

## Research Article

# r-Costar Pair of Contravariant Functors

**S. Al-Nofayee**

*Department of Mathematics, Taif University, P.O. Box 439, Hawiah 21974, Saudi Arabia*

Correspondence should be addressed to S. Al-Nofayee, [alnofayee@hotmail.com](mailto:alnofayee@hotmail.com)

Received 8 June 2012; Accepted 16 July 2012

Academic Editor: Feng Qi

Copyright © 2012 S. Al-Nofayee. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We generalize r-costar module to r-costar pair of contravariant functors between abelian categories.

## 1. Introduction

For a ring  $A$ , a fixed right  $A$ -module  $M$ , and  $D = \text{End}_A(M)$ , let  $\text{fgd-tl}(M_A)$  denote the class of torsionless right  $R$ -modules whose  $M$ -dual are finitely generated over  $D$  and  $\text{fg-tl}({}_D M)$  denote the class of finitely generated torsionless left  $D$ -modules.  $M$  is called costar module if

$$\text{Hom}_A(-, M) : \text{fgd-tl}(M_A) \rightleftarrows \text{fg-tl}({}_D M) : \text{Hom}_D(-, M) \quad (1.1)$$

is a duality. Costar modules were introduced by Colby and Fuller in [1].  $M$  is said to be an r-costar module provided that any exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0, \quad (1.2)$$

such that  $X$  and  $Y$  are  $M$ -reflexive, remains exact after applying the functor  $\text{Hom}_A(-, M)$  if and only if  $Z$  is  $M$ -reflexive. The notion of r-costar module was introduced by Liu and Zhang in [2]. We say that a right  $A$ -module  $X$  is  $n$ -finitely  $M$ -copresented if there exists a long exact sequence

$$0 \longrightarrow X \longrightarrow M^{k_0} \longrightarrow M^{k_1} \longrightarrow \dots \longrightarrow M^{k_{n-1}} \quad (1.3)$$

such that  $k_i$  are positive integers for  $0 \leq i \leq n - 1$ . The class of all  $n$ -finitely  $M$ -copresented modules is denoted by  $n\text{-cop}(M)$ . We say that a right  $A$ -module  $M$  is a finitistic  $n$ -self-cotilting module provided that any exact sequence

$$0 \longrightarrow X \longrightarrow M^m \longrightarrow Z \longrightarrow 0, \quad (1.4)$$

such that  $Z \in n\text{-cop}(M)$  and  $m$  is a positive integer, remains exact after applying the functor  $\text{Hom}_A(-, M)$  and  $n\text{-cop}(Q) = (n + 1)\text{-cop}(Q)$ . Finitistic  $n$ -self-cotilting modules were introduced by Breaz in [3].

In [4] Castaño-Iglesias generalizes the notion of costar module to Grothendieck categories. Pop in [5] generalizes the notion of finitistic  $n$ -self-cotilting module to finitistic  $n$ - $F$ -cotilting object in abelian categories and he describes a family of dualities between abelian categories. Breaz and Pop in [6] generalize a duality exhibited in [3, Theorem 2.8] to abelian categories.

In this work we continue this kind of study and generalizes the notion of  $r$ -costar module to  $r$ -costar pair of contravariant functors between abelian categories, by generalizing the work in [2]. We use the same technique of proofs of that paper.

## 2. Preliminaries

Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  and  $G: \mathcal{D} \rightarrow \mathcal{C}$  be additive and contravariant left exact functors between two abelian categories  $\mathcal{C}$  and  $\mathcal{D}$ . It is said that they are adjoint on the right if there are natural isomorphisms

$$\eta_{X,Y}: \text{Hom}_{\mathcal{C}}(X, G(Y)) \longrightarrow \text{Hom}_{\mathcal{D}}(Y, F(X)), \quad (2.1)$$

for  $X \in \mathcal{C}$  and  $Y \in \mathcal{D}$ . Then they induce two natural transformations  $\delta: 1_{\mathcal{C}} \rightarrow GF$  and  $\delta': 1_{\mathcal{D}} \rightarrow FG$  defined by  $\delta_X = \eta_{X, F(X)}^{-1}(1_{F(X)})$  and  $\delta'_Y = \eta_{G(Y), Y}^{-1}(1_{G(Y)})$ . Moreover the following identities are satisfied for each  $X \in \mathcal{C}$  and  $Y \in \mathcal{D}$ :

$$F(\delta_X) \circ \delta'_{F(X)} = 1_{F(X)}, \quad G(\delta'_Y) \circ \delta'_{G(Y)} = 1_{G(Y)}. \quad (2.2)$$

The pair  $(F, G)$  is called a duality if there are functorial isomorphisms  $GF \simeq 1_{\mathcal{C}}$  and  $FG \simeq 1_{\mathcal{D}}$ . An object  $X$  of  $\mathcal{C}$  ( $Y \in \mathcal{D}$ ) is called  $F$ -reflexive (resp.,  $G$ -reflexive) in case  $\delta_X$  (resp.,  $\delta'_Y$ ) is an isomorphism. By  $\text{Ref}(F)$  we will denote the full subcategory of all  $F$ -reflexive objects. As well by  $\text{Ref}(G)$  we will denote the full subcategory of all  $G$ -reflexive objects. It is clear that the functors  $F$  and  $G$  induce a duality between the categories  $\text{Ref}(F)$  and  $\text{Ref}(G)$ .

We say that the pair  $(F, G)$  of left exact contravariant functors is  $r$ -costar provided that any exact sequence

$$0 \longrightarrow Q \longrightarrow U \longrightarrow V \longrightarrow 0, \quad (2.3)$$

with  $Q, U \in \text{Ref}(F)$  remains exact after applying the functor  $F$  if and only if  $V \in \text{Ref}(F)$ .

An object  $U$  is called  $V$ -finitely generated if there is an epimorphism  $V^n \rightarrow U \rightarrow 0$ , for some positive integer  $n$ . We denote by  $\text{gen}(V)$  the subcategory of all  $V$ -finitely generated

objects.  $\text{add}(V)$  denotes the class of all summands of finite direct sums of copies of  $V$ . We will denote by  $\text{proj}(\mathfrak{D})$  the full subcategory of all projective objects in  $\mathfrak{D}$ .

From now on we suppose that  $\mathfrak{D}$  has enough projectives that is, for every object  $X \in \mathfrak{D}$  there is a projective object  $P \in \mathfrak{D}$  and an epimorphism  $P \rightarrow X \rightarrow 0$ . It is clear that we can construct a projective resolution for any object  $X$ . Suppose we have a projective resolution of  $X$

$$P: \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow X \rightarrow 0. \tag{2.4}$$

This gives rise to the sequence

$$0 \rightarrow G(X) \rightarrow G(P_0) \rightarrow G(P_1) \rightarrow G(P_2) \rightarrow \cdots, \tag{2.5}$$

and the cochain complex  $G(P)$ , which we can compute its cohomology at the  $n$ th spot (the kernel of the map from  $G(P_n)$  modulo the image of the map to  $G(P_n)$ ) and denote it by  $H^n(G(P))$ . We define  $R^n G(X) = H^n(G(P))$  as the  $n$ th right derived functor of  $G$ . For the functor  $G$  we define  ${}^{\perp}T_G^{i \geq n} = \{X \in \mathfrak{D} : R^i G(X) = 0 \text{ for every } i \geq n\}$ .

Let

$$0 \rightarrow Q \xrightarrow{f} U \rightarrow V \rightarrow 0 \tag{2.6}$$

be an exact sequence in  $\mathfrak{C}$ . Applying the functor  $F$  we get the exact sequence

$$0 \rightarrow F(V) \rightarrow F(U) \xrightarrow{p} X \rightarrow 0, \tag{2.7}$$

where  $X = \text{Im}(F(f))$ . Let  $F(f) = j \circ p$  be the canonical decomposition of  $F(f)$ , where  $j: X \rightarrow F(Q)$  is the inclusion map. Applying the functor  $G$  to the sequence (2.7), we have the following exact sequence

$$0 \rightarrow GX \xrightarrow{G(p)} GF(U) \rightarrow GF(V). \tag{2.8}$$

Now if we put  $\alpha = G(j) \circ \delta_Q$ , then

$$G(p) \circ \alpha = G(p) \circ G(j) \circ \delta_Q = G(j \circ p) \circ \delta_Q = GF(f) \circ \delta_Q = \delta_U \circ f. \tag{2.9}$$

So we have the following commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Q & \xrightarrow{f} & U & \longrightarrow & V \longrightarrow 0 \\
 \downarrow & & \downarrow \alpha & & \downarrow \delta_U & & \downarrow \delta_V \\
 0 & \longrightarrow & GX & \xrightarrow{G(p)} & GF(U) & \longrightarrow & GF(V)
 \end{array} \tag{2.10}$$

### 3. r-Costar Pair of Contravariant Functors

We will fix all the notations and terminologies used in previous section.

**Proposition 3.1.** *Let  $(F, G)$  be a pair of left exact contravariant functors which are adjoint on the right. Assume that  $\text{Ref}(G) \subseteq {}^{\perp}T_G^{i \geq 1}$  and  ${}^{\perp}T_G^{i \geq 0} = 0$ . Then  $(F, G)$  is an r-costar.*

*Proof.* Let

$$0 \longrightarrow Q \xrightarrow{f} U \longrightarrow V \longrightarrow 0 \quad (3.1)$$

be an exact sequence with  $Q, U \in \text{Ref}(F)$ . Assume that we have the exact sequence

$$0 \longrightarrow F(V) \longrightarrow F(U) \longrightarrow F(Q) \longrightarrow 0, \quad (3.2)$$

after applying the functor  $F$ . Applying the functor  $G$  to the last sequence, we get an exact sequence

$$0 \longrightarrow GF(Q) \longrightarrow GF(U) \longrightarrow GF(V) \longrightarrow R^1G(F(Q)) = 0, \quad (3.3)$$

since  $F(Q) \in \text{Ref}(G) \subseteq {}^{\perp}T_G^{i \geq 1}$ . Hence we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q & \longrightarrow & U & \longrightarrow & V & \longrightarrow & 0 \\ \downarrow & & \downarrow \delta_Q & & \downarrow \delta_U & & \downarrow \delta_V & & \downarrow \\ 0 & \longrightarrow & GF(Q) & \longrightarrow & GF(U) & \longrightarrow & GF(V) & \longrightarrow & 0 \end{array} \quad (3.4)$$

Since  $Q, U \in \text{Ref}(F)$ ,  $\delta_Q$  and  $\delta_U$  are isomorphisms. Now it is clear that  $\delta_V$  is an isomorphism which means that  $V \in \text{Ref}(F)$ .

Conversely, suppose that  $V \in \text{Ref}(F)$ . Applying the functor  $F$  to the sequence (3.1), we get an exact sequence

$$0 \longrightarrow F(V) \longrightarrow F(U) \longrightarrow X \longrightarrow 0, \quad (3.5)$$

where  $X = \text{Im}(F(f))$ . Hence we can get the exact sequence

$$0 \longrightarrow X \xrightarrow{j} F(Q) \longrightarrow Y \longrightarrow 0, \quad (3.6)$$

for some  $Y \in \mathcal{D}$ , and  $j$  is the inclusion map. Applying the functor  $G$  to the sequence (3.5), we have the following exact commutative diagram (see diagram (2.10))

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q & \longrightarrow & U & \longrightarrow & V & \longrightarrow & 0 \\ \downarrow & & \downarrow \alpha & & \downarrow \delta_U & & \downarrow \delta_V & & \downarrow \\ 0 & \longrightarrow & GX & \longrightarrow & GF(U) & \longrightarrow & GF(V) & \longrightarrow & R^1G(X) \longrightarrow R^1G(F(U)) = 0 \end{array} \quad (3.7)$$

where  $\alpha = G(j) \circ \delta_Q$ . Note that  $\delta_U$  and  $\delta_V$  are isomorphisms, since  $Q, U \in \text{Ref}(F)$ . It is clear from the diagram that  $R^1G(X) = 0$ . Now  $R^iG(X) = 0$  for all  $i \geq 2$ , by dimension shifting, since  $F(U), F(V) \in \text{Ref}(G) \subseteq {}^\perp T_G^{i \geq 1}$ . Hence  $X \in {}^\perp T_G^{i \geq 1}$ . Now applying the functor  $G$  to sequence (3.6), we get the long exact sequence

$$\begin{aligned} 0 \longrightarrow G(Y) \longrightarrow GF(Q) \xrightarrow{G(j)} G(X) \longrightarrow R^1G(Y) \\ \longrightarrow R^1G(F(Q)) \longrightarrow R^1G(X) \longrightarrow R^2G(Y) \longrightarrow R^2G(F(Q)) \longrightarrow \dots \end{aligned} \tag{3.8}$$

Above we conclude that  $X \in {}^\perp T_G^{i \geq 1}$  and by assumptions  $F(Q) \in \text{Ref}(G) \subseteq {}^\perp T_G^{i \geq 1}$ , thus  $R^1G(X) = 0$  and  $R^2G(F(Q)) = 0$ . Hence by dimension shifting  $Y \in {}^\perp T_G^{i \geq 2}$ . Now consider the following part from sequence (3.8)

$$0 \longrightarrow G(Y) \longrightarrow GF(Q) \xrightarrow{G(j)} G(X) \longrightarrow R^1G(Y). \tag{3.9}$$

Note that  $\alpha = G(j) \circ \delta_Q$  in diagram (3.7) is an isomorphism, since  $\delta_U$  and  $\delta_V$  are isomorphisms. Hence  $G(j)$  is an isomorphism, since  $\delta_Q$  is an isomorphism, so from sequence (3.9),  $R^1G(Y) = 0 = G(Y)$ . We conclude that  $Y \in {}^\perp T_G^{i \geq 0}$ . Since  ${}^\perp T_G^{i \geq 0} = 0$  by assumptions,  $Y = 0$  and hence from sequence (3.6)  $X \cong F(Q)$  canonically. Therefore the functor  $F$  preserves the exactness of the exact sequence

$$0 \longrightarrow Q \longrightarrow U \longrightarrow V \longrightarrow 0 \tag{3.10}$$

in  $\text{Ref}(F)$ . We conclude that the pair  $(F, G)$  is an  $r$ -costar. □

**Corollary 3.2.** *Let  $(F, G)$  be a pair of left exact contravariant functors which are adjoint on the right. Assume that  $\text{Ref}(G) = {}^\perp T_G^{i \geq 1}$ . Then  $(F, G)$  is an  $r$ -costar.*

*Proof.* Let  $X \in {}^\perp T_G^{i \geq 0}$ , then  $X \in {}^\perp T_G^{i \geq 1} = \text{Ref}(G)$ . Hence  $X \cong FG(X) = 0$ . □

**Proposition 3.3.** *Let  $V$  be a  $G$ -reflexive generator in  $\mathfrak{D}$  and  $U = G(V)$ . Let  $(F, G)$  be an  $r$ -costar pair. If  $\text{Ref}(G) \subseteq g(V)$ , then for any  $X \in \text{Ref}(F)$ , there is an infinite exact sequence*

$$0 \longrightarrow X \longrightarrow U_1 \longrightarrow \dots \longrightarrow U_n \longrightarrow \dots, \tag{3.11}$$

*which remains exact after applying the functor  $F$ , where  $U_i \in \text{add}(U)$  for each  $i$ .*

*Proof.* Let  $X \in \text{Ref}(F)$ . Then  $F(X) \in \text{Ref}(G)$ , so by assumption there is an exact sequence

$$V^n \longrightarrow F(X) \longrightarrow 0. \tag{3.12}$$

Applying the functor  $G$  we have an exact sequence

$$0 \longrightarrow X \longrightarrow U^n \longrightarrow Y \longrightarrow 0, \tag{3.13}$$

for some  $Y \in \mathcal{C}$ . Since  $(F, G)$  is an  $r$ -costar pair, the last sequence is exact after applying the functor  $F$ , that is we have an exact sequence

$$0 \longrightarrow F(Y) \longrightarrow F(U^n) \longrightarrow F(X) \longrightarrow 0. \quad (3.14)$$

Applying the functor  $G$  again we get the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & U^n & \longrightarrow & Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \delta_X & & \delta_{U^n} & & \delta_Y & & \\ 0 & \longrightarrow & GF(X) & \longrightarrow & GF(U^n) & \longrightarrow & GF(Y) & & \end{array} \quad (3.15)$$

Since  $X, U^n \in \text{Ref}(F), Y \in \text{Ref}(F)$ . By repeating the process to  $Y$ , and so on, we finally obtain the desired exact sequence.  $\square$

**Proposition 3.4.** *Let  $V$  be a  $G$ -reflexive and a projective generator in  $\mathcal{D}$  and  $U = G(V)$ . Let  $(F, G)$  be an  $r$ -costar pair and suppose that  $\text{Ref}(G) \subseteq \text{gen}(V)$ . Then  $\text{Ref}(G) \subseteq {}^{\perp}T_G^{i \geq 1}$ .*

*Proof.* Let  $X \in \text{Ref}(G)$ , then  $G(X) \in \text{Ref}(F)$  and hence by Proposition 3.3, there is an infinite exact sequence

$$0 \longrightarrow G(X) \longrightarrow U_1 \longrightarrow \cdots \longrightarrow U_n \longrightarrow \cdots, \quad (3.16)$$

which remains exact after applying the functor  $F$ , where  $U_i \in \text{add}(U)$  for each  $i$ . So we have an exact sequence

$$\cdots \longrightarrow F(U_n) \longrightarrow \cdots \longrightarrow F(U_1) \longrightarrow FG(X) \longrightarrow 0 \quad (3.17)$$

Again the last sequence remains exact after applying the functor  $G$ , since we get a sequence isomorphic to sequence (3.16), because  $G(X), U_i$ , for each  $i$ , are  $F$ -reflexive. We obtain that  $\text{Ref}(G) \subseteq {}^{\perp}T_G^{i \geq 1}$  by dimension shifting.  $\square$

Suppose we have the following exact sequence in  $\mathcal{D}$

$$0 \longrightarrow X \longrightarrow P_2 \longrightarrow P_1 \longrightarrow Y \longrightarrow 0, \quad (3.18)$$

where  $P_2, P_1$  are projective objects in  $\mathcal{D}$  and  $Y \in {}^{\perp}T_G^{i \geq 1}$ . Applying the functor  $G$  we get the following exact sequence

$$0 \longrightarrow G(Y) \longrightarrow G(P_1) \longrightarrow G(P_2) \longrightarrow G(X) \longrightarrow R^1G(Y) = 0. \quad (3.19)$$

Applying the functor  $F$  we get the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X & \longrightarrow & P_2 & \longrightarrow & P_1 \\
 \downarrow & & \downarrow \delta_X & & \downarrow \delta_{P_2} & & \downarrow \delta_{P_1} \\
 0 & \longrightarrow & FG(X) & \longrightarrow & FG(P_2) & \longrightarrow & FG(P_1)
 \end{array} \tag{3.20}$$

If  $\text{proj}(\mathfrak{D}) \subseteq \text{Ref}(G)$ , then it is clear that  $X \in \text{Ref}(G)$ .

**Proposition 3.5.** *Let  $V$  be a  $G$ -reflexive generator in  $\mathfrak{D}$ . Let  $(F, G)$  be an  $r$ -costar pair and suppose that  $\text{proj}(\mathfrak{D}) \subseteq \text{Ref}(G) \subseteq \text{gen}(V)$ . Then  ${}^{\perp}T_G^{i \geq 0} = 0$ .*

*Proof.* For any  $X \in {}^{\perp}T_G^{i \geq 0}$ , we can build the following exact sequence in  $\mathfrak{D}$

$$0 \longrightarrow Y \longrightarrow P_2 \longrightarrow P_1 \longrightarrow X \longrightarrow 0, \tag{3.21}$$

where  $P_2, P_1$  are projective objects and  $Y$  an object in  $\mathfrak{D}$ . By the argument before the proposition it is clear that  $Y \in \text{Ref}(G)$  and hence  $G(Y) \in \text{Ref}(F)$ . Applying the functor  $G$  we get the following exact sequence

$$0 \longrightarrow G(X) = 0 \longrightarrow G(P_1) \longrightarrow G(P_2) \longrightarrow G(Y) \longrightarrow R^1G(X) = 0. \tag{3.22}$$

Applying the functor  $F$  we get the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & Y & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & X & \longrightarrow & 0 \\
 \downarrow & & \downarrow \delta_Y & & \downarrow \delta_{P_2} & & \downarrow \delta_{P_1} & & \downarrow & & \\
 0 & \longrightarrow & FG(Y) & \longrightarrow & FG(P_2) & \longrightarrow & FG(P_1) & \longrightarrow & 0 & & 
 \end{array} \tag{3.23}$$

Thus it is clear that  $X \cong 0$ . □

Now we are able to give the following characterization of  $r$ -costar pair.

**Theorem 3.6.** *Let  $(F, G)$  be a pair of left exact contravariant functor which are adjoint on the right. Suppose that  $V$  be a  $G$ -reflexive projective generator in  $\mathfrak{D}$  and  $\text{proj}(\mathfrak{D}) \subseteq \text{Ref}(G) \subseteq \text{gen}(V)$ . Then  $(F, G)$  is an  $r$ -costar if and only if  $\text{Ref}(G) \subseteq {}^{\perp}T_G^{i \geq 1}$  and  ${}^{\perp}T_G^{i \geq 0} = 0$ .*

*Proof.* By Propositions 3.4, 3.1, and 3.5. □

**Corollary 3.7.** *Let  $(F, G)$  be a pair of left exact contravariant functor which are adjoint on the right. Suppose that  $V$  be a  $G$ -reflexive projective generator in  $\mathfrak{D}$  and  $U = G(V)$ . If  $\text{proj}(\mathfrak{D}) \subseteq \text{Ref}(G) \subseteq \text{gen}(V)$ , then the following are equivalent.*

- (1)  $(F, G)$  is an  $r$ -costar.
- (2) For any exact sequence

$$0 \longrightarrow X \longrightarrow U^n \longrightarrow Y \longrightarrow 0, \tag{3.24}$$

with  $X \in \text{Ref}(F)$ , then  $Y \in \text{Ref}(F)$  if and only if the exact sequence remains exact after applying the functor  $F$ .

*Proof.* (1) $\Rightarrow$ (2) follows from the definition of  $r$ -costar pair.

(2) $\Rightarrow$ (1) the proof goes the same as the proofs of Propositions 3.3, 3.4, 3.5, and Theorem 3.6.  $\square$

## References

- [1] R. R. Colby and K. R. Fuller, "Costar modules," *Journal of Algebra*, vol. 242, no. 1, pp. 146–159, 2001.
- [2] H. Liu and S. Zhang, " $r$ -costar modules," *International Electronic Journal of Algebra*, vol. 8, pp. 167–176, 2010.
- [3] S. Breaz, "Finitistic  $n$ -self-cotilting modules," *Communications in Algebra*, vol. 37, no. 9, pp. 3152–3170, 2009.
- [4] F. Castaño-Iglesias, "On a natural duality between Grothendieck categories," *Communications in Algebra*, vol. 36, no. 6, pp. 2079–2091, 2008.
- [5] F. Pop, "Natural dualities between abelian categories," *Central European Journal of Mathematics*, vol. 9, no. 5, pp. 1088–1099, 2011.
- [6] S. Breaz and F. Pop, "Dualities induced by right adjoint contravariant functors," *Studia Universitatis Babeş-Bolyai Mathematica*, vol. 55, no. 1, pp. 75–83, 2010.





# Hindawi

Submit your manuscripts at  
<http://www.hindawi.com>

