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Research Article

Taylor's Expansion Revisited: A General Formula for the Remainder

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We give a new approach to Taylor's remainder formula, via a generalization of Cauchy's generalized mean value theorem, which allows us to include the well-known Schölomilch, Lebesgue, Cauchy, and the Euler classic types, as particular cases.

1. Introduction

Taylor's polynomial is a central tool in any elementary course in mathematical analysis. Nowadays, its importance is centred on its applications, for instance, to asymptotic analysis or to obtain satisfactory numerical or integral inequalities (see, e.g., [1–5]). The core of these results comes from manipulations on the explicit formula of the remainder, that is, the error estimation when considering the Taylor's polynomial expansion instead of the function.

In this paper, we provide a new explicit formula for the remainder that generalizes classic ones, namely, Schölomilch, Lebesgue, Cauchy, and Euler's remainders.

Inspired by the explicit expression for an arbitrary polynomial

$$x \longrightarrow p(x), \quad \forall x \in \mathbb{R}, \tag{1.1}$$

B. Taylor (1712) used to write

$$p(x) = \sum_{k=0}^{n} \alpha_k (x-a)^k, \tag{1.2}$$

where a is a real parameter and the coefficients α_k are given by their derivatives of k-order

$$\alpha_k := \frac{p^{(k)}(a)}{k!}, \quad k = 0, 1, \ldots, n. \tag{1.3}$$

Therefore, in a heuristic way, he introduced, three centuries ago, the expression

$$f(x) := \sum_{k=0}^{+\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \tag{1.4}$$

for an arbitrary function f. Later, but during the same century, A. L. Cauchy gave the name of *analytic* to a type of functions which stands for their series expansions. (It is well known that Cauchy worked to introduce the concept of convergence of series.)

An obvious problem is to calculate the formula for the remainder in an explicit form (not only to know that there exists a polynomial p such that

$$f(x) - p(x) = o(|x-a|^n) \quad \text{if } x \longrightarrow a, \tag{1.5}$$

that is, the function f has a contact of order greater that n in a neighborhood of a point a). It is also well known that this explicit expression provides an upper bound for the error when we consider p instead of f near to a.

After Taylor, authors such as Euler, Lagrange, Cauchy, or Scholömilch have considered functions satisfying

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R(x), \tag{1.6}$$

where $R(x)$ measures the error when it is not possible to represent f in an analytic form (e.g., when f has derivatives up to the n-order, but no further). If $a = 0$, then the above expression is called the McLaurin series of f.

In this paper, we are interested in Taylor's polynomials to obtain a new and more general explicit form for the remainder $R(x)$.

In Section 3.1, via slight modifications on Cauchy's general mean value theorem (CGMVT) for functions f with continuous derivatives of order n on $[a,b]$, we obtain a very general expression for each of the corresponding classic expressions for the remainder as particular cases (Section 3.2).

2. Notation and Preliminaries

Throughout the paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{N}$ the set of the positive integers, $[a,b]$ is a closed and bounded interval with endpoints a and b, $\mathcal{C}^n([a,b])$ denotes the class of all real functions with continuous derivative of n-order defined on $[a,b]$, and the extreme cases: $\mathcal{C}^0([a,b]) := \mathcal{C}([a,b])$ = the class of all continuous functions defined on $[a,b]$ and

$$\mathcal{C}^\infty([a,b]) := \{f \in \mathcal{C}^n([a,b]) : \forall n \in \mathbb{N}\}. \tag{2.1}$$

For given $f \in \mathcal{C}^n([a,b])$ and $x_0 \in]a,b[$, we denote by

$$p_{f,x_0,n}(x) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k, \quad \forall x \in [a,b] \tag{2.2}$$

for Taylor's polynomial of n-order centred at x_0 of the function f. For the sake of simplicity, we refer to it as p. A function f is said to be *analytic* at $x_0 \in]a,b[$ if there exists $\delta > 0$ such that

$$f(x) := \sum_{k=0}^{+\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k, \quad \forall x \in]x_0-\delta, x_0+\delta[. \tag{2.3}$$

3. A General Formula for the Remainder

3.1. A General Taylor's Theorem

The classic technique for obtaining Taylor's polynomial with a remainder that consists of applying a more general result than the CGMVT is widely known.

Proposition 3.1 (n-CGMVT). *Let $f, g \in \mathcal{C}^n([a,b])$ such that $f^{(n+1)}$ and $g^{(n+1)}$ exist and are continuous on the open interval $]a,b[$. Then, there exists $\xi \in]a,b[$ such that*

$$\left[f(b) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k\right] g^{(n+1)}(\xi) = f^{(n+1)}(\xi)\left[g(b) - \sum_{k=0}^{n} \frac{g^{(k)}(a)}{k!}(b-a)^k\right]. \tag{3.1}$$

Note that with this notation, the CGMVT is the corresponding 0-CGMVT. Now, a slight modification in the hypothesis of the above proposition allows a more general statement.

Lemma 3.2 (n-m-GMVCT). *Let $f \in \mathcal{C}^n([a,b])$ and $g \in \mathcal{C}^m([a,b])$. If $f^{(n+1)}, g^{(m+1)} \in \mathcal{C}(]a,b[)$, then there exists $\xi \in]a,b[$ such that*

$$\left[f(b) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k\right] g^{(m+1)}(\xi) = f^{(n+1)}(\xi)\left[g(b) - \sum_{k=0}^{m} \frac{g^{(k)}(a)}{k!}(b-a)^k\right]. \tag{3.2}$$

Proof. We consider auxiliary functions

$$\begin{aligned} F(x) &:= \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(b-x)^k, \\ G(x) &:= \sum_{k=0}^{m} \frac{g^{(k)}(x)}{k!}(b-x)^k, \quad \forall x \in [a,b]. \end{aligned} \tag{3.3}$$

They satisfy the CGMVT for functions in $C^1([a,b])$. Moreover, we have the following identities:

$$F(b) = f(b), \qquad G(b) = g(b),$$
$$F'(x) = \frac{f^{(n+1)}(x)}{n!}(b-x)^n, \qquad G'(x) = \frac{g^{(m+1)}(x)}{m!}(b-x)^m. \tag{3.4}$$

Therefore, the result immediately follows. □

Theorem 3.3 (of Taylor). *Let $f \in C^n([a,b])$ and $h \in C([a,b])$, having no zeros in $[a,b]$ (i.e., $h(x) \neq 0$ for all x in $[a,b]$). Suppose that there exists $f^{(n+1)} \in C(]a,b[)$. Then, there is $\xi \in]a,b[$ such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\xi)}{h(\xi)} \frac{(x-\xi)^n}{(x-\xi)^m} \frac{m!}{n!} \int_a^x \frac{(x-s)^m}{m!} h(s)ds. \qquad [*]$$

Proof. Put $g : [a,b] \to \mathbb{R}$ satisfying the following conditions:

$$g^{(m+1)}(x) := h(x) \quad \text{if } a \leq x \leq b,$$
$$g^{(k)}(a) = 0 \quad \text{if } 0 \leq k \leq m. \tag{3.5}$$

Now, using the n-m-CGMVT, we obtain $[*]$. □

3.2. *Particular Cases*

In this subsection, we show how the remainder formula $[*]$ can be reduced to each particular case. Firstly, if we define the function h with a constant real value, namely $\alpha \in \mathbb{R}$, then Schölomilch's version for the remainder follows (see [6]):

$$R(x) = f^{(n+1)}(\xi) \frac{(x-\xi)^n}{(x-\xi)^m} \frac{m!}{n!} \frac{(x-a)^{m+1}}{(m+1)!}. \qquad [S]$$

(This formula is usually obtained directly from the n-CGMVT doing $g(x) := (x-a)^{n+1}$, because $g^{(k)}(a) = 0$, $0 \leq k \leq n$, and $g^{(n+1)}(x) = (n+1)!$).

When $n = m$ in $[S]$, this formula gives the Lagrange remainder type (see [6–9]):

$$R(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}. \qquad [L]$$

Using newly $[S]$, if we do $m = 0$, the Cauchy formula appears for the remainder (see [10]):

$$R(x) = \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n(x-a). \qquad [C]$$

Now, we return to $[*]$, and putting $h(x) := f^{(n+1)}(x)$ and $n = m$, we obtain Euler's integral expression (see [11]):

$$\int_a^x \frac{(x-s)^n}{n!} f^{(n+1)}(s)ds. \qquad [E]$$

Of course, doing $a = 0$ in $[*]$, we have a general McLaurin type series:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n+1)}(\xi)}{h(\xi)} \frac{(x-\xi)^n}{(x-\xi)^m} \frac{m!}{n!} \int_0^x \frac{(x-s)^m}{m!} h(s)ds. \qquad [* - \text{McL}]$$

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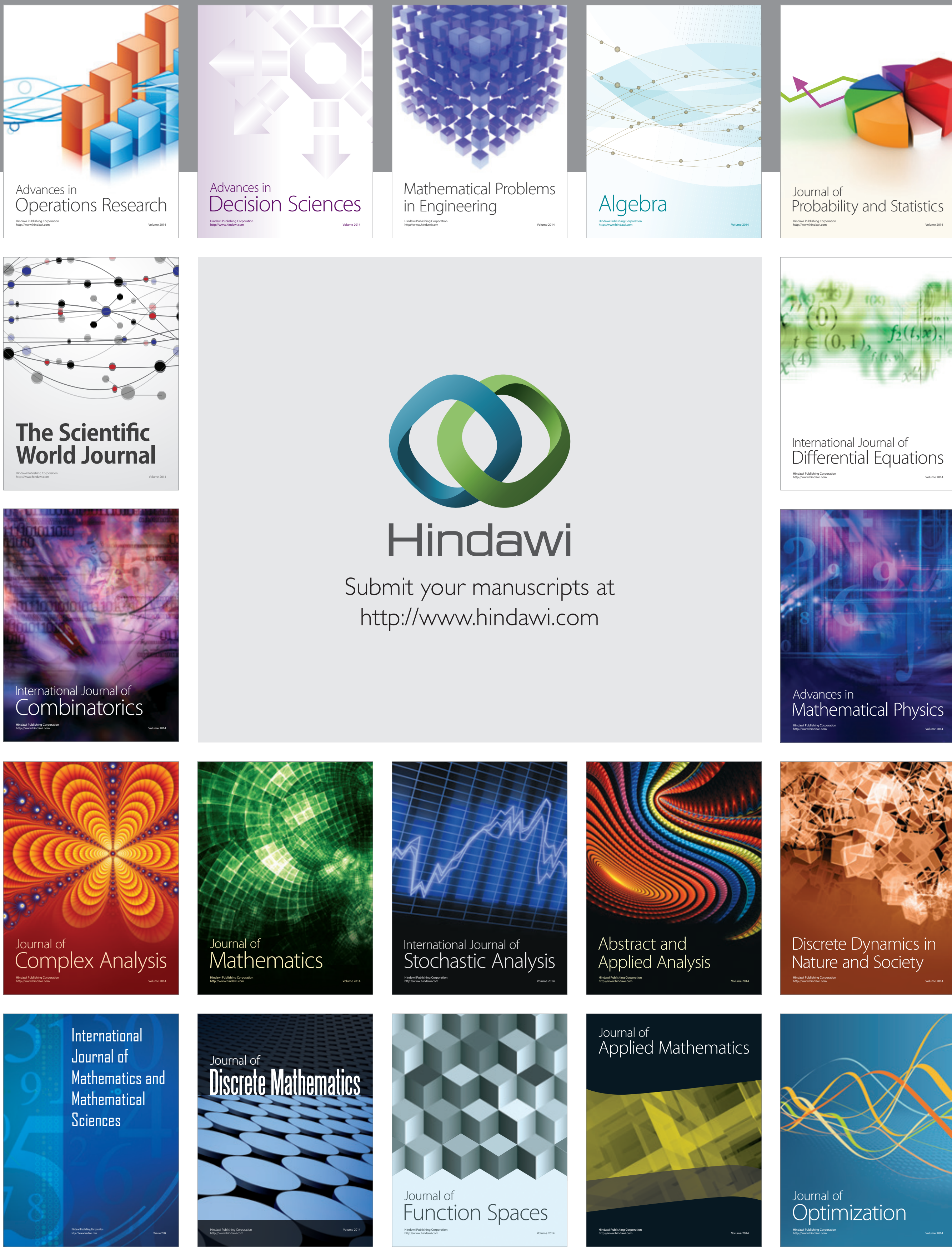
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