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Research Article

Generalizations of the Simpson-Like Type Inequalities for Co-Ordinated s-Convex Mappings in the Second Sense

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A generalized identity for some partial differentiable mappings on a bidimensional interval is obtained, and, by using this result, the author establishes generalizations of Simpson-like type inequalities for coordinated s-convex mappings in the second sense.

1. Introduction

In recent years, a number of authors have considered error estimate inequalities for some known and some new quadrature formulas. Sometimes they have considered generalizations of the Simpson-like type inequality which gives an error bound for the well-known Simpson rule.

Theorem 1.1. *Let $f : \mathbb{I} \subset [0,\infty) \to \mathbb{R}$ be a four-time continuous differentiable mapping on $[a,b]$ and $\|f^{(4)}\|_\infty = \sup_{x\in[a,b]}|f^{(4)}(x)| < \infty$. Then, the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^4}{2880}\left\|f^{(4)}\right\|_\infty. \tag{1.1}$$

It is well known that the mapping f is neither four times differentiable nor is the fourth derivative $f^{(4)}$ bounded on (a,b), then we cannot apply the classical Simpson quadrature formula.

For recent results on Simpson type inequalities, you may see the papers [1–5].

In [2, 6–8], Dragomir et al. and Park considered among others the class of mappings which are s-convex on the coordinates.

In the sequel, in this paper let $\Delta = [a,b] \times [c,d]$ be a bidimensional interval in $\mathbb{R}^2$ with $a < b$ and $c < d$.

Definition 1.2. A mapping $f : \Delta \to \mathbb{R}$ will be called s-convex in the second sense on Δ if the following inequality:

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda^s f(x,y) + (1-\lambda)^s f(z,w), \tag{1.2}$$

holds, for all $(x,y), (z,w) \in \Delta$, $\lambda \in [0,1]$ and $s \in [0,1]$.

Modification for convex and s-convex mapping on Δ, which are also known as co-ordinated convex, s-convex mapping, and s-r-convex, respectively, were introduced by Dragomir, Sarikaya [5, 9, 10], and Park [4, 8, 11, 12].

Definition 1.3. A mapping $f : \Delta \to \mathbb{R}$ will be called coordinated s-convex in the second sense on Δ if the partial mappings

$$\begin{aligned} f_y &: [a,b] \longrightarrow \mathbb{R}, \quad f_y(u) = f(u,y), \\ f_x &: [c,d] \longrightarrow \mathbb{R}, \quad f_x(v) = f(x,v), \end{aligned} \tag{1.3}$$

are s-convex in the second sense, for all $x \in [a,b]$, $y \in [c,d]$, and $s \in [0,1]$ [5, 9, 10].

A formal definition for coordinated s-convex mappings may be stated as follow [8].

Definition 1.4. A mapping $f : \Delta \to \mathbb{R}$ will be called coordinated s-convex in the second sense on Δ if the following inequality:

$$\begin{aligned} &f(tx + (1-t)z, \lambda y + (1-\lambda)w) \\ &\quad \leq t^s\lambda^s f(x,y) + (1-t)^s\lambda^s f(z,y) + t^s(1-\lambda)^s f(x,w) + (1-t)^s(1-\lambda)^s f(z,w), \end{aligned} \tag{1.4}$$

holds, for all $t,\ \lambda \in [0,1]$, $(x,y), (z,w) \in \Delta$, and $s \in [0,1]$.

In [2], S.S. Dragomir established the following theorem.

Theorem 1.5. *Let $f : \Delta \to \mathbb{R}$ be convex on the coordinates on Δ. Then, one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dy\,dx \\ &\leq \frac{1}{4}\left[f(a,c) + f(b,c) + f(a,d) + f(b,d)\right]. \end{aligned} \tag{1.5}$$

In [13], Hwang et al. gave a refinement of Hadamard's inequality on the coordinates and they proved some inequalities for coordinated convex mappings.

In [1, 6, 14], Alomari and Darus proved inequalities for coordinated s-convex mappings.

In [15], Latif and Alomari defined coordinated h-convex mappings, established some inequalities for co-ordinated h-convex mappings and proved inequalities involving product of convex mappings on the coordinates.

In [3], Özdemir et al. gave the following theorems:

Theorem 1.6. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on Δ. If $\partial^2 f/\partial t\partial\lambda$ is convex on the coordinates on Δ, then the following inequality holds:*

$$\begin{aligned}
&\Bigg|\frac{1}{9}\left[f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\
&\quad\left.+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right]+\frac{1}{36}\{f(a,c)+f(a,d)+f(b,c)+f(b,d)\}\\
&\quad+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx-A\Bigg|\\
&\quad\le\left(\frac{5}{72}\right)^2(b-a)(d-c)\\
&\quad\times\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right\},
\end{aligned}\tag{1.6}$$

where

$$\begin{aligned}
A&=\frac{1}{b-a}\int_a^b\left\{\frac{f(x,c)+4f(x,(c+d)/2)+f(x,d)}{6}\right\}dx\\
&\quad+\frac{1}{d-c}\int_c^d\left\{\frac{f(a,y)+4f((a+b)/2,y)+f(b,y)}{6}\right\}dy.
\end{aligned}\tag{1.7}$$

Theorem 1.7. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on Δ. If $\partial^2 f/\partial t\partial\lambda$ is bounded, that is,*

$$\begin{aligned}
&\left\|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right\|_\infty\\
&\quad=\sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|<\infty
\end{aligned}\tag{1.8}$$

for all $(t,\lambda)\in[0,1]^2$, then the following inequality holds:

$$\begin{aligned}
&\Bigg|\frac{1}{9}\left[f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right]\\
&\quad+\frac{1}{36}\{f(a,c)+f(a,d)+f(b,c)+f(b,d)\}
\end{aligned}$$

$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx - A\Bigg|$$
$$\leq \left(\frac{5}{72}\right)^2(b-a)(d-c)\left\|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right\|_\infty, \tag{1.9}$$

where A is defined in Theorem 1.6.

In [3], Özdemir et al. proved a new equality and, by using this equality, established some inequalities on coordinated convex mappings.

In this paper the author give a generalized identity for some partial differentiable mappings on a bidimensional interval and, by using this result, establish a generalizations of Simpson-like type inequalities for coordinated s-convex mappings in the second sense.

2. Main Results

To prove our main results, we need the following lemma.

Lemma 2.1. *Let $f : \Delta \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a,b]\times[c,d] \subset \mathbb{R}^2$. If $(\partial^2 f/\partial t\partial\lambda) \in L_1(\Delta)$, then, for $r_1, r_2 \geq 2$ and $h_1, h_2 \in (0,1)$ with $(1/r_1) \leq h_1 \leq (r_1 - 1/r_1)$ and $(1/r_2) \leq h_2 \leq ((r_2-1)/r_2)$, the following equality holds:*

$$\begin{aligned}
I(f)(h_1,h_2,r_1,r_2) \overset{\text{let}}{=} & \left\{\frac{(r_1-2)(r_2-2)}{r_1r_2}\right\} f(h_1a+(1-h_1)b, h_2c+(1-h_2)d)\\
&+\left\{\frac{(r_1-2)}{r_1r_2}\right\}\{f(h_1a+(1-h_1)b,c)+f(h_1a+(1-h_1)b,d)\}\\
&+\left\{\frac{(r_2-2)}{r_1r_2}\right\}\{f(a,h_2c+(1-h_2)d)+f(b,h_2c+(1-h_2)d)\}\\
&+\frac{1}{r_1r_2}\{f(a,c)+f(a,d)+f(b,c)+f(b,d)\}\\
&-\frac{1}{r_2(b-a)}\int_a^b\{f(x,c)+(r_2-2)f(x,h_2c+(1-h_2)d)+f(x,d)\}dx\\
&-\frac{1}{r_1(d-c)}\int_c^d\{f(a,y)+(r_1-2)f(h_1a+(1-h_1)b,y)+f(b,y)\}dy\\
&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\\
=\ & (b-a)(d-c)\iint_0^1 p(h_1,r_1,t)q(h_2,r_2,\lambda)\\
&\times\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)dtd\lambda,
\end{aligned}\tag{2.1}$$

where

$$p(h_1,r_1,t)=\begin{cases} t-\dfrac{1}{r_1}, & t\in[0,h_1],\\ t-\dfrac{r_1-1}{r_1}, & t\in(h_1,1],\end{cases}$$
$$q(h_2,r_2,\lambda)=\begin{cases} \lambda-\dfrac{1}{r_2}, & \lambda\in[0,h_2],\\ \lambda-\dfrac{r_2-1}{r_2}, & \lambda\in(h_2,1].\end{cases} \tag{2.2}$$

Proof. By the definitions of $p(h_1,r_1,t)$ and $q(h_2,r_2,\lambda)$, we can write

$$\begin{aligned} I &= \iint_0^1 p(h_1,r_1,t)q(h_2,r_2,\lambda)\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)dtd\lambda \\ &= \int_0^1 q(h_2,r_2,\lambda)\left[\int_0^1 p(h_1,r_1,t)\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)dt\right]d\lambda \\ &= \int_0^1 q(h_2,r_2,\lambda)\left[\int_0^{h_1}\left(t-\frac{1}{r_1}\right)\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)dt\right. \\ &\qquad\left.+\int_{h_1}^1\left(t-\frac{r_1-1}{r_1}\right)\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)dt\right]d\lambda \\ &= \int_0^1 q(h_2,r_2,\lambda)[I_{11}+I_{12}]d\lambda, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} I_{11} &= \int_0^{h_1}\left(t-\frac{1}{r_1}\right)\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)dt, \\ I_{12} &= \int_{h_1}^1\left(t-\frac{r_1-1}{r_1}\right)\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)dt. \end{aligned} \tag{2.4}$$

By integration by parts, we have

$$\begin{aligned} I_{11} &= \int_0^{h_1}\left(t-\frac{1}{r_1}\right)\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)dt \\ &= \frac{1}{a-b}\left[\left(h_1-\frac{1}{r_1}\right)\frac{\partial f}{\partial\lambda}(h_1a+(1-h_1)b,\lambda c+(1-\lambda)d)\right. \\ &\qquad+\frac{1}{r_1}\frac{\partial f}{\partial\lambda}(b,\lambda c+(1-\lambda)d) \\ &\qquad\left.-\int_0^{h_1}\frac{\partial f}{\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)dt\right], \end{aligned} \tag{2.5}$$

$$\begin{aligned}
I_{12} &= \int_{h_1}^{1}\left(t-\frac{r_1-1}{r_1}\right)\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)dt \\
&= \frac{1}{a-b}\left[\left(\frac{1}{r_1}\right)\frac{\partial f}{\partial\lambda}(a,\lambda c+(1-\lambda)d)\right. \\
&\qquad -\left(h_1-\frac{r_1-1}{r_1}\right)\frac{\partial f}{\partial\lambda}(h_1a+(1-h_1)b,\lambda c+(1-\lambda)d) \\
&\qquad \left.-\int_{h_1}^{1}\frac{\partial f}{\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)dt\right].
\end{aligned} \tag{2.6}$$

By using the equalities (2.5) and (2.6) in (2.3), we have

$$\begin{aligned}
I &= \left(\frac{1}{a-b}\right)\left\{\left(\frac{r_1-2}{r_1}\right)\int_0^1 q(h_2,r_2,\lambda)\frac{\partial f}{\partial\lambda}(h_1a+(1-h_1)b,\lambda c+(1-\lambda)d)d\lambda\right. \\
&\qquad +\left(\frac{1}{r_1}\right)\int_0^1 q(h_2,r_2,\lambda)\frac{\partial f}{\partial\lambda}(b,\lambda c+(1-\lambda)d)d\lambda \\
&\qquad +\left(\frac{1}{r_1}\right)\int_0^1 q(h_2,r_2,\lambda)\frac{\partial f}{\partial\lambda}(a,\lambda c+(1-\lambda)d)d\lambda \\
&\qquad \left.-\int_0^1 q(h_2,r_2,\lambda)\int_0^1\frac{\partial f}{\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)d\lambda dt\right\} \\
&= \left(\frac{1}{a-b}\right)\left\{\left(\frac{r_1-2}{r_1}\right)I_{21}+\left(\frac{1}{r_1}\right)I_{22}+\left(\frac{1}{r_1}\right)I_{23}-I_{24}\right\},
\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}
I_{21} &= \int_0^1 q(h_2,r_2,\lambda)\frac{\partial f}{\partial\lambda}(h_1a+(1-h_1)b,\lambda c+(1-\lambda)d)d\lambda, \\
I_{22} &= \int_0^1 q(h_2,r_2,\lambda)\frac{\partial f}{\partial\lambda}(b,\lambda c+(1-\lambda)d)d\lambda, \\
I_{23} &= \int_0^1 q(h_2,r_2,\lambda)\frac{\partial f}{\partial\lambda}(a,\lambda c+(1-\lambda)d)d\lambda, \\
I_{24} &= \int_0^1 q(h_2,r_2,\lambda)\int_0^1\frac{\partial f}{\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)d\lambda dt.
\end{aligned} \tag{2.8}$$

Note that

$$\begin{aligned}
&\text{(i)}\int_0^{h_2}\left(\lambda-\frac{1}{r_2}\right)\frac{\partial f}{\partial\lambda}(h_1a+(1-h_1)b,\lambda c+(1-\lambda)d)d\lambda \\
&\quad= \frac{1}{c-d}\left\{\left(h_2-\frac{1}{r_2}\right)f(h_1a+(1-h_1)b,h_2c+(1-h_2)d)\right. \\
&\qquad +\frac{1}{r_2}f(h_1\text{a}+(1-h_1)b,d) \\
&\qquad \left.-\int_0^{h_2}f(h_1a+(1-h_1)b,\lambda c+(1-\lambda)d)d\lambda\right\},
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
\text{(ii)} \int_{h_2}^{1} \left(\lambda - \frac{r_2 - 1}{r_2}\right) \frac{\partial f}{\partial \lambda}(h_1 a + (1 - h_1)b, \lambda c + (1 - \lambda)d)\, d\lambda
&= \frac{1}{c - d}\Bigg\{ \left(\frac{1}{r_2}\right) f(h_1 a + (1 - h_1)b, c) \\
&\quad - \left(h_2 - \frac{r_2 - 1}{r_2}\right) f(h_1 a + (1 - h_1)b, h_2 c + (1 - h_2)d) \\
&\quad - \int_{h_2}^{1} f(h_1 a + (1 - h_1)b, \lambda c + (1 - \lambda)d)\, d\lambda \Bigg\}.
\end{aligned} \tag{2.10}$$

By the equalities (2.9) and (2.10), we have

$$\begin{aligned}
I_{21} &= \int_0^1 q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda}(h_1 a + (1 - h_1)b, \lambda c + (1 - \lambda)d)\, d\lambda \\
&= \int_0^{h_2} \left(\lambda - \frac{1}{r_2}\right) \frac{\partial f}{\partial \lambda}(h_1 a + (1 - h_1)b, \lambda c + (1 - \lambda)d)\, d\lambda \\
&\quad + \int_{h_2}^{1} \left(\lambda - \frac{r_2 - 1}{r_2}\right) \frac{\partial f}{\partial \lambda}(h_1 a + (1 - h_1)b, \lambda c + (1 - \lambda)d)\, d\lambda \\
&= \frac{1}{c - d}\Bigg\{ \frac{1}{r_2} f(h_1 a + (1 - h_1)b, c) + \frac{1}{r_2} f(h_1 a + (1 - h_1)b, d) \\
&\quad + \frac{r_2 - 2}{r_2} f(h_1 a + (1 - h_1)b, h_2 c + (1 - h_2)d) \\
&\quad - \int_0^1 f(h_1 a + (1 - h_1)b, \lambda c + (1 - \lambda)d)\, d\lambda \Bigg\}.
\end{aligned} \tag{2.11}$$

By the similar way, we get the following:

$$\begin{aligned}
I_{22} &= \int_0^1 q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda}(b, \lambda c + (1 - \lambda)d)\, d\lambda \\
&= \frac{1}{c - d}\Bigg\{ \frac{1}{r_2} f(b, c) + \frac{1}{r_2} f(b, d) + \left(\frac{r_2 - 2}{r_2}\right) f(b, h_2 c + (1 - h_2)d) \\
&\quad - \int_0^1 f(b, \lambda c + (1 - \lambda)d)\, d\lambda \Bigg\},
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
I_{23} &= \int_0^1 q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda}(a, \lambda c + (1 - \lambda)d)\, d\lambda \\
&= \frac{1}{c - d}\Bigg\{ \frac{1}{r_2} f(a, c) + \frac{1}{r_2} f(a, d) + \left(\frac{r_2 - 2}{r_2}\right) f(a, h_2 c + (1 - h_2)d) \\
&\quad - \int_0^1 f(a, \lambda c + (1 - \lambda)d)\, d\lambda \Bigg\},
\end{aligned} \tag{2.13}$$

$$
\begin{aligned}
I_{24} &= \int_0^1 q(h_2, r_2, \lambda) \int_0^1 \frac{\partial f}{\partial \lambda}(ta + (1-t)b, \lambda c + (1-\lambda)d)\, d\lambda\, dt \\
&= \frac{1}{c-d}\left\{ \frac{1}{r_2}\int_0^1 f(ta + (1-t)b, c)dt + \frac{1}{r_2}\int_0^1 f(ta + (1-t)b, d)dt \right. \\
&\qquad + \left(\frac{r_2 - 2}{r_2}\right)\int_0^1 f(ta + (1-t)b, h_2 c + (1-h_2)d)\mathrm{d}t \\
&\qquad \left. - \iint_0^1 f(ta + (1-t)b, \lambda c + (1-\lambda)d)\, d\lambda\, dt \right\}.
\end{aligned}
\tag{2.14}
$$

By the equalities (2.7) and (2.11)–(2.14) and using the change of the variables $x = ta + (1-t)b$ and $y = \lambda c + (1-\lambda)d$ for $(t, \lambda) \in [0,1]^2$, then multiplying both sides with $(b-a)(d-c)$, we have the required result (2.1), which completes the proof. □

Remark 2.2. Lemma 2.1 is a generalization of the results which proved by Sarikaya, Set, Özdemir, and Dragomir [3, 5, 9, 10].

Theorem 2.3. *Let $f : \Delta \to \mathbb{R}^2$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d] \subset \mathbb{R}^2$. If $\partial^2 f / \partial t \partial \lambda$ is in $L_1(\Delta)$ and is a coordinated s-convex mapping in the second sense on Δ, then, for $r_1, r_2 \geq 2$ and $h_1, h_2 \in (0,1)$ with $(1/r_1) \leq h_1 \leq ((r_1 - 1)/r_1)$, and $(1/r_2) \leq h_2 \leq ((r_2 - 1)/r_2)$ the following inequality holds:*

$$
\begin{aligned}
&\frac{1}{(b-a)(d-c)}\left| I(f)(h_1, h_2, r_1, r_2) \right| \\
&\quad \leq \mu_1(r,s)\left\{ \mu_2(r,s)\left| \frac{\partial^2 f}{\partial t \partial \lambda}(a,c) \right| + \nu_2(r,s)\left| \frac{\partial^2 f}{\partial t \partial \lambda}(a,d) \right| \right\} \\
&\qquad + \nu_1(r,s)\left\{ \mu_2(r,s)\left| \frac{\partial^2 f}{\partial t \partial \lambda}(b,c) \right| + \nu_2(r,s)\left| \frac{\partial^2 f}{\partial t \partial \lambda}(b,d) \right| \right\},
\end{aligned}
\tag{2.15}
$$

where

$$
\begin{aligned}
\mu_1(r,s) &= M(r_1, s) + N(h_1, s), \\
\mu_2(r,s) &= M(r_2, s) + N(h_2, s), \\
\nu_1(r,s) &= M(r_1, s) + N(1 - h_1, s), \\
\nu_2(r,s) &= M(r_2, s) - N(1 - h_2, s)
\end{aligned}
\tag{2.16}
$$

for

$$
\begin{aligned}
M(r,s) &= \frac{2 + 2(r-1)^{s+2} + r^{s+1}(s - r + 2)}{(s+1)(s+2)r^{s+2}}, \\
N(h,s) &= \frac{h^{s+1}((2h-1)s + 2(h-1))}{(s+1)(s+2)}.
\end{aligned}
\tag{2.17}
$$

Proof. From Lemma 2.1 and by the coordinated s-convexity in the second sense of $\partial^2 f/\partial t\partial\lambda$, we can write

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\left|I(f)(h_1,h_2,r_1,r_2)\right| \\
&\quad\le \iint_0^1 \left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right| \\
&\qquad\times\left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right| dt d\lambda \\
&\quad\le \iint_0^1 \left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right| \\
&\qquad\times\left\{t^s\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right| + t^s(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|\right. \\
&\qquad\quad\left. +(1-t)^s\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right| + (1-t)^s(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right\} dt d\lambda \\
&\quad= \left\{\int_0^1 \left|p(h_1,r_1,t)\right| t^s dt\right\}\left\{\int_0^1 \left|q(h_2,r_2,\lambda)\right|\lambda^s d\lambda\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|\right. \\
&\qquad\qquad\left. + \int_0^1 \left|q(h_2,r_2,\lambda)\right|(1-\lambda)^s d\lambda\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|\right\} \\
&\qquad+ \left\{\int_0^1 \left|p(h_1,r_1,t)\right|(1-t)^s dt\right\}\left\{\int_0^1 \left|q(h_2,r_2,\lambda)\right|\lambda^s d\lambda\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|\right. \\
&\qquad\qquad\left. + \int_0^1 \left|q(h_2,r_2,\lambda)\right|(1-\lambda)^s d\lambda\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right\}.
\end{aligned} \tag{2.18}$$

Note that

$$\begin{aligned}
&\text{(i)} \int_0^1 \left|p(h_1,r_1,t)\right| t^s dt = \mu_1(r,s), \\
&\text{(ii)} \int_0^1 \left|p(h_1,r_1,t)\right|(1-t)^s dt = \nu_1(r,s), \\
&\text{(iii)} \int_0^1 \left|q(h_2,r_2,\lambda)\right|\lambda^s d\lambda = \mu_2(r,s), \\
&\text{(iv)} \int_0^1 \left|q(h_2,r_2,\lambda)\right|(1-\lambda)^s d\lambda = \nu_2(r,s).
\end{aligned} \tag{2.19}$$

By (2.18) and (2.19), we get the inequality (2.15) by the simple calculations. □

Remark 2.4. In Theorem 2.3,

(i) if we choose $h_1 = h_2 = 1/2$, $r_1 = r_2 = 6$, and $s = 1$ in (2.15), then we get

$$\left|I(f)\left(\frac{1}{2},\frac{1}{2},6,6\right)\right| \leq \left(\frac{5}{72}\right)^2 M(b-a)(d-c), \tag{2.20}$$

(ii) if we choose $h_1 = h_2 = 1/2$, $r_1 = r_2 = 2$, and $s = 1$ in (2.15), then we get

$$\left|I(f)\left(\frac{1}{2},\frac{1}{2},2,2\right)\right| \leq \left(\frac{1}{8}\right)^2 M(b-a)(d-c), \tag{2.21}$$

where

$$M = \left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right| + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right| + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right| + \left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|, \tag{2.22}$$

which implies that Theorem 2.3 is a generalization of Theorem 1.6.

Theorem 2.5. *Let $f : \Delta \to \mathbb{R}^2$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d] \subset \mathbb{R}^2$. If $\partial^2 f/\partial t\partial\lambda$ is bounded, that is,*

$$\begin{aligned}\left\|\frac{\partial^2 f}{\partial t\partial\lambda}\right\|_\infty &\overset{\text{let}}{=} \left\|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right\|_\infty \\ &= \sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right| < \infty\end{aligned} \tag{2.23}$$

for all $(t,\lambda) \in [0,1]^2$, then, for $r_1, r_2 \geq 2$ and $h_1, h_2 \in (0,1)$ with $(1/r_1) \leq h_1 \leq ((r_1-1)/r_1)$ and $(1/r_2) \leq h_2 \leq ((r_2-1)/r_2)$, the following inequality holds:

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\left|I(f)(h_1,h_2,r_1,r_2)\right| \\ &\quad\leq \left\{\frac{1}{2} - h_1 + h_1^2 + \frac{(2-r_1)}{r_1^2}\right\}\left\{\frac{1}{2} - h_2 + h_2^2 + \frac{(2-r_2)}{r_2^2}\right\}\left\|\frac{\partial^2 f}{\partial t\partial\lambda}\right\|_\infty.\end{aligned} \tag{2.24}$$

Proof. From Lemma 2.1, using the property of modulus and the boundedness of $\partial^2 f/\partial t\partial s$, we get

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\left|I(f)(h_1,h_2,r_1,r_2)\right| \\ &\quad\leq \left\|\frac{\partial^2 f}{\partial t\partial\lambda}\right\|_\infty \iint_0^1 \left|p(h_1,r_1,t)q(h_2,r_2,\lambda))\right| dt\,d\lambda.\end{aligned} \tag{2.25}$$

By the simple calculations, we have

$$\text{(i)} \int_0^1 |p(h_1, r_1, t)| dt = \frac{1}{2} - h_1 + h_1^2 + \frac{(2 - r_1)}{r_1^2}, \tag{2.26}$$

$$\text{(ii)} \int_0^1 |q(h_2, r_2, \lambda)| dt = \frac{1}{2} - h_2 + h_2^2 + \frac{(2 - r_2)}{r_2^2}. \tag{2.27}$$

By using the inequality (2.25) and the equalities (2.26)-(2.27), the assertion (2.24) holds. □

Remark 2.6. In Theorem 2.5,

(i) if we choose $h_1 = h_2 = \dfrac{1}{2}$ and $r_1 = r_2 = 6$, then we get

$$\left| I(f)\left(\frac{1}{2}, \frac{1}{2}, 6, 6\right) \right| \leq \left(\frac{5}{36}\right)^2 \left\| \frac{\partial^2 f}{\partial t \partial \lambda} \right\|_\infty (b - a)(d - c), \tag{2.28}$$

(ii) if we choose $h_1 = h_2 = 1/2$ and $r_1 = r_2 = 2$, then we get

$$\left| I(f)\left(\frac{1}{2}, \frac{1}{2}, 2, 2\right) \right| \leq \left(\frac{1}{4}\right)^2 \left\| \frac{\partial^2 f}{\partial t \partial \lambda} \right\|_\infty (b - a)(d - c), \tag{2.29}$$

which implies that Theorem 2.5 is a generalization of Theorem 1.7.

The following theorem is a generalization of Theorem 1.6.

Theorem 2.7. *Let $f : \Delta = [a, b] \times [c, d] \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a partial differentiable mapping on $\Delta = [a, b] \times [c, d]$. If $|\partial^2 f/\partial t \partial \lambda|^q (q > 1)$ is in $L_1(\Delta)$ and is a coordinated s-convex mapping in the second sense on Δ, then, for $r_1, r_2 \geq 2$ and $h_1, h_2 \in (0, 1)$ with $(1/r_1) \leq h_1 \leq ((r_1 - 1)/r_1)$ and $(1/r_2) \leq h_2 \leq ((r_2 - 1)/r_2)$, the following inequality holds:*

$$\begin{aligned} &\frac{1}{(b - a)(d - c)} |I(f)(h_1, h_2, r_1, r_2)| \leq \mu_3^{1/p} \nu_3^{1/p} \\ &\quad \times \left\{ \frac{|\partial^2 f/\partial t \partial \lambda(a, c)|^q + |\partial^2 f/\partial t \partial \lambda(a, d)|^q + |\partial^2 f/\partial t \partial \lambda(b, c)|^q + |\partial^2 f/\partial t \partial \lambda(b, d)|^q}{(s + 1)^2} \right\}^{1/p}, \end{aligned} \tag{2.30}$$

where

$$\mu_3 = \frac{2 + (r_1 - r_1 h_1 - 1)^{p+1} + (r_1 h_1 - 1)^{p+1}}{r_1^{p+1}(p+1)},$$
$$\nu_3 = \frac{2 + (r_2 - r_2 h_2 - 1)^{p+1} + (r_2 h_2 - 1)^{p+1}}{r_2^{p+1}(p+1)}. \tag{2.31}$$

Proof. From Lemma 2.1, we can write

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\left|I(f)(h_1,h_2,r_1,r_2)\right| \\
&\quad\le \iint_0^1 \left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right| \\
&\qquad\times \left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right| dt d\lambda \\
&\quad\le \left\{\iint_0^1 \left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right|^p dt d\lambda\right\}^{1/p} \\
&\qquad\times \left\{\iint_0^1 \left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dt d\lambda\right\}^{1/q}.
\end{aligned} \tag{2.32}$$

Hence, by the inequality (2.32) and the coordinated s-convexity in the second sense of $|\partial^2 f/\partial t\partial\lambda|^q$, it follows that

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\left|I(f)(h_1,h_2,r_1,r_2)\right| \\
&\quad\le \left\{\iint_0^1 \left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right|^p dt d\lambda\right\}^{1/p} \\
&\qquad\times \left\{\frac{|\partial^2 f/\partial t\partial\lambda(a,c)|^q + |\partial^2 f/\partial t\partial\lambda(a,d)|^q + |\partial^2 f/\partial t\partial\lambda(b,c)|^q + |\partial^2 f/\partial t\partial\lambda(b,d)|^q}{(s+1)^2}\right\}^{1/q}.
\end{aligned} \tag{2.33}$$

Note that

$$\text{(i)} \int_0^1 \left|p(h_1,r_1,t)\right|^p dt = \frac{2 + (r_1 - r_1 h_1 - 1)^{p+1} + (r_1 h_1 - 1)^{p+1}}{r_1^{p+1}(p+1)}, \tag{2.34}$$

$$\text{(ii)} \int_0^1 \left|q(h_2,r_2,\lambda)\right|^p d\lambda = \frac{2 + (r_2 - r_2 h_2 - 1)^{p+1} + (r_2 h_2 - 1)^{p+1}}{r_2^{p+1}(p+1)}. \tag{2.35}$$

By the inequality (2.33) and the equalities (2.34) and (2.35), the assertion (2.30) holds. □

Remark 2.8. In Theorem 2.7,

(i) if we choose $h_1 = h_2 = 1/2$, $r_1 = r_2 = 6$, and $s = 1$, then we get

$$\left|I(f)\left(\frac{1}{2},\frac{1}{2},6,6\right)\right| \leq \left\{\frac{2\left(1+2^{p+1}\right)}{6^{p+1}(p+1)}\right\}^{2/p}(b-a)(d-c)M_q^{1/q}, \tag{2.36}$$

(ii) if we choose $h_1 = h_2 = 1/2$, $r_1 = r_2 = 2$, and $s = 1$, then we get

$$\left|I(f)\left(\frac{1}{2},\frac{1}{2},2,2\right)\right| \leq \left\{\frac{1}{2^p(p+1)}\right\}^{2/p}(b-a)(d-c)M_q^{1/q}, \tag{2.37}$$

(iii) if we choose $h_1 = h_2 = 1/2$, $r_1 = r_2 = 6$, $s = 1$, and $q = 1$, then we get

$$\left|I(f)\left(\frac{1}{2},\frac{1}{2},6,6\right)\right| \leq \left(\frac{5}{36}\right)^2(b-a)(d-c)M_1, \tag{2.38}$$

where

$$M_q = \frac{\left|\partial^2 f/\partial t\partial\lambda(a,c)\right|^q + \left|\partial^2 f/\partial t\partial\lambda(a,d)\right|^q + \left|\partial^2 f/\partial t\partial\lambda(b,c)\right|^q + \left|\partial^2 f/\partial t\partial\lambda(b,d)\right|^q}{4}. \tag{2.39}$$

Theorem 2.9. *Let $f : \Delta \to \mathbb{R}^2$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d] \subset \mathbb{R}^2$. If $|\partial^2 f/\partial t\partial\lambda|^q (q \geq 1)$ is in $L_1(\Delta)$ and is a coordinated s-convex mapping in the second sense on Δ, then, for $r_1, r_2 \geq 2$ and $h_1, h_2 \in (0,1)$ with $(1/r_1) \leq h_1 \leq ((r_1-1)/r_1)$ and $(1/r_2) \leq h_2 \leq ((r_2-1)/r_2)$, the following inequality holds:*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\left|I(f)(h_1,h_2,r_1,r_2)\right| \\
&\quad\leq \left\{\left(\frac{1}{2}-h_1+h_1^2+\frac{(2-r_1)}{r_1^2}\right)\left(\frac{1}{2}-h_2+h_2^2+\frac{(2-r_2)}{r_2^2}\right)\right\}^{1-(1/q)} \\
&\qquad\times\left\{\mu_1\left(\mu_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\nu_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q\right)\right. \\
&\qquad\qquad\left.+\nu_1\left(\mu_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+\nu_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)\right\}^{1/q},
\end{aligned} \tag{2.40}$$

where μ_i and $\nu_i (i = 1,2)$ are as given in Theorem 2.3.

Proof. From Lemma 2.1, we can write

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\left|I(f)(h_1,h_2,r_1,r_2)\right| \\
&\quad\leq \iint_0^1 \left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right| \\
&\qquad\times\left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right| dt d\lambda \\
&\quad\leq \left\{\iint_0^1 \left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right| dt d\lambda\right\}^{1-(1/q)} \\
&\qquad\times\left\{\iint_0^1 \left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right|\right. \\
&\qquad\qquad\left.\times\left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dt d\lambda\right\}^{1/q}.
\end{aligned} \tag{2.41}$$

By the simple calculations, we have

$$\text{(i)} \int_0^1 \left|p(h_1,r_1,t)\right| dt = \frac{1}{2}-h_1+h_1^2+\frac{(2-r_1)}{r_1^2}, \tag{2.42}$$

$$\text{(ii)} \int_0^1 \left|q(h_2,r_2,\lambda)\right| d\lambda = \frac{1}{2}-h_2+h_2^2+\frac{(2-r_2)}{r_2^2}. \tag{2.43}$$

Since $|\partial^2 f/\partial t\partial\lambda|^q$ is a coordinated s-convex mapping in the second sense on $\Delta = [a,b]\times[c,d]$, we have that, for $t\in[0,1]$,

$$\begin{aligned}
&\iint_0^1 \left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dt d\lambda \\
&\quad\leq \iint_0^1 \left|p(h_1,r_1,t)q(h_2,r_2,\lambda)\right| \\
&\qquad\times\left\{t^s\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q + t^s(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q\right. \\
&\qquad\qquad\left.+(1-t)^s\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q + (1-t)^s(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right\} dt d\lambda
\end{aligned}$$

$$
\begin{aligned}
&= \left\{\int_0^1 |p(h_1,r_1,t)| t^s dt\right\}\left\{\left(\int_0^1 |q(h_2,r_2,\lambda)|\lambda^s d\lambda\right)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\right. \\
&\quad \left. +\left(\int_0^1 |q(h_2,r_2,\lambda)|(1-\lambda)^s d\lambda\right)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q\right\} \\
&\quad + \left\{\int_0^1 |p(h_1,r_1,t)|(1-t)^s dt\right\}\left\{\left(\int_0^1 |q(h_2,r_2,s)|\lambda^s d\lambda\right)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q\right. \\
&\qquad \left. +\left(\int_0^1 |q(h_2,r_2,\lambda)|(1-\lambda)^s d\lambda\right)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right\} \\
&= \mu_1\left\{\mu_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q + \nu_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q\right\} \\
&\quad + \nu_1\left\{\mu_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q + \nu_2\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right\}.
\end{aligned}
\tag{2.44}
$$

By (2.41)–(2.44), the assertion (2.40) holds. □

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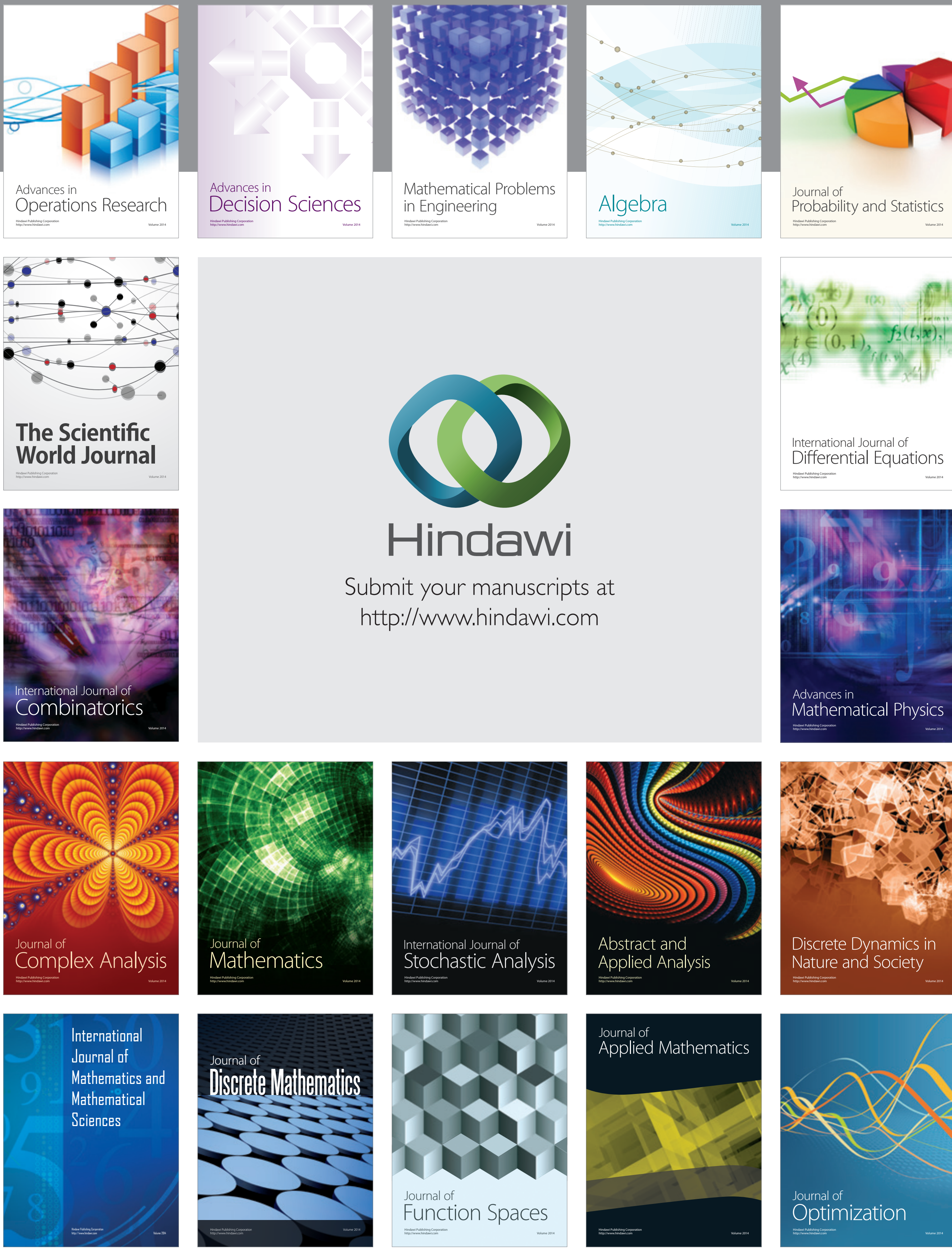

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