Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 431872, 11 pages
doi:10.1155/2012/431872

Research Article

An Extension of Generalized (ψ, φ)-Weak Contractions

Tran Van An,[1] Kieu Phuong Chi,[1] Erdal Karapınar,[2] and Tran Duc Thanh[1]

[1] *Department of Mathematics, Vinh University, 182 Le Duan, Vinh City, Vietnam*
[2] *Department of Mathematics, Atilim University, İncek, 06836 Ankara, Turkey*

Correspondence should be addressed to Erdal Karapınar, erdalkarapinar@yahoo.com

Received 6 February 2012; Accepted 23 May 2012

Academic Editor: A. Zayed

We prove a fixed-point theorem for a class of maps that satisfy generalized (ψ, φ)-weak contractions depending on a given function. An example is given to illustrate our extensions.

1. Introduction

Because fixed-point theory has a wide array of applications in many areas such as economics, computer science, and engineering, it plays evidently a crucial role in nonlinear analysis. One of the cornerstones of this theory is the Banach fixed-point theorem, also known as the Banach contraction mapping theorem [1], which can be stated as follows.

Let $T : X \to X$ be a contraction on a compete metric space $(X; d)$; that is, there is a nonnegative real number $k < 1$ such that $d(T(x), T(y)) \leq kd(x, y)$ for all $x, y \in X$. Then the map T admits one and only one point $x^* \in X$ such that $Tx^* = x^*$. Moreover, this fixed point is the limit of the iterative sequence $x_{n+1} = T(x_n)$ for $n = 0, 1, 2, \ldots$, where x_0 is an arbitrary starting point in X. This theorem attracted a lot of attention because of its importance in the field. Many authors have started studying on fixed-point theory to explore some new contraction mappings to generalize the Banach contraction mapping theorem. In particular, Boyd and Wong [2] introduced the notion of Φ-contractions. In 1997 Alber and Guerre-Delabriere [3] defined the φ-weak contraction which is a generalization of Φ-contractions (see also [4–8]).

On the other hand, the notion of T-contractions introduced and studied by the authors of the interesting papers in [9–11]. Following this trend, we explore in this paper another extension of (ψ, φ)-weak contractions in the context of T-contractions.

2. Preliminaries

Let (X, d) be a metric space. Boyd and Wong [2] introduced the notion of Φ-contraction as follows. A map $T : X \to X$ is called a Φ-contraction if there exists an upper semicontinuous function $\Phi : [0, +\infty) \to [0, +\infty)$ such that

$$d(Tx, Ty) \leq \Phi(d(x, y)) \tag{2.1}$$

for all $x, y \in X$. The concept of the φ-weak contraction was defined by Alber and Guerre-Delabriere [3] as a generalization of Φ-contraction under the setting of Hilbert spaces and obtained fixed-point results. A map $T : X \to X$ is a φ-weak contraction, if there exists a function $\varphi : [0, +\infty) \to [0, +\infty)$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \tag{2.2}$$

for all $x, y \in X$ provided that the function φ satisfies the following condition:

$$\varphi(t) = 0 \quad \text{iff } t = 0. \tag{2.3}$$

Later Rhoades [7] proved analogs of the result in [3] in the context of metric spaces.

Theorem 2.1. *Let (X, d) be a complete metric space. Let $\varphi : [0, +\infty) \to [0, +\infty)$ be a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$. If $T : X \to X$ is a φ weak contraction, then T has a unique fixed point.*

In [12], Dutta and Choudhury proved an extension of Rhoades.

Theorem 2.2. *Let (X, d) be a complete metric space, and let $T : X \to X$ be a self-mapping satisfying*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)), \quad \forall x, y \in X, \tag{2.4}$$

where $\psi, \varphi : [0, +\infty) \to [0, +\infty)$ are continuous and nondecreasing functions with $\varphi(t) = \psi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Zhang and Song [8] improved Theorem 2.1 and gave the following result which states the existence of common fixed points of certain maps in metric spaces.

Theorem 2.3. *Let (X, d) be a complete metric space, and let $f, g : X \to X$ be self-mappings satisfying*

$$d(fx, gy) \leq M(x, y) - \varphi(M(x, y)), \quad \forall x, y \in X, \tag{2.5}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,fy)+d(y,gx)}{2}\right\} \tag{2.6}$$

and $\varphi : [0,+\infty) \to [0,+\infty)$ *are lower semicontinuous functions with* $\varphi(t) = 0$ *if and only if* $t = 0$. *Then* f, g *have a unique common fixed point.*

Combining the theorems above with the results of Dutta and Choudhury [12], Đoricorić [13] obtained the following theorem.

Theorem 2.4. *Let* (X,d) *be a complete metric space, and let* $T, S : X \to X$ *be self-mappings satisfying*

$$\psi(d(fx,gy)) \le \psi(M(x,y)) - \varphi(M(x,y)), \quad \forall x,y \in X, \tag{2.7}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(y,fx)}{2}\right\}, \tag{2.8}$$

$\psi : [0,+\infty) \to [0,+\infty)$ *is a continuous and nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$, *and* $\varphi : [0,+\infty) \to [0,+\infty)$ *is a lower semicontinuous function with* $\varphi(t) = 0$ *if and only if* $t = 0$. *Then* f, g *have a unique common fixed point.*

The notion of the T-contraction is defined in ([10, 11]) as follows.

Definition 2.5. Let T and S be two self-mappings on a metric space (X,d). The mapping S is said to be a T-*contraction* if there exists $k \in (0,1)$ such that

$$d(TSx, TSy) \le k d(Tx, Ty), \quad \forall x,y \in X. \tag{2.9}$$

It can be easily seen that if T is the identity map, then the T-contraction coincides with the usual contraction.

Example 2.6. Let $X = (0,\infty)$ with the usual metric $d(x,y) = |x-y|$ induced by $(\mathbb{R},d)$. Consider the following self-mappings $T(x) = 1/x$ and $Sx = 3x$ on X. It is clear that S is not a contraction. On the contrary,

$$d(TSx, TSy) = \left|\frac{1}{3x} - \frac{1}{3y}\right| = \frac{1}{3}\left|\frac{1}{y} - \frac{1}{x}\right| \le \frac{1}{3} d(Tx, Ty), \quad \forall x,y \in X. \tag{2.10}$$

Definition 2.7 (see, e.g., [9, 11]). Let (X,d) be a metric space. If $\{y_n\}$ is a convergent sequence whenever $\{Ty_n\}$ is convergent, then $T : X \to X$ is called *sequentially convergent*.

The aim of this work is to give a proper extension of Đoricorić's result of using the concept of T-contraction, that is, the contraction depending on a given function. We will show the existence of a common fixed point for a class of certain maps.

3. Main Results

We start this section by recalling the following two classes of functions.

Let Ψ denote the set of all functions $\psi : [0,\infty) \to [0,\infty)$ which satisfy

(i) ψ is continuous and nondecreasing,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Similarly Φ denotes the set of all functions $\varphi : [0,\infty) \to [0,\infty)$ which satisfy

(i) φ is lower semi continuous,

(ii) $\varphi(t) = 0$ if and only if $t = 0$.

It is easy to see that $\psi_1(t) = t$, $\psi_2(t) = t/(t+1)$, $\psi_3(t) = t^2$ belong to Ψ and $\varphi_1(t) = \min\{t,1\}$, $\varphi_2(t) = \ln(1+t)$ belong to Φ.

We are ready to state our main theorem that is a proper extension of Theorem 2.4.

Theorem 3.1. *Let (X,d) be a complete metric space and $T : X \to X$ an injective, continuous, and sequentially convergent mapping. Let $f,g : X \to X$ be self-mappings. If there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$\psi(d(Tfx,Tgy)) \leq \psi(M(Tx,Ty)) - \varphi(M(Tx,Ty)), \tag{3.1}$$

for all $x,y \in X$, where

$$M(Tx,Ty) = \max\left\{ d(Tx,Ty), d(Tx,Tfx), d(Ty,Tgy), \frac{d(Tx,Tgy)+d(Ty,Tfx)}{2} \right\}, \tag{3.2}$$

then f,g have a unique common fixed point.

Proof. We will follow the lines in the proof of the main result in [13]. By injection of T, we easily check that $M(Tx,Ty) = 0$ if and only if $x = y$ is a common fixed point of f and g. Let $x_0 \in X$. We define two iterative sequences $\{x_n\}$ and $\{y_n\}$ in the following way:

$$x_{2n+2} = fx_{2n+1}, \quad x_{2n+1} = gx_{2n}\ , \quad y_n = Tx_n, \quad \forall\ n = 0,1,2,\ldots. \tag{3.3}$$

We prove $\{y_n\}$ is a Cauchy sequence. For this purpose, we first claim that $\lim_{n\to\infty} d(y_{n+1},y_n) = 0$. It follows from property of φ that if n is odd

$$\begin{aligned} \psi(d(y_{n+1},y_n)) &= \psi(d(Tx_{n+1},Tx_n)) = \psi(d(Tfx_n,Tgx_{n-1})) \\ &\leq \psi(M(Tx_n,Tx_{n-1})) - \varphi(M(Tx_n,Tx_{n-1})) \\ &\leq \psi(M(Tx_n,Tx_{n-1})), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned}M(Tx_n,Tx_{n-1}) &= \max\Big\{d(Tx_n,Tx_{n-1}),d(Tfx_n,Tx_n),d(Tgx_{n-1},Tx_{n-1}),\\&\qquad\frac{d(Tgx_{n-1},Tx_n)+d(Tfx_n,Tx_{n-1})}{2}\Big\}\\&=\max\Big\{d(y_n,y_{n-1}),d(y_{n+1},y_n),d(y_n,y_{n-1}),\frac{d(y_{n-1},y_{n+1})}{2}\Big\}\\&\le\max\Big\{d(y_n,y_{n-1}),d(y_{n+1},y_n),\frac{d(y_{n-1},y_n)+d(y_n,y_{n+1})}{2}\Big\}.\end{aligned}\tag{3.5}$$

Hence, we have

$$\psi(d(y_{n+1},y_n))\le\psi\left(\max\Big\{d(y_n,y_{n-1}),d(y_{n+1},y_n),\frac{d(y_{n-1},y_n)+d(y_n,y_{n+1})}{2}\Big\}\right).\tag{3.6}$$

If $d(y_n,y_{n+1})>d(y_{n-1},y_n)\ge 0$ then $M(Tx_n,Tx_{n-1})=d(y_n,y_{n+1})$, hence

$$\psi(d(y_n,y_{n+1}))\le\psi(d(y_n,y_{n+1}))-\varphi(d(y_n,y_{n+1}))\tag{3.7}$$

and which contradicts with $d(y_n,y_{n+1})>0$ and the property of φ. Thus, it follows from (3.5) that

$$d(y_{n+1},y_n)\le M(Tx_n,Tx_{n-1})=d(y_n,y_{n-1}).\tag{3.8}$$

If n is even then by the same argument above, we obtain

$$d(y_{n+1},y_n)\le M(Tx_{n-1},Tx_n)=d(y_n,y_{n-1}).\tag{3.9}$$

Therefore,

$$d(y_{n+1},y_n)\le M(Tx_n,Tx_{n-1})=d(y_n,y_{n-1})\tag{3.10}$$

for all n and $\{d(y_n,y_{n+1})\}$ is a nonincreasing sequence of nonnegative real numbers. Hence, there exists $r\ge 0$ such that

$$\lim_{n\to\infty}d(y_n,y_{n+1})=\lim_{n\to\infty}M(Tx_n,Tx_{n-1})=r.\tag{3.11}$$

By the lower semicontinuity of φ, we have

$$\varphi(r)\le\lim_{n\to\infty}\inf\,\varphi(M(Tx_n,Tx_{n-1})).\tag{3.12}$$

Taking the upper limits as $n \to \infty$ on either side of

$$\psi(d(y_n, y_{n+1})) \le \psi(M(Tx_n, Tx_{n-1})) - \varphi(M(Tx_n, Tx_{n-1})), \tag{3.13}$$

we get

$$\psi(r) \le \psi(r) - \lim_{n\to\infty} \inf \varphi(M(Tx_n, Tx_{n-1})) \le \psi(r) - \varphi(r), \tag{3.14}$$

that is, $\varphi(r) \le 0$. By the property of φ, this implies that $\varphi(r) = 0$. It follows that $r = 0$ and

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{3.15}$$

It is implied from (3.10) that

$$\lim_{n\to\infty} M(Tx_n, Tx_{n-1}) = 0. \tag{3.16}$$

Now, we claim that $\{y_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, it is sufficient to prove that $\{y_{2n}\}$ is a Cauchy sequence. Suppose on the contrary that $\{y_{2n}\}$ is not a Cauchy sequence. Then, there exist $\varepsilon > 0$ and subsequences $\{y_{2n(k)}\}$ and $\{y_{2m(k)}\}$ of $\{y_{2n}\}$ such that $n(k)$ is the smallest index for which

$$n(k) > m(k) > k, \qquad d(y_{2m(k)}, y_{2n(k)}) > \varepsilon. \tag{3.17}$$

This means that

$$d(y_{2m(k)}, y_{2n(k)-2}) < \varepsilon. \tag{3.18}$$

From (3.18) and the triangle inequality, we get

$$\begin{aligned}
\varepsilon &\le d(y_{2m(k)}, y_{2n(k)}) \\
&\le d(y_{2m(k)}, y_{2n(k)-2}) + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}) \\
&< \varepsilon + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}).
\end{aligned} \tag{3.19}$$

Letting $k \to \infty$ and using (3.15), we get

$$\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}) = \varepsilon. \tag{3.20}$$

By the fact

$$\begin{aligned}
\left| d(y_{2m(k)}, y_{2n(k)+1}) - d(y_{2m(k)}, y_{2n(k)}) \right| &\le d(y_{2n(k)}, y_{2n(k)+1}) \\
\left| d(y_{2m(k)-1}, y_{2n(k)}) - d(y_{2m(k)}, y_{2n(k)}) \right| &\le d(y_{2m(k)-1}, y_{2m(k)})
\end{aligned} \tag{3.21}$$

and using (3.15) and (3.20), we obtain

$$\lim_{k\to\infty} d\left(y_{2m(k)-1}, y_{2n(k)}\right) = \lim_{k\to\infty} d\left(y_{2m(k)}, y_{2n(k)+1}\right) = \varepsilon. \tag{3.22}$$

Moreover, from

$$\left|d\left(y_{2m(k)-1}, y_{2n(k)+1}\right) - d\left(y_{2m(k)-1}, y_{2n(k)}\right)\right| \leq d\left(y_{2n(k)}, y_{2n(k)+1}\right) \tag{3.23}$$

and combining with (3.15) and (3.22), we conclude that

$$\lim_{k\to\infty} d\left(y_{2m(k)-1}, y_{2n(k)+1}\right) = \varepsilon. \tag{3.24}$$

Now, by the definition of $M(Tx, Ty)$ and from (3.10), (3.15), and (3.20)–(3.24), we can deduce that

$$\lim_{k\to\infty} M\left(Tx_{2m(k)-1}, Tx_{2n(k)}\right) = \varepsilon. \tag{3.25}$$

Due to (3.1), we have

$$\begin{aligned} \psi\left(d\left(y_{2m(k)}, y_{2n(k)+1}\right)\right) &= \psi\left(d\left(Tx_{2m(k)}, Tx_{2n(k)+1}\right)\right) = \psi\left(d\left(Tfx_{2m(k)-1}, Tgx_{2n(k)}\right)\right) \\ &\leq \psi\left(M\left(Tx_{2m(k)-1}, Tx_{2n(k)}\right)\right) - \varphi\left(M\left(Tx_{2m(k)-1}, Tx_{2n(k)}\right)\right). \end{aligned} \tag{3.26}$$

Letting $k \to \infty$ and using (3.22) and (3.25), we have

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon). \tag{3.27}$$

It is a contradiction to $\varphi(t) > 0$ for every $t > 0$. This proves that $\{y_n\}$ is a Cauchy sequence.

Since X is a complete metric space, there exists $u \in X$ such that $\lim_{n\to\infty} y_n = u$. Since T is sequentially convergent, we can deduce that $\{x_n\}$ converges to $v \in X$. By the continuity of T, we infer that

$$u = \lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_n = Tv. \tag{3.28}$$

We will show that $v = fv = gv$. Indeed, suppose that $v \neq fv$, since T is injective, we have $u = Tv \neq Tfv$. Hence, $d(Tv, Tfv) > 0$. Since

$$\begin{gathered} \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} y_{2n} = u, \\ \lim_{n\to\infty} d\left(y_{2n}, y_{2n+1}\right) = 0, \end{gathered} \tag{3.29}$$

we can seek $N_0 \in \mathbb{N}$ such that for any $n \geq N_0$

$$d(y_{2n+1}, u) < \frac{d(Tv, Tfv)}{4}, \qquad d(y_{2n}, u) < \frac{d(Tv, T\text{f}v)}{4}, \qquad d(y_{2n}, y_{2n+1}) < \frac{d(Tv, Tfv)}{4}. \tag{3.30}$$

Then, we have

$$\begin{aligned}
d(Tv, Tfv) \leq M(Tv, Tx_{2n}) &= \max\left\{ d(Tv, Tx_{2n}), d(Tv, Tfv), d(Tx_{2n}, Tgx_{2n}), \right. \\
&\qquad \left. \frac{d(Tv, Tgx_{2n}) + d(Tx_{2n}, Tfv)}{2} \right\} \\
&= \max\left\{ d(u, y_{2n}), d(Tv, Tfv), d(y_{2n}, y_{2n+1}), \right. \\
&\qquad \left. \frac{d(u, y_{2n+1}) + d(y_{2n}, Tfv)}{2} \right\} \\
&\leq \max\left\{ d(u, y_{2n}), d(Tv, Tfv), d(y_{2n}, y_{2n+1}), \right. \\
&\qquad \left. \frac{d(u, y_{2n+1}) + d(y_{2n}, Tv) + d(Tv, Tfv)}{2} \right\} \\
&\leq \max\left\{ \frac{d(Tv, Tfv)}{4}, d(Tv, Tfv), \frac{d(Tv, Tfv)}{4}, \right. \\
&\qquad \left. \frac{d(Tv, Tfv)/4 + d(Tv, Tfv)/4 + d(Tv, Tfv)}{2} \right\} \\
&\leq \max\left\{ d(Tv, Tfv), \frac{3}{4} d(Tv, Tfv) \right\} = d(Tv, Tfv).
\end{aligned} \tag{3.31}$$

Therefore, $M(Tv, Tx_{2n}) = d(Tv, Tfv)$ for every $n \geq N_0$. Since

$$\begin{aligned}
\psi(d(Tfv, y_{2n+1})) = \psi(d(Tfv, Tx_{2n+1})) &= \psi(d(Tfv, Tgx_{2n})) \\
&\leq \psi(M(Tv, Tx_{2n})) - \varphi(M(Tv, Tx_{2n})) \\
&= \psi(d(Tv, Tfv)) - \varphi(d(Tv, Tfv))
\end{aligned} \tag{3.32}$$

and letting $n \to \infty$, we arrive at

$$\psi(d(Tfv, Tv)) \leq \psi(d(Tv, Tfv)) - \varphi(d(Tv, Tfv)). \tag{3.33}$$

We get a contradiction. Hence, $v = fv$. By the same argument, we get $v = gv$.

Let $w \in X$ such that $w = fw = gw$. Then, we have

$$M(Tv,Tw) = \max\left\{d(Tv,Tw), d(Tv,Tfv), d(Tw,Tgw), \frac{d(Tv,Tgw)+d(Tfv,Tw)}{2}\right\}$$
$$= \max\left\{d(Tv,Tw), \frac{d(Tv,Tw)+d(Tv,Tw)}{2}\right\} = d(Tw,Tv). \tag{3.34}$$

Thus

$$\psi(d(Tv,Tw)) = \psi\big(d(Tfv,Tgw)\big) \le \psi(M(Tv,Tw)) - \varphi(M(Tv,Tw))$$
$$= \psi(d(Tv,Tw)) - \varphi(d(Tv,Tw)). \tag{3.35}$$

This implies that $d(Tv,Tw) = 0$, or $Tv = Tw$. Since T is injective, we have $w = v$. The theorem is proved. □

Remark 3.2. (1) In Theorem 3.1, if we choose $Tx = x$ for all $x \in X$, then we get Theorem 2.4.

(2) In Theorem 3.1, if we fix $\psi(t) = t$ for all t, then we obtain another extension of Theorem 2.3.

(3) In Theorem 3.1, if we choose $f = g$, then we get the uniqueness and existence of fixed point of generalized φ-weak T-contractions.

The following example shows that Theorem 3.1 is a proper extension of Theorem 2.4.

Example 3.3. Let $X = [1,+\infty)$ and d be the usual metric in X. Consider the maps $f(x) = g(x) = 4\sqrt{x}$. It is easy to see that 16 is the unique fixed point of f and g. We claim that f and g are not generalized φ-weak contraction. Indeed, if there exist lower semicontinuous functions $\psi, \varphi : [0,\infty) \to [0,\infty)$ with $\psi(t) > 0$, $\varphi(t) > 0$ for $t \in (0,\infty)$ and $\varphi(0) = \psi(0) = 0$, such that

$$\psi(d(fx,gy)) \le \psi(M(x,y)) - \varphi(M(x,y)), \quad \forall x,y \in X, \tag{3.36}$$

then

$$\psi(4|\sqrt{x} - \sqrt{y}|) \le \psi(M(x,y)) - \varphi(M(x,y)), \quad \forall x,y \in X, \tag{3.37}$$

where $M(x,y) = \max\{d(x,y), d(fx,x), d(gy,y), (1/2)[d(gx,y) + d(fy,x)]\}$. For $x = 4$ and $y = 1$, we obtain

$$M(x,y) = \max\left\{3,4,3,\frac{7}{2}\right\} = 4. \tag{3.38}$$

It follows from (3.37) that

$$\psi(4) \le \psi(4) - \varphi(4). \tag{3.39}$$

Hence, $\varphi(4) \le 0$. We arrive at a contradiction with $\varphi(t) > 0$ for $t \in (0,\infty)$.

Consider the map $Tx = \ln x + 1$, for all $x \in X$. It is easy to see that T is injective, continuous, and sequentially convergent. Let $\psi(t) = t$ and $\varphi(t) = t/3$, for all $t \in [0,+\infty)$. Now, we show that f and g are generalized φ-weak T-contractions. It reduces to check the following inequality:

$$\left|\ln 4\sqrt{x} - \ln 4\sqrt{y}\right| \le \frac{2}{3} M(Tx,Ty), \quad \forall x,y \in [1,+\infty). \tag{3.40}$$

We have

$$\left|\ln 4\sqrt{x} - \ln 4\sqrt{y}\right| = \frac{1}{2}\left|\ln \frac{x}{y}\right| \tag{3.41}$$

$$\begin{aligned} M(Tx,Ty) = {} & \max\Bigg\{ \left|\ln x - \ln y\right|, \left|\ln 4\sqrt{x} - \ln x\right|, \left|\ln 4\sqrt{y} - \ln y\right| \\ & \frac{\left|\ln 4\sqrt{y} - \ln x\right| + \left|\ln 4\sqrt{x} - \ln y\right|}{2} \Bigg\} \\ \ge {} & \left|\ln x - \ln y\right| = \left|\ln \frac{x}{y}\right|. \end{aligned} \tag{3.42}$$

It follows from (3.41) and (3.42) that

$$\left|\ln 4\sqrt{x} - \ln 4\sqrt{y}\right| \le \frac{1}{2} M(Tx,Ty) \tag{3.43}$$

for every $x, y \in X$. This proves that (3.40) is true.

By the same method used in the proof of Theorem 3.1, we get the following theorem.

Theorem 3.4. *Let (X,d) be a complete metric space and $T : X \to X$ an injective, continuous, and sequentially convergent mapping. Let $f,g : X \to X$ be self-mappings. If there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$\psi(d(fTx,gTy)) \le \psi(d(Tx,Ty)) - \varphi(d(Tx,Ty)) \tag{3.44}$$

for all $x, y \in X$, then f, g have a unique common fixed point.

Proof. It follows from the proof of Theorem 3.1 with necessary modifications. □

References

[1] S. Banach, "Sur les operations dans les ensembles abstraits et leur application aux equations itegrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.
[2] D. W. Boyd and S. W. Wong, "On nonlinear contractions," *Proceedings of the American Mathematical Society*, vol. 20, pp. 458–464, 1969.

[3] Y. I. Alber and S. Guerre-Delabriere, "Principle of weakly contractive maps in Hilbert spaces," in *New Results in Operator Theory and its Applications*, I. Gohberg and Y. Lyubich, Eds., vol. 98 of *Operator Theory, Advances and Applications*, pp. 7–22, Birkhäuser, Basel, Switzerland, 1997.

[4] E. Karapinar, "Weak ϕ-contraction on partial metric spaces," *Journal of Computational Analysis and Applications*, vol. 14, no. 2, pp. 206–210, 2012.

[5] E. Karapinar and K. Sadarangani, "Fixed point theory for cyclic $(\varphi - \psi)$-contractions," *Fixed Point Theory and Applications*, vol. 2011, article 69, 2011.

[6] E. Karapınar, "Fixed point theory for cyclic weak ϕ-contraction," *Applied Mathematics Letters*, vol. 24, no. 6, pp. 822–825, 2011.

[7] B. E. Rhoades, "Some theorems on weakly contractive maps," *Nonlinear Analysis*, vol. 47, no. 4, pp. 2683–2693, 2001.

[8] Q. Zhang and Y. Song, "Fixed point theory for generalized φ-weak contractions," *Applied Mathematics Letters*, vol. 22, no. 1, pp. 75–78, 2009.

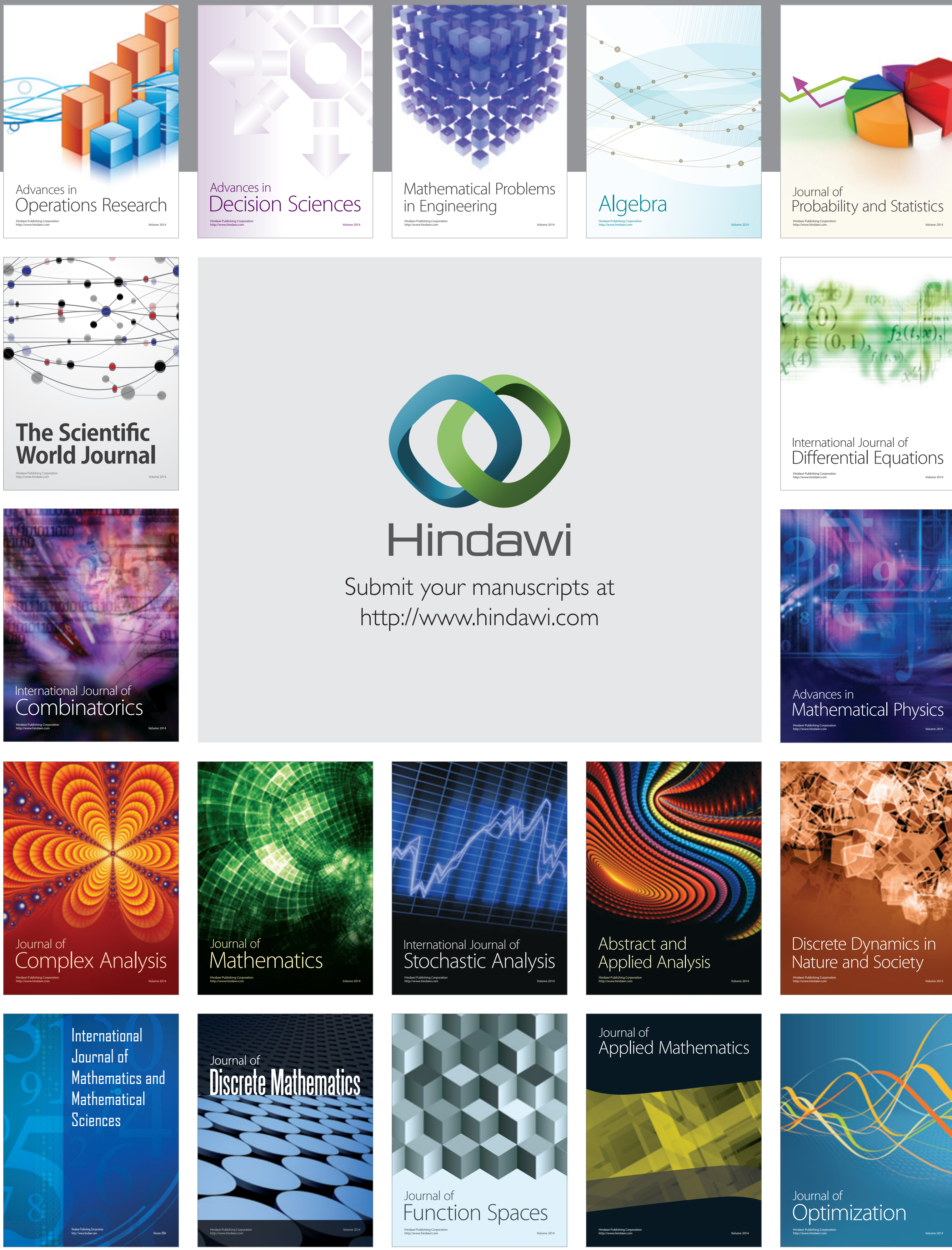
[9] K. P. Chi, "On a fixed point theorem for certain class of maps satisfying a contractive condition depended on an another function," *Lobachevskii Journal of Mathematics*, vol. 30, no. 4, pp. 289–291, 2009.

[10] K. P. Chi and H. T. Thuy, "A fixed point theorem in 2-metric spaces for a class of maps that satisfy a contractive condition dependent on an another function," *Lobachevskii Journal of Mathematics*, vol. 31, no. 4, pp. 338–346, 2010.

[11] S. Moradi and M. Omid, "A fixed-point theorem for integral type inequality depending on another function," *International Journal of Mathematical Analysis*, vol. 4, no. 29–32, pp. 1491–1499, 2010.

[12] P. N. Dutta and B. S. Choudhury, "A generalisation of contraction principle in metric spaces," *Fixed Point Theory and Applications*, vol. 2008, Article ID 406368, 8 pages, 2008.

[13] D. Đorić, "Common fixed point for generalized (ψ, φ)-weak contractions," *Applied Mathematics Letters*, vol. 22, no. 12, pp. 1896–1900, 2009.


Advances in Operations Research
Advances in Decision Sciences
Mathematical Problems in Engineering
Algebra
Journal of Probability and Statistics
The Scientific World Journal
International Journal of Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
International Journal of Stochastic Analysis
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
Journal of Applied Mathematics
Journal of Optimization