge $zw$ such that $z \in V(P)$ and $w \in V(C)\backslash\{w_1\}$. Clearly, $z \notin \{x_1, x_2\}$. Further, we can argue as in Case 2.2.5.1.

*Case* 2.2.5.2.3 ($d_C(x_2) \geq 2$). Since $d_C(x_1) \geq d_C(x_2)$, we have $d_C(x_1) \geq 2$. Hence $|Q| \geq \overline{p} + 2 = \delta + 3$, contradicting (3.114). □

*Proof of Theorem 1.1.* Let $G$ be a graph satisfying the hypothesis of Theorem 1.1, which is equivalent to

$$q \leq \delta^2 + \delta - 1. \tag{3.119}$$

Since

$$q = \frac{1}{2}\sum_{u \in V(G)} d(u) \geq \frac{\delta n}{2}, \tag{3.120}$$

we have $\delta n/2 \leq \delta^2 + \delta - 1$ which is equivalent to

$$\delta \geq \frac{n-1}{2} - \frac{1}{2} + \frac{1}{\delta}. \tag{3.121}$$

If $n$ is even, that is, $n = 2t$ for some integer $t$, then

$$\delta \geq \frac{2t-1}{2} - \frac{1}{2} + \frac{1}{\delta} = t - 1 + \frac{1}{\delta}, \tag{3.122}$$

implying that $\delta \geq t = n/2$. By Theorem A, $G$ is hamiltonian. Let $n$ is odd, that is, $n = 2t + 1$ for some integer $t$. Then $\delta \geq t - 1/2 + 1/\delta$ implying that $\delta \geq t \geq (n-1)/2$. Recalling that $G$ is hamiltonian when $\delta > (n-1)/2$, we can assume that $\delta = (n-1)/2$. By Theorem C, either $G$ is hamiltonian or containers at least $\delta^2 + \delta$ edges, contradicting (3.119). Theorem 1.1 is proved. □

*Proof of Theorem 1.2.* Let $G$ be a 2-connected graph. The hypothesis of Theorem 1.2 is equivalent to

$$q \leq \begin{cases} 8 & \text{when } \delta = 2, \\ \dfrac{3(\delta-1)(\delta+2)-1}{2} & \text{when } \delta \geq 3. \end{cases} \tag{3.123}$$

*Case* 1 ($\delta = 2$ and $q \leq 8$). Let $C$ be a longest cycle in $G$ and $P = x_1\vec{P}x_2$ a longest path in $G \backslash C$ of length $\overline{p}$. If $\overline{p} = 0$, then $C$ is a dominating cycle and we are done. Let $\overline{p} \geq 1$. Since $\kappa \geq 2$, there is a path $Q = \xi\vec{Q}\eta$ such that $V(Q) \cap V(C) = \{\xi, \eta\}$ and $|Q| \geq 3$. Further, since $C$ is extreme, we have $|C| = |\xi\vec{C}\eta| + |\eta\vec{C}\xi| \geq 2|Q| \geq 6$ and therefore, $q \geq |C| + |Q| \geq 9$, contradicting the hypothesis.

*Case* 2 ($\delta \geq 3$ and $q \leq (3(\delta-1)(\delta+2)-1)/2$). Since

$$q = \frac{1}{2}\sum_{u \in V(G)} d(u) \geq \frac{\delta n}{2}, \tag{3.124}$$

we have $\delta n/2 \leq (3(\delta-1)(\delta+2)-1)/2$, which is equivalent to

$$\delta \geq \frac{n-2}{3} - \frac{1}{3} + \frac{7}{3\delta}. \tag{3.125}$$

If $n = 3t$ for some integer $t$, then

$$\delta \geq \frac{3t-2}{3} - \frac{1}{3} + \frac{7}{3\delta} = t - 1 + \frac{7}{3\delta}, \tag{3.126}$$

implying that $\delta \geq t = n/3 > (n-2)/3$. Next, if $n = 3t + 1$ for some integer $t$, then

$$\delta \geq \frac{3t-1}{3} - \frac{1}{3} + \frac{7}{3\delta} = t - \frac{2}{3} + \frac{7}{3\delta}, \tag{3.127}$$

implying that $\delta \geq t = (n-1)/3 > (n-2)/3$. Finally, if $n = 3t + 2$ for some integer $t$, then

$$\delta \geq \frac{3t}{3} - \frac{1}{3} + \frac{7}{3\delta} = t - \frac{1}{3} + \frac{7}{3\delta}, \tag{3.128}$$

implying that $\delta \geq t = (n-2)/3$. So, $\delta \geq (n-2)/3$, in any case. By Lemma 2.4, each longest cycle in $G$ is a dominating cycle. □

## Acknowledgment

The author would like to thank the anonymous referee for careful reading of the paper and valuable suggestions.

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