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*Research Article*

# Hyperbolically Bi-Lipschitz Continuity for $1/|w|^2$-Harmonic Quasiconformal Mappings

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We study the class of $1/|w|^2$-harmonic $K$-quasiconformal mappings with angular ranges. After building a differential equation for the hyperbolic metric of an angular range, we obtain the sharp bounds of their hyperbolically partial derivatives, determined by the quasiconformal constant $K$. As an application we get their hyperbolically bi-Lipschitz continuity and their sharp hyperbolically bi-Lipschitz coefficients.

## 1. Introduction

Let $\Omega$ and $\Omega'$ be two domains of hyperbolic type in the complex plane $\mathbb{C}$. A $C^2$ sense-preserving homeomorphism $f$ of $\Omega$ onto $\Omega'$ is said to be a *$\rho$-harmonic mapping* if it satisfies the Euler-Lagrange equation

$$f_{z\overline{z}} + (\log \rho)_w(f) f_z f_{\overline{z}} = 0, \tag{1.1}$$

where $w = f(z)$ and $\rho(w)|dw|^2$ is a smooth metric in $\Omega'$. If $\rho$ is a constant then $f$ is said to be *euclidean harmonic*. A euclidean harmonic mapping defined on a simply connected domain is of the form $f = h + \overline{g}$, where $h$ and $g$ are two analytic functions in $\Omega$. For a survey of harmonic mappings, see [1–3].

In this paper we study the class of $1/|w|^2$-harmonic mappings. This class of mappings seems very particular but it includes the class of so-called logharmonic mappings. In fact, a *logharmonic mapping* is a solution of the nonlinear elliptic partial differential equation

$$\overline{f_{\overline{z}}} = \left(a\frac{\overline{f}}{f}\right) f_z, \tag{1.2}$$

where $a(z)$ is analytic and $|a(z)| < 1$ (see [4–6] for more details). By differentiating (1.2) in $\overline{z}$, we have that

$$\overline{f_{z\overline{z}} + \left(\log \frac{1}{|w|^2}\right)_w \circ f f_z f_{\overline{z}}} = a\frac{\overline{f}}{f}\left(f_{z\overline{z}} + \left(\log \frac{1}{|w|^2}\right)_w \circ f f_z f_{\overline{z}}\right). \tag{1.3}$$

Hence, it follows that a logharmonic mapping is a $1/|w|^2$-harmonic mapping.

If a $\rho$-harmonic mapping $f$ also satisfies the condition that $|f_{\overline{z}}(z)| \leq k|f_z(z)|$ holds for every $z \in \Omega$, then it is called a *$\rho$-harmonic $K$-quasiconformal mapping* (for simplicity, a *harmonic quasiconformal mapping* or *H.Q.C mapping*), where $K = (1+k)/(1-k)$ .

Let $\lambda_\Omega(z)|dz|$ denote the hyperbolic metric of a simply connected region $\Omega$ with gaussian curvature $-4$. For a harmonic quasiconformal mapping $f$ of $\Omega$ onto $\Omega'$, we call the quantity

$$\|\partial f\| = \frac{\lambda_{\Omega'} \circ f}{\lambda_\Omega}|f_z| \tag{1.4}$$

the *hyperbolically partial derivative* of $f$. If $f$ is a harmonic quasiconformal mapping of $\Omega_1$ onto $\Omega_2$ and $\varphi$ is a conformal mapping of $\Omega_0$ onto $\Omega_1$ then $f \circ \varphi$ is also a harmonic quasiconformal mapping. We have

$$\|\partial(f \circ \varphi)\| = \frac{\lambda_{\Omega_2}(f \circ \varphi(\zeta))}{\lambda_{\Omega_0}(\zeta)}\left|(f \circ \varphi)_\zeta\right| = \frac{\lambda_{\Omega_2}(f(z))}{\lambda_{\Omega_1}(z)}|f_z| = \|\partial f\|, \tag{1.5}$$

where $z = \varphi(\zeta)$. Hence, we always fix the domain of a harmonic quasiconformal mapping to be the unit disk $D$ when studying its hyperbolically partial derivative.

The hyperbolic distance $d_h(z_1, z_2)$ between $z_1$ and $z_2$ is defined by $\inf_\gamma \int_\gamma \lambda_\Omega(z)|dz|$, where $\gamma$ runs through all rectifiable curves in $\Omega$ which connect $z_1$ and $z_2$. A harmonic quasiconformal mapping $f$ of $\Omega$ onto $\Omega'$ is said to *be hyperbolically $L_1$-Lipschitz* ($L_1 > 0$) if

$$d_h(f(z_1), f(z_2)) \leq L_1 d_h(z_1, z_2), \quad z_1, z_2 \in \Omega. \tag{1.6}$$

The constant $L_1$ is said to be the *hyperbolically Lipschitz coefficient* of $f$. If there also exists a constant $L_2 > 0$ such that

$$L_2 d_h(z_1, z_2) \ \leq d_h(f(z_1), f(z_2)), \quad z_1, z_2 \in \Omega, \tag{1.7}$$

then $f$ is said to be *hyperbolically* $(L_2, L_1)$*-bi-lipschitz*. We also call the array $(L_2, L_1)$ the *hyperbolically bi-lipschitz coefficient* of $f$.

Under differently restrictive conditions of the ranges of euclidean harmonic quasiconformal mappings, recent papers [7–13] obtained their euclidean Lipschitz and *bi-Lipschitz* continuity. In [8], Kalaj obtained the following.

**Theorem A.** *Let $\Omega$ and $\Omega'$ be two Jordan domains, let $\alpha \in (0,1]$ and let $f : \Omega \mapsto \Omega'$ be a euclidean harmonic quasiconformal mapping. If $\partial\Omega$ and $\partial\Omega' \in C^{1,\alpha}$, then $f$ is euclidean Lipschitz. In particular, if $\Omega'$ is convex, then $f$ is euclidean bi-lipschitz.*

Recently, the hyperbolically Lipschitz or *bi-lipschitz* continuity of euclidean harmonic quasiconformal mappings also excited much interest (see [14–17]). In [14], Chen and Fang proved the following.

**Theorem B.** *Let $f$ be a euclidean harmonic $K$-quasiconformal mapping of $\Omega$ onto a convex domain $\Omega'$. Then $f$ is hyperbolically $(1/K, K)$-bi-lipschitz.*

Theorems A and B tell us that an euclidean harmonic quasiconformal mapping with a convex range has both euclidean and hyperbolically *bi-lipschitz* continuity. Naturally, we want to ask whether a general $\rho$-harmonic quasiconformal mapping also has similar Lipschitz or *bi-lipschitz* continuity. In this paper we study the corresponding question for the class of $1/|w|^2$-harmonic quasiconformal mappings.

To this question, Examples 5.1, and 5.2 show that if the metric $\rho$ is not necessary to be smooth in the range of a $\rho$-harmonic quasiconformal mapping $f$, then $f$ generally does not need to have euclidean and hyperbolically Lipschitz continuity even if its range is convex. Hence, we only consider the case that $\rho$ is smooth, that is, $1/|w|^2$ does not vanish in the range of a $1/|w|^2$-harmonic quasiconformal mapping in this paper. Kalaj and Mateljević (see Theorem 4.4 of [18]) showed the following.

**Theorem C.** *Let $\varphi$ be analytic in $\Omega'$ and $f$ a $|\varphi|$-harmonic quasiconformal mapping of the $C^{1,\alpha}$ domain $\Omega$ onto the $C^{1,\alpha}$ Jordan domain $\Omega'$. If $M = \|(\log\varphi)'\|_\infty < \infty$, then $f$ is euclidean Lipschitz.*

Let $|\varphi(w)|$ be equal to $1/|w|^2$, where $w \in \Omega'$. If the closure of the range $\Omega'$ does not include the origin, then $M = \|(\log\varphi)'\|_\infty = \|1/|w|\|_\infty$ is finite. So by Theorem C a $1/|w|^2$-harmonic quasiconformal mapping with such a range $\Omega'$ has euclidean Lipschitz continuity. Example 5.3 shows that if the origin is a boundary point of $\partial\Omega'$ then a $1/|w|^2$-harmonic quasiconformal mapping does not need to have euclidean Lipschitz continuity. However, Example 5.3 also shows that there is a different result when we consider its hyperbolically Lipschitz continuity. In this paper we will study the hyperbolically Lipschitz or *bi-lipschitz* continuity of a $1/|w|^2$-harmonic quasiconformal mapping with an angular range and its sharp hyperbolically Lipschitz coefficient determined by the constant of quasiconformality. The main result of this paper is the sharp bounds of their hyperbolically partial derivatives. The key of this paper is to build a differential equation for the hyperbolic metric of an angular domain, which is different for using a differential inequality when we studied the class of euclidean harmonic quasiconformal mappings in [14]. The rest of this paper is organized as follows.

In Section 2, using a property of hyperbolic metric of the upper half plane $\mathbb{H}$, we first build a differential equation for the hyperbolic metric of an angular domain with the origin of $\mathbb{C}$ as its vertex (see Lemma 2.1). The two-order differential equation (2.4) is important to derive the upper and lower bounds of the hyperbolically partial derivative of a $1/|w|^2$-harmonic quasiconformal mappings with an angular range.

In Section 3, by combining the well-known Ahlfors-Schwarz lemma and its opposite type given by Mateljević [19] with the differential inequality (2.4), we obtain the upper and lower bounds of the hyperbolically partial derivatives $\|\partial f\|$ of $1/|w|^2$-harmonic $K$-quasiconformal mappings with angular ranges (see Theorem 3.1). We also show that both the upper and lower bounds of $\|\partial f\|$ are sharp.

In Section 4, the hyperbolically $K$-*bi-lipschitz* continuity of a $1/|w|^2$-harmonic $K$-quasiconformal mapping with an angular range is obtained by the sharp inequality (3.2) (see Theorem 4.1). The hyperbolically *bi-lipschitz* coefficients $(1/K, K)$ are sharp.

At last, some auxiliary examples are given. In order to show the sharpness of Theorems 3.1 and 4.1, we present two examples satisfying that the inequalities (3.2) no longer hold for two classes of $1/|w|^2$-harmonic quasiconformal mappings with nonangular ranges (see Examples 5.4 and 5.5).

## 2. A Differential Equation for the Hyperbolic Metric of an Angular Domain

Let $\lambda_H(w)|dw|$ be the hyperbolic metric of the upper half plane $\mathbb{H}$ with gaussian curvature $-4$. Then

$$\lambda_H(w)|dw| = \frac{i}{w-\overline{w}}|dw|, \quad (\log\lambda_H)_w = -\frac{1}{w-\overline{w}}, \quad (\log\lambda_H)_{w\overline{w}} = \frac{1}{(w-\overline{w})^2}. \tag{2.1}$$

Hence, the hyperbolic metric $\lambda_H(w)|dw|$ of $\mathbb{H}$ satisfies that

$$(\log\lambda_H)_{ww} + \frac{(\log\lambda_H)_w}{w} + \frac{\overline{w}}{w}\lambda_H^2 = 0. \tag{2.2}$$

By the relation that $(\log\lambda_H)_w = (\lambda_H)_w/\lambda_H$, the differential equation (2.2) becomes

$$\frac{(\lambda_H)_{ww}}{\lambda_H} = \left(\frac{(\lambda_H)_w}{\lambda_H}\right)^2 - \frac{(\lambda_H)_w}{w\lambda_H} - \frac{\overline{w}}{w}(\lambda_H)^2. \tag{2.3}$$

Using the differential equation (2.3) of the hyperbolic metric of $\mathbb{H}$ we obtain the following.

**Lemma 2.1.** *Let $A$ be an angular domain with the origin of the complex plane $\mathbb{C}$ as its vertex. Then for every $\zeta \in A$ the hyperbolic metric $\lambda_A(\zeta)|d\zeta|$ of $A$ satisfies the following differential equation*

$$(\log\lambda_A)_{\zeta\zeta} + \frac{(\log\lambda_A)_\zeta}{\zeta} + \frac{\overline{\zeta}}{\zeta}(\lambda_A)^2 = 0. \tag{2.4}$$

*Proof.* Let $A_\theta$ be the angular domain $\{z \in \mathbb{C} \mid 0 < \arg z < \theta,\ \theta \in (0, 2\pi]\}$ with 0 as its vertex and $\lambda_{A_\theta}(z)|dz|$ as its hyperbolic metric with gaussian curvature $-4$. Let $f$ be a conformal mapping of $A_\theta$ onto $\mathbb{H}$. Then by the fact that a hyperbolic metric is a conformal invariant it follows that

$$\lambda_{A_\theta}(z) = \lambda_H \circ f\left|f'\right|. \tag{2.5}$$

Hence by the chain rule [20] we get

$$(\log \lambda_{A_\theta})_z = \frac{(\lambda_H)_w \circ f f'}{\lambda_H \circ f} + \frac{1}{2}\frac{f''}{f'},$$
$$(\log \lambda_{A_\theta})_{zz} = \frac{\lambda_H \circ f\left[(\lambda_H)_{ww} \circ f f'^2 + (\lambda_H)_w \circ f f''\right] - \left[(\lambda_H)_w \circ f f'\right]^2}{(\lambda_H \circ f)^2} + \left(\frac{f''}{2f'}\right)'. \tag{2.6}$$

From the relations (2.5) and (2.6) we get

$$\begin{aligned}(\log \lambda_{A_\theta})_{zz} &+ \frac{(\log \lambda_{A_\theta})_z}{z} + \frac{\overline{z}}{z}(\lambda_{A_\theta})^2 \\ &= \frac{(\lambda_H)_{ww}}{\lambda_H} \circ f f'^2 + \frac{(\lambda_H)_w}{\lambda_H} \circ f f'' - \left[\frac{(\lambda_H)_w}{\lambda_H} \circ f\right]^2 f'^2 \\ &\quad + \frac{1}{2}\left(\frac{f''}{f'}\right)' + \frac{(\lambda_H)_w}{\lambda_H} \circ f \frac{f'}{z} + \frac{1}{2z}\frac{f''}{f'} + \frac{\overline{z}}{z}(\lambda_H \circ f|f'|)^2.\end{aligned} \tag{2.7}$$

Using (2.3) we can simplify the previous relation as

$$\begin{aligned}(\log \lambda_{A_\theta})_{zz} &+ \frac{(\log \lambda_{A_\theta})_z}{z} + \frac{\overline{z}}{z}(\lambda_{A_\theta})^2 \\ &= \frac{(\lambda_H)_w}{\lambda_H} \circ f\left(f'' + \frac{f'}{z} - \frac{f'^2}{f}\right) + \frac{1}{2}\left(\frac{f''}{f'}\right)' + \frac{1}{2z}\frac{f''}{f'} + (\lambda_H \circ f)^2\left(\frac{\overline{z}}{z}|f'|^2 - \frac{\overline{f}}{f}f'^2\right),\end{aligned} \tag{2.8}$$

where $w = f(z)$.

Let $f(z) = z^\alpha$, $\alpha \in [1/2, 1) \cup (1, \infty)$. Then $f$ is a conformal mapping of $A_\theta$ onto the upper half plane $\mathbb{H}$ and the following relations

$$f'' + \frac{f'}{z} - \frac{f'^2}{f} = 0, \quad \frac{1}{2}\left(\frac{f''}{f'}\right)' + \frac{1}{2z}\frac{f''}{f'} = 0, \quad \frac{\overline{z}}{z}|f'|^2 - \frac{\overline{f}}{f}f'^2 = 0 \tag{2.9}$$

hold for every $z \in A_\theta$. Hence, it follows from the above relations (2.8) and (2.9) that

$$(\log \lambda_{A_\theta})_{zz} + \frac{(\log \lambda_{A_\theta})_z}{z} + \frac{\overline{z}}{z}(\lambda_{A_\theta})^2 = 0. \tag{2.10}$$

Let $A$ be an arbitrary angular domain only satisfying that its vertex is the origin of $\mathbb{C}$. Then there exists a rotation transformation $z = g(\zeta) = e^{i\theta_0}\zeta$, $\zeta \in A$ with $0 \le \theta_0 \le 2\pi$ such that $g$ conformally maps $A$ onto $A_\theta$. Hence,

$$\lambda_A(\zeta) = \lambda_{A_\theta}(g(\zeta)), \quad (\log \lambda_A(\zeta))_\zeta = e^{i\theta_0}(\log \lambda_\theta(z))_z, \quad (\log \lambda_A(\zeta))_{\zeta\zeta} = e^{2i\theta_0}(\log \lambda_\theta(z))_{zz}. \tag{2.11}$$

Thus by the relation (2.10) the following differential equation:

$$(\log \lambda_A)_{\zeta\zeta} + \frac{(\log \lambda_A)_\zeta}{\zeta} + \frac{\overline{\zeta}}{\zeta}(\lambda_A)^2 = 0 \tag{2.12}$$

holds for every $\zeta \in A$. □

## 3. Sharp Bounds for Hyperbolically Partial Derivatives

In order to study the hyperbolically *bi-lipschitz* continuity of a $1/|w|^2$-harmonic $K$-quasiconformal mapping, we will first derive the bounds, determined by the quasiconformal constant $K$, of its hyperbolically partial derivative.

To do so we need the well-known Ahlfors-Schwarz lemma [21] and its opposite type given by Mateljević [19] as follows.

**Lemma A.** *If $\rho > 0$ is a $C^2$ metric density on $D$ for which the gaussian curvature satisfies $K_\rho \geq -4$ and if $\rho(z)$ tends to $+\infty$ when $|z|$ tends to $1^-$, then $\lambda_D \leq \rho$.*

Kalaj [7] obtained the following.

**Lemma B.** *Let $\Omega$ be a convex domain in C. If $f$ is a euclidean harmonic $K$-quasiconformal mapping of the unit disk onto $\Omega$, satisfying $f(0) = a$, then*

$$\left|f_z\right| \geq \frac{1}{2(1+k)}\delta_\Omega, \quad z \in D, \tag{3.1}$$

*where $\delta_\Omega = d(a, \partial\Omega) = \inf\{|f - a| : f \in \partial\Omega\}$ and $k = (K-1)/(K+1)$.*

**Theorem 3.1.** *Let $A$ be an angular domain with the origin of the complex plane $\mathbb{C}$ as its vertex. If $f$ is a $1/|w|^2$-harmonic $K$-quasiconformal mapping of the unit disk $D$ onto $A$, then for every $z \in D$ its hyperbolically partial derivative satisfies the following inequality:*

$$\frac{K+1}{2K} \leq \|\partial f\| \leq \frac{K+1}{2}. \tag{3.2}$$

*Moreover, the upper and lower bound is sharp.*

*Proof.* Let $A$ be an angular domain with the origin of the complex plane $\mathbb{C}$ as its vertex and $f$ a $1/|w|^2$-harmonic $K$-quasiconformal mapping of $D$ onto $A$. Let $k = (K-1)/(K+1)$. From the assumptions we have that $f$ does not vanish on $D$. So $\log f$ is harmonic in $\Omega$. Hence, we have that $(\log f)_z$ does not vanish by Lewy Theorem [22]. So $f_z$ also does not vanish. Suppose that $\sigma(z) = (1-k)\lambda_A(f(z))|f_z|$, $z \in D$. Therefore $\sigma(z) > 0$ for every point $z \in D$. Thus we obtain

$$(\Delta \log \sigma)(z) = 4\left[(\log \lambda_A \circ f)_{z\overline{z}}(z) + \left(\log\left|f_z\right|\right)_{z\overline{z}}\right]. \tag{3.3}$$

By the chain rule [20] we get

$$4(\log \lambda_A \circ f)_{z\overline{z}}(z) = 4\Big\{ ((\log \lambda_A)_{w\overline{w}} \circ f)\left(|f_z|^2 + |f_{\overline{z}}|^2\right) \\ +2\Re\big[((\log \lambda_A)_{ww} \circ f) f_z f_{\overline{z}}\big] + 2\Re\big[(\log \lambda_A)_w \circ f f_{z\overline{z}}\big]\Big\}. \tag{3.4}$$

By Euler-Lagrange equation we have that a $1/|w|^2$-harmonic mapping $f$ satisfies

$$f_{z\overline{z}} - \frac{f_z f_{\overline{z}}}{f} = 0. \tag{3.5}$$

Since $f_z$ does not vanish, we have from (3.5) that

$$(\log|f_z|)_{z\overline{z}} = 0. \tag{3.6}$$

Using the relations (3.3), (3.4), (3.5), and (3.6) we have

$$(\Delta \log \sigma)(z) = 4\left\{ (\log \lambda_A)_{w\overline{w}} \circ f\left(|f_z|^2 + |f_{\overline{z}}|^2\right) + 2\Re\left[\left[(\log \lambda_A)_{ww} + \frac{(\log \lambda_A)_w}{w}\right] \circ f f_z f_{\overline{z}}\right]\right\}. \tag{3.7}$$

By the differential equation at Lemma 2.1 the above relation becomes

$$(\Delta \log \sigma)(z) = 4\left\{ (\log \lambda_A)_{w\overline{w}} \circ f\left(|f_z|^2 + |f_{\overline{z}}|^2\right) - 2\Re\left[(\lambda_A \circ f)^2 \frac{\overline{f} f_z f_{\overline{z}}}{f}\right]\right\}. \tag{3.8}$$

So we get

$$-\frac{\Delta \log \sigma}{\sigma^2} = \frac{-4}{(1-k)^2}\left[\frac{\Delta \log \lambda_A}{4(\lambda_A)^2} \circ f \frac{|f_z|^2 + |f_{\overline{z}}|^2}{|f_z|^2} - 2\Re \frac{\overline{f} f_{\overline{z}}}{f \overline{f_z}}\right]. \tag{3.9}$$

By (1.2) it is clear that $|f_{\overline{z}}/f_z| = |a|$. Hence, it follows from (3.9) and the inequality $|a| \leq k$ that

$$K_\sigma = -\frac{\Delta \log \sigma}{\sigma^2} \leq -\frac{4}{(1-k)^2}\left(1 + |a|^2 - 2|a|\right) = -4\frac{(1-|a|)^2}{(1-k)^2} \leq -4. \tag{3.10}$$

Thus by Ahlfors-Schwarz Lemma [21, P13] it follows that $\sigma \leq \lambda_D$, that is,

$$\|\partial f\| = \frac{\lambda_A \circ f}{\lambda_D}|f_z| \leq \frac{K+1}{2}. \tag{3.11}$$

Let $F = w|w|^{K-1}$, $w \in \mathbb{H}$. Then $F$ is a $1/|w|^2$-harmonic $K$-quasiconformal mapping of $\mathbb{H}$ onto itself. Moreover, we also have

$$\|\partial F\| = \frac{\lambda_H \circ F}{\lambda_H}|F_w| = \frac{K+1}{2}. \tag{3.12}$$

Choosing $L$ to be a conformal mapping of $D$ onto $\mathbb{H}$, we have that $F \circ L$ is $1/|w|^2$-harmonic $K$-quasiconformal mapping of $D$ onto $\mathbb{H}$. Thus by (1.5) the equality (3.12) becomes that

$$\|\partial(F \circ L)\| = \frac{K+1}{2}. \tag{3.13}$$

Therefore the upper bound at (3.2) is sharp.

Next we will prove the lower bound of $\|\partial f\|$. Suppose that $f$ is a $1/|w|^2$-harmonic $K$-quasiconformal mapping of $D$ onto $A$. Let $\delta = (1+k)\lambda_A(f)|f_z|$.

Hence, we have

$$(\Delta \log \delta)(z) = 4\left[(\log \lambda_A \circ f)_{z\overline{z}}(z) + \left(\log|f_z|\right)_{z\overline{z}}\right]. \tag{3.14}$$

Combining Lemma 2.1 with the relations (3.4), (3.5), (3.6), and (3.14) we have

$$-\frac{\Delta \log \delta}{\delta^2} = \frac{-4}{(1+k)^2}\left[\frac{\Delta \log \lambda_A}{4(\lambda_A)^2} \circ f \frac{|f_z|^2 + |f_{\overline{z}}|^2}{|f_z|^2} - 2\Re \frac{\overline{f} f_{\overline{z}}}{f \overline{f_z}}\right]. \tag{3.15}$$

Hence, it follows from the inequality $|a| \leq k$ and (3.15) that

$$K_\delta = -\frac{\Delta \log \delta}{\delta^2} \geq -\frac{4}{(1+k)^2}\left(1 + |a|^2 + 2|a|\right) = -4\frac{(1+|a|)^2}{(1+k)^2} \geq -4. \tag{3.16}$$

Since the mapping $\log w$ maps $A$ onto a strip domain $S$, we have that $\log f$ is an euclidean harmonic mapping of $D$ onto $S$. So it follows from Lemma B that $|(\log f)_z| \geq C_0$, where $C_0$ is a positive constant. Thus we have $\lambda_A(f)|f_z| = \lambda_S(\log f)|(\log f)_z| \to +\infty$ as $|z| \to 1^-$. Thus it follows from Lemma A that

$$\|\partial f\| = \frac{\lambda_A \circ f}{\lambda_D}|f_z| \geq \frac{K+1}{2K}. \tag{3.17}$$

Let $F = w|w|^{1/K-1}$, $w \in \mathbb{H}$. Then $F$ is a $1/|w|^2$-harmonic $K$-quasiconformal mapping of $\mathbb{H}$ onto itself. Moreover, we also have

$$\|\partial F\| = \frac{\lambda_H \circ F}{\lambda_H}|F_w| = \frac{K+1}{2K}. \tag{3.18}$$

Choosing $L$ to be a conformal mapping of $D$ onto $\mathbb{H}$, we have that $F \circ L$ is $1/|w|^2$-harmonic $K$-quasiconformal mapping of $D$ onto $\mathbb{H}$. Thus by (1.5) it shows that

$$\|\partial(F \circ L)\| = \frac{K+1}{2K}. \tag{3.19}$$

Therefore the positive lower bound at (3.2) is also sharp. □

## 4. Sharp Coefficients of Hyperbolically Lipschitz Continuity

As an application of Theorem 3.1, we have the following main result in this paper.

**Theorem 4.1.** *Let $A$ be an angular domain with the origin of the complex plane $\mathbb{C}$ as its vertex. If $f$ is a $1/|w|^2$-harmonic $K$-quasiconformal mapping of the unit disk $D$ onto $A$, then $f$ is hyperbolically $(1/K, K)$-bi-lipschitz. Moreover, both the coefficients $K$ and $1/K$ are sharp.*

*Proof.* Let $\gamma$ be the hyperbolic geodesic between $z_1$ and $z_2$, where $z_1$ and $z_2$ are two arbitrary points in $D$. Then it follows that

$$\int_{f(\gamma)} \lambda_A(w)|dw| \le \int_\gamma \lambda_A(f(z))L_f(z)|dz| \le \frac{2K}{K+1}\int_\gamma \frac{\lambda_A(f(z))\left|f_z(z)\right|}{\lambda_D(z)}\lambda_D(z)|dz|, \tag{4.1}$$

where $w = f(z)$. By the inequality of (3.2) and the definition of a hyperbolic geodesic, we obtain from the above inequality that

$$d_h(f(z_1), f(z_2)) \le \int_{f(\gamma)} \lambda_A(w)|dw| \le K\int_\gamma \lambda_D(z)|dz| = Kd_h(z_1, z_2). \tag{4.2}$$

Hence, $f$ is hyperbolically $K$-Lipschitz.

Let $F = w|w|^{K-1}$, $w \in \mathbb{H}$. Then $F$ is a $1/|w|^2$-harmonic $K$-quasiconformal mapping of $\mathbb{H}$ onto itself. Let $z_1 = i$ and $z_2 = iy$, $y > 1$ be two points in $\mathbb{H}$. Then $F(z_1) = i$ and $F(z_2) = iy^K$. Thus $d_h(z_1, z_2) = \log y$ and $d_h(F(z_1), F(z_2)) = K\log y$. So the equality

$$d_h(F(z_1), F(z_2)) = Kd_h(z_1, z_2) \tag{4.3}$$

holds. Choosing $L$ to be a conformal mapping of $D$ onto $\mathbb{H}$, we have that $\phi = F \circ L$ is $1/|w|^2$-harmonic $K$-quasiconformal mapping of $D$ onto $\mathbb{H}$. Let $\phi(\zeta_1) = z_1$ and $\phi(\zeta_2) = z_2$. Thus by the fact that the hyperbolic distance is a conformal invariant it follows from (1.5) that

$$d_h(\phi(\zeta_1), \phi(\zeta_2)) = Kd_h(L(\zeta_1), L(\zeta_2)) = Kd_h(\zeta_1, \zeta_2). \tag{4.4}$$

Thus the coefficient $K$ is sharp.

Let $f(\gamma) \subset A$ be the hyperbolic geodesic connected $f(z_1)$ with $f(z_2)$. By the assumption that $\lambda_A|f_z|$ tends to $+\infty$ as $|z| \to 1^-$, we have that the inequality (3.2) also holds. Hence, we also have

$$d_h(f(z_1), f(z_2)) = \int_{f(\gamma)} \lambda_A(w)|dw| \geq \frac{1}{K}\int_{\gamma} \lambda_D(z)|dz| \geq \frac{1}{K} d_h(z_1, z_2), \tag{4.5}$$

where $w = f(z)$. Thus $f$ is hyperbolically $(1/K, K)$-*bi-lipschitz*.

Let $G = w|w|^{1/K-1}$, $w \in \mathbb{H}$. Let $z_1 = i$, and $z_2 = iy$, $y > 1$ be two points in $\mathbb{H}$. Then $G(z_1) = i$ and $G(z_2) = iy^{1/K}$. Thus $d_h(z_1, z_2) = \log y$ and $d_h(G(z_1), G(z_2)) = (1/K)\log y$. So the equality

$$d_h(G(z_1), G(z_2)) = \frac{d_h(z_1, z_2)}{K} \tag{4.6}$$

holds. Choosing $L$ to be a conformal mapping of $D$ onto $\mathbb{H}$, we have that $\psi = G \circ L$ is $1/|w|^2$-harmonic $K$-quasiconformal mapping of $D$ onto $\mathbb{H}$. Let $\psi(\zeta_1) = z_1$ and $\psi(\zeta_2) = z_2$. Thus by the fact that the hyperbolic distance is a conformal invariant it shows that

$$d_h(\psi(\zeta_1), \psi(\zeta_2)) = \frac{d_h(L(\zeta_1), L(\zeta_2))}{K} = \frac{d_h(\zeta_1, \zeta_2)}{K}. \tag{4.7}$$

Thus the coefficient $1/K$ is also sharp. The proof of Theorem 4.1 is complete. □

## 5. Auxiliary Examples

*Example 5.1.* Suppose that $f = z|z|^{1/K-1}$, $K > 1$. Let $D^* = \{z \mid 0 < |z| < 1\}$ be the punctured unit disk and $D = \{z \mid |z| < 1\}$ the unit disk. Then $1/|w|^2$ is a smooth metric on $D^*$ but not smooth on $D$. We have that $f$ is a $1/|w|^2$-harmonic $K$-quasiconformal mapping of $D^*$ onto itself. If a $\rho$-harmonic mapping is not necessary to be smooth, then $f$ is also a $1/|w|^2$-harmonic $K$-quasiconformal mapping of $D$ onto itself. Moreover, it follows that

$$\begin{gathered}
\lim_{z\to 0}\left|\frac{\lambda_D(f(z))}{\lambda_D(z)} f_z\right| = \lim_{z\to 0}\frac{1-r^2}{1-r^{2/K}}\frac{(1/K+1)|z|^{1/K-1}}{2} = \infty, \\
\lim_{z\to 0}\frac{|f(z)-f(0)|}{|z-0|} = \lim_{z\to 0}|z|^{1/K-1} = \infty, \\
\lim_{z\to 0}\frac{d_h(f(0), f(z))}{d_h(0,z)} = \lim_{z\to 0}\frac{\log\left(\left(1+|z|^{1/K}\right)/\left(1-|z|^{1/K}\right)\right)}{\log((1+|z|)/(1-|z|))} = \lim_{z\to 0}|z|^{1/K-1} = \infty, \\
\lim_{z\to 0}\left|\frac{\lambda_{D^*}(f(z))}{\lambda_{D^*}(z)} f_z\right| = \lim_{z\to 0}\frac{|z|\log(1/|z|)}{|z|^{1/K}\log\left(1/|z|^{1/k}\right)}\frac{(1/K+1)|z|^{1/k-1}}{2} = \frac{K+1}{2}.
\end{gathered} \tag{5.1}$$

*Example 5.2.* Suppose that $f = z|z|^{K-1}, K > 1$. We have that $f$ is a $1/|w|^2$-harmonic $K$-quasiconformal mapping of $D^*$ onto itself. If a $\rho$-harmonic mapping is not necessary to be smooth, then $f$ is also a $1/|w|^2$-harmonic $K$-quasiconformal mapping of $D$ onto itself. Similar to Example 5.1, it follows that

$$\begin{gathered}
\lim_{z\to 0}\left|\frac{\lambda_D(f(z))}{\lambda_D(z)}f_z\right| = \lim_{z\to 0}\frac{K+1}{2}\frac{1-r^2}{1-r^{2K}}r^{K-1} = 0, \\
\lim_{z\to 0}\frac{|z-0|}{|f(z)-f(0)|} = \lim_{z\to 0}|z|^{1-K} = \infty, \\
\lim_{z\to 0}\frac{d_h(0,z)}{d_h(f(0),f(z))} = \lim_{z\to 0}\frac{\log((1+|z|)/(1-|z|))}{\log\left(\left(1+|z|^K\right)/\left(1-|z|^K\right)\right)} = \lim_{z\to 0}|z|^{1-K} = \infty, \\
\lim_{z\to 0}\left|\frac{\lambda_{D^*}(f(z))}{\lambda_{D^*}(z)}f_z\right| = \lim_{z\to 0}\frac{|z|\log(1/|z|)}{|z|^K\log\left(1/|z|^K\right)}\frac{K+1}{2}|z|^{K-1} = \frac{K+1}{2K}.
\end{gathered} \tag{5.2}$$

*Example 5.3.* Suppose that $f(z) = z|z|^{K-1}$, $K > 1$. Then $f$ is a $|\varphi|$-harmonic $K$-quasiconformal mapping of the upper half plane $H$ onto itself, here $\varphi(w) = 1/w^2$. Moreover,

$$\begin{gathered}
\lim_{|z|\to\infty}|f_z| = \lim_{|z|\to\infty}\frac{K+1}{2K}|z|^{K-1} = +\infty, \quad |(\log\varphi(w))_w| = \left|\frac{\varphi_w}{\varphi}\right| = \left|\frac{1}{w}\right| \longrightarrow \infty, \quad w \longrightarrow 0, \\
\lim_{z\to\infty}\frac{|f(z)|}{|z|} = \lim_{z\to\infty}|z|^{K-1} = \infty, \quad \|\partial f\| = \frac{K+1}{2}.
\end{gathered} \tag{5.3}$$

*Example 5.4.* Let $\Omega^* = \mathbb{C}\setminus\overline{D}\bigcup\{\infty\}$ and $K > 1$. Let $\varphi(w) = 1/w^2$, $w \in \Omega^*$. Then $f = z|z|^{1/K-1}$ is a $|\varphi|$-harmonic $K$-quasiconformal mapping of $\Omega^*$ onto itself and satisfies that

$$\begin{gathered}
|(\log\varphi(w)_w)| = \left|\frac{\varphi_w}{\varphi}\right| = \left|\frac{1}{w}\right| \le 1, \\
\lim_{z\to 0}\|\partial f\| = \lim_{z\to\infty}\frac{r^2-1}{r^{2/K-1}}\frac{(1/K+1)|z|^{1/K-1}}{2} = \infty, \\
\lim_{r\to\infty}\frac{\log((1+1/r^{1/K})/(1-1/r^{1/K}))}{\log((1+1/r)/(1-1/r))} = \infty.
\end{gathered} \tag{5.4}$$

*Example 5.5.* Let $\mathbb{U}^+$ be the right half plane. Let $\widetilde{\Omega} = \mathbb{U}^+\setminus[1,+\infty)$. Then $\widetilde{\Omega}$ is not an angular domain. The hyperbolic metric $\lambda_{\widetilde{\Omega}}(z)|dz|$ with gaussian curvature $-4$ is given by

$$\lambda_{\widetilde{\Omega}}(z)|dz| = \frac{1}{\sqrt{z^2+1}+\sqrt{\sqrt{z^2+1}}}\left|\frac{z}{\sqrt{z^2+1}}\right||dz|. \tag{5.5}$$

Let $f(z) = z|z|^{1/K-1}$, $z \in \widetilde{\Omega}$, where $K > 1$. Then $f$ is a $1/|w|^2$-harmonic $K$-quasiconformal mapping of $\widetilde{\Omega}$ onto itself. Moreover, we have

$$\lim_{z\to 0}\|\partial f\| = \lim_{z\to 0}\frac{K+1}{2K}\frac{\left|\sqrt{z^2+1}\right|}{\left|\sqrt{w^2+1}\right|}\frac{\sqrt{z^2+1}+\overline{\sqrt{z^2+1}}}{\sqrt{w^2+1}+\overline{\sqrt{w^2+1}}}|z|^{2(1/K-1)} = \infty, \tag{5.6}$$

where $w = f(z)$. Let $g(z) = z|z|^{K-1}$, $z \in \widetilde{\Omega}$, where $K > 1$. Then $g$ is a $1/|w|^2$-harmonic $K$-quasiconformal mapping of $\widetilde{\Omega}$ onto itself. Moreover, we have

$$\lim_{z\to 0}\|\partial g\| = \lim_{z\to 0}\frac{K+1}{2}\frac{\left|\sqrt{z^2+1}\right|}{\left|\sqrt{\xi^2+1}\right|}\frac{\sqrt{z^2+1}+\overline{\sqrt{z^2+1}}}{\sqrt{\xi^2+1}+\overline{\sqrt{\xi^2+1}}}|z|^{2(K-1)} = 0, \tag{5.7}$$

where $\xi = g(z)$.

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