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Research Article

A Kind of Compact Quantum Semigroups

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We show that the quantum family of all maps from a finite space to a finite-dimensional compact quantum semigroup has a canonical quantum semigroup structure.

1. Introduction

According to the Gelfand duality, the category of compact Hausdorff spaces and continuous maps and the category of commutative unital C*-algebras and unital *-homomorphisms are dual. In this duality, any compact space X corresponds to $\mathcal{C}(X)$, the C*-algebra of all continuous complex valued maps on X, and any commutative unital C*-algebra corresponds to its maximal ideal space. Thus as the fundamental concept in noncommutative topology, a noncommutative unital C*-algebra A is considered as the algebra of continuous functions on a symbolic compact noncommutative space $\mathfrak{Q}A$. In this correspondence, *-homomorphisms $\Phi : A \to B$ interpret as symbolic continuous maps $\mathfrak{Q}\Phi : \mathfrak{Q}B \to \mathfrak{Q}A$. Since the coordinates observable of a quantum mechanical systems are noncommutative, some-times noncommutative spaces are called quantum spaces.

Woronowicz [1] and Sołtan [2] have defined a quantum space $\mathfrak{Q}C$ of all maps from $\mathfrak{Q}B$ to $\mathfrak{Q}A$ and showed that C exists under appropriate conditions on A and B. In [3], we considered the functorial properties of this notion. In this paper, we show that if $\mathfrak{Q}A$ is a compact finite dimensional (i.e., A is unital and finitely generated) quantum semigroup, and if $\mathfrak{Q}B$ is a finite commutative quantum space (i.e., B is a finite dimensional commutative C*-algebra), then $\mathfrak{Q}C$ has a canonical quantum semigroup structure. In the other words, we construct the noncommutative version of semigroup $\mathfrak{F}(X, S)$ described as follows.

Let X be a finite space and S be a compact semigroup. Then the space $\mathfrak{F}(X, S)$ of all maps from X to S is a compact semigroup with compact-open topology and pointwise multiplication.

In Section 2, we define quantum families of all maps and compact quantum semigroups. In Section 3, we state and prove our main result; also we consider a result about quantum semigroups with counits. At last, in Section 4, we consider some examples.

2. Quantum Families of Maps and Quantum Semigroups

All C*-algebras in this paper have unit and all C*-algebra homomorphisms preserve the units. For any C*-algebra A, I_A and 1_A denote the identity homomorphism from A to A, and the unit of A, respectively. For C*-algebras A, B, $A \otimes B$ denotes the spatial tensor product of A and B. If $\Phi : A \to B$ and $\Phi' : A' \to B'$ are *-homomorphisms, then $\Phi \otimes \Phi' : A \otimes A' \to B \otimes B'$ is the *-homomorphism defined by $\Phi \otimes \Phi'(a \otimes a') = \Phi(a) \otimes \Phi'(a')$ $(a \in A, a' \in A')$.

Let X, Y, and Z be three compact Hausdorff spaces and $\mathcal{C}(Y, X)$ be the space of all continuous maps from Y to X with compact open topology. Consider a continuous map $f : Z \to \mathcal{C}(Y, X)$. Then the pair (Z, f) is a continuous family of maps from Y to X indexed by f with parameters in Z. On the other hand, by topological exponential law we know that f is characterized by a continuous map $\tilde{f} : Y \times Z \to X$ defined by $\tilde{f}(y, z) = f(z)(y)$. Thus $(Z, \tilde{f})$ can be considered as a family of maps from Y to X. Now, by Gelfand's duality we can simply translate this system to noncommutative language.

Definition 2.1 (see [1, 2]). Let A and B be unital C*-algebras. By a quantum family of maps from $\mathfrak{Q}B$ to $\mathfrak{Q}A$, we mean a pair (C, Φ), containing a unital C*-algebra C and a unital *-homomorphism $\Phi : A \to B \otimes C$.

Now, suppose instead of parameter space Z we use $\mathcal{C}(Y, X)$ (note that in general this space is not locally compact). Then the family

$$\mathrm{Id} : \mathcal{C}(Y, X) \longrightarrow \mathcal{C}(Y, X) \qquad \left(\widetilde{\mathrm{Id}} : \mathcal{C}(Y, X) \times Y \longrightarrow X\right) \tag{2.1}$$

of all maps from Y to X has the following universal property.

For every family $\tilde{f} : Z \times Y \to X$ of maps from Y to X, there is a unique map $f : Z \to \mathcal{C}(Y, X)$ such that the following diagram is commutative:

$$\begin{array}{ccc} Z \times Y & \xrightarrow{\quad \tilde{f} \quad} & X \\ \big\downarrow {\scriptstyle f \times Id_Y} & & \big\| \\ C(Y, X) \times Y & \xrightarrow{\quad \widetilde{Id} \quad} & X \end{array} \tag{2.2}$$

Thus, we can make the following definition in noncommutative setting.

Definition 2.2 (see [1, 2]). With the assumptions of Definition 2.1, (C, Φ) is called a quantum family of all maps from $\mathfrak{Q}B$ to $\mathfrak{Q}A$ if for every unital C*-algebra D and any unital *-homomorphism $\Psi : A \to B \otimes D$, there is a unique unital *-homomorphism $\Gamma : C \to D$ such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\Phi} & B \otimes C \\ \Big\| & & \Big\downarrow I_B \otimes \Gamma \\ A & \xrightarrow{\Psi} & B \otimes D \end{array} \tag{2.3}$$

By the universal property of Definition 2.2, it is clear that if (C, Φ) and (C', Φ') are two quantum families of all maps from $\mathfrak{Q}B$ to $\mathfrak{Q}A$, then there is a *-isometric isomorphism between C and C'.

Proposition 2.3. *Let A be a unital finitely generated C*-algebra and B be a finite dimensional C*-algebra. Then the quantum family of all maps from $\mathfrak{Q}B$ to $\mathfrak{Q}A$ exists.*

Proof. See [1] or [2]. □

Definition 2.4 (see [2, 4–6]). A pair (A, Δ) consisting of a unital C*-algebra A and a unital *-homomorphism $\Delta : A \to A \otimes A$ is called a compact quantum semigroup if Δ is a coassociative comultiplication: $(\Delta \otimes \mathrm{I}_A)\Delta = (\mathrm{I}_A \otimes \Delta)\Delta$.

A *-homomorphism $\Delta : A \to A \otimes A$ induces a binary operation on the dual space A^* defined by $\tau\sigma = (\tau \otimes \sigma)\Delta$ for $\tau, \sigma \in A^*$. Now, suppose that S is a compact Hausdorff topological semigroup. Using the canonical identity $\mathcal{C}(S) \otimes \mathcal{C}(S) \cong \mathcal{C}(S \times S)$, we define a *-homomorphism $\Delta : \mathcal{C}(S) \to \mathcal{C}(S) \otimes \mathcal{C}(S)$ by $\Delta(f)(s, s') = f(ss')$ for $f \in \mathcal{C}(S)$ and $s, s' \in S$. Then Δ is a coassociative comultiplication on $\mathcal{C}(S)$ and thus $(\mathcal{C}(S), \Delta)$ is a compact quantum semigroup. Conversely, if (A, Δ) is a compact quantum semigroup such that A is abelian, then the character space of A, with the binary operation induced by Δ, is a compact Hausdorff topological semigroup [7]. It is well known that a compact semigroup with cancellation property is a compact group [8, Proposition 3.2]. Analogous cancellation properties for quantum semigroups are defined as follows.

Definition 2.5. Let (A, Δ) be a compact quantum semigroup.

(i) (see [5]) (A, Δ) has left (resp., right) cancellation property if the linear span of $\{(b \otimes 1)\Delta(a) : a, b \in A\}$ (resp., $\{(1 \otimes b)\Delta(a) : a, b \in A\}$) is dense in $A \otimes A$.

(ii) (see [5]) (A, Δ) has weak left cancellation property if, whenever $\tau, \sigma \in A^*$ are such that $(\tau a)\sigma = 0$ for all $a \in A$, we must have $\tau = 0$ or $\sigma = 0$. Similarly, (A, Δ) has weak right cancellation property if, whenever $\tau(\sigma a) = 0$ for all $a \in A$, we must have $\tau = 0$ or $\sigma = 0$.

(iii) (see [2]) A left (resp., right) counit for (A, Δ), is a character ϵ on A (a unital *-homomorphism $\epsilon : A \to \mathbb{C}$), satisfying $(\epsilon \otimes \mathrm{I}_A)\Delta = \mathrm{I}_A$ (resp., $(\mathrm{I}_A \otimes \epsilon)\Delta = \mathrm{I}_A$). A left and right counit is called (two-sided) counit.

In the above definition the functionals τa and $a\tau$ are defined by $\tau a(x) = \tau(ax)$ and $a\tau(x) = \tau(xa)$.

Remark 2.6. In [4], counits are characters on special dense subalgebras of compact quantum groups. These subalgebras are constructed from finite dimensional unitary representations of compact quantum groups. In this paper we mainly deal with quantum semigroups and since it is not natural to define unitary representations for (quantum) semigroups, we use the above notion for counits.

It is clear that the left (resp., right) cancellation property implies weak left (resp., weak right) cancellation property. The converse is partially satisfied [5, Theorem 3.2]:

Theorem 2.7. *Let* (A, Δ) *be a compact quantum semigroup. Then* (A, Δ) *has both left and right cancellation properties if and only if it has both weak left and weak right cancellation properties.*

Definition 2.8 (see [4, 5, 8]). A compact quantum semigroup with both left and right cancellation properties is called compact quantum group.

Again consider compact semigroup S and its corresponding compact quantum semigroup $(\mathcal{C}(S), \Delta)$ defined above. Using Proposition 3.2 of [8], it is easily proved that S is a compact group if and only if $(\mathcal{C}(S), \Delta)$ is a compact quantum group.

3. The Results

In this section, we state and prove the main result.

Theorem 3.1. *Let* (A, Δ) *be a compact quantum semigroup with finitely generated* A*,* B *be a finite dimensional commutative* C^**-algebra, and* (C, Φ) *be the quantum family of all maps from* $\mathfrak{Q}B$ *to* $\mathfrak{Q}A$*. Consider the unique unital* **-homomorphism* $\Gamma : C \to C \otimes C$ *such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\quad\Phi\quad} & B\otimes C \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle I_B\otimes\Gamma} \\ A\otimes A & & B\otimes C\otimes C \\ \downarrow{\scriptstyle\Phi\otimes\Phi} & & \uparrow{\scriptstyle m\otimes I_{C\otimes C}} \\ B\otimes C\otimes B\otimes C & \xrightarrow{I_B\otimes F\otimes I_C} & B\otimes B\otimes C\otimes C \end{array} \tag{3.1}$$

is commutative, where $F : C\otimes B \to B\otimes C$ *is the flip map, that is,* $c\otimes b \mapsto b\otimes c$ $(b \in B, c \in C)$*, and* $m : B\otimes B \to B$ *is the multiplication* **-homomorphism of* B*, that is,* $m(b\otimes b') = bb'$ $(b, b' \in B)$*. Then* (C, Γ) *is a compact quantum semigroup.*

Proof. We must prove that $(\mathrm{I}_C \otimes \Gamma)\Gamma = (\Gamma \otimes \mathrm{I}_C)\Gamma$, and for this, by the universal property of quantum families of maps, it is enough to prove that

$$(\mathrm{I}_B \otimes \mathrm{I}_C \otimes \Gamma)(\mathrm{I}_B \otimes \Gamma)\Phi = (\mathrm{I}_B \otimes \Gamma \otimes \mathrm{I}_C)(\mathrm{I}_B \otimes \Gamma)\Phi. \tag{3.2}$$

Note that by the commutativity of (3.1), we have

$$(\mathrm{I}_B \otimes \Gamma)\Phi = (m \otimes \mathrm{I}_{C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_C)(\Phi \otimes \Phi)\Delta. \tag{3.3}$$

Let us begin from the left hand side of (3.2):

$$\begin{aligned}
&(\mathrm{I}_B \otimes \mathrm{I}_C \otimes \Gamma)(\mathrm{I}_B \otimes \Gamma)\Phi \\
&\quad = (\mathrm{I}_B \otimes \mathrm{I}_C \otimes \Gamma)(m \otimes \mathrm{I}_{C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_C)(\Phi \otimes \Phi)\Delta \\
&\quad = (m \otimes \mathrm{I}_C \otimes \Gamma)(\mathrm{I}_B \otimes F \otimes \mathrm{I}_C)(\Phi \otimes \Phi)\Delta \\
&\quad = (m \otimes \mathrm{I}_{C\otimes C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_{C\otimes C})(\mathrm{I}_{B\otimes C\otimes B} \otimes \Gamma)(\Phi \otimes \Phi)\Delta \\
&\quad = (m \otimes \mathrm{I}_{C\otimes C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_{C\otimes C})(\mathrm{I}_{B\otimes C\otimes B} \otimes \Gamma)(\Phi \otimes \mathrm{I}_{B\otimes C})(\mathrm{I}_A \otimes \Phi)\Delta \\
&\quad = (m \otimes \mathrm{I}_{C\otimes C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_{C\otimes C})(\Phi \otimes \mathrm{I}_{B\otimes C\otimes C})(\mathrm{I}_A \otimes \mathrm{I}_B \otimes \Gamma)(\mathrm{I}_A \otimes \Phi)\Delta \\
&\quad = (m \otimes \mathrm{I}_{C\otimes C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_{C\otimes C})(\Phi \otimes \mathrm{I}_B \otimes \Gamma)(\mathrm{I}_A \otimes \Phi)\Delta \\
&\quad = (m \otimes \mathrm{I}_{C\otimes C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_{C\otimes C})(\Phi \otimes [(\mathrm{I}_B \otimes \Gamma)\Phi])\Delta \\
&\quad = (m \otimes \mathrm{I}_{C\otimes C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_{C\otimes C})(\Phi \otimes [(m \otimes \mathrm{I}_{C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_C)(\Phi \otimes \Phi)\Delta])\Delta \\
&\quad = (m \otimes \mathrm{I}_{C\otimes C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_{C\otimes C})(\mathrm{I}_{B\otimes C} \otimes m \otimes \mathrm{I}_{C\otimes C})(\mathrm{I}_{B\otimes C\otimes B} \otimes F \otimes \mathrm{I}_C)(\Phi \otimes \Phi \otimes \Phi)(\mathrm{I}_A \otimes \Delta)\Delta.
\end{aligned} \tag{3.4}$$

For the right hand side of (3.2), we have

$$\begin{aligned}
&(\mathrm{I}_B \otimes \Gamma \otimes \mathrm{I}_C)(\mathrm{I}_B \otimes \Gamma)\Phi \\
&\quad = (\mathrm{I}_B \otimes \Gamma \otimes \mathrm{I}_C)(m \otimes \mathrm{I}_{C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_C)(\Phi \otimes \Phi)\Delta \\
&\quad = (m \otimes \Gamma \otimes \mathrm{I}_C)(\mathrm{I}_B \otimes F \otimes \mathrm{I}_C)(\Phi \otimes \Phi)\Delta \\
&\quad = (m \otimes \mathrm{I}_{C\otimes C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_C \otimes \mathrm{I}_C)(\mathrm{I}_B \otimes \mathrm{I}_C \otimes F \otimes \mathrm{I}_C)(\mathrm{I}_B \otimes \Gamma \otimes \mathrm{I}_B \otimes \mathrm{I}_C)(\Phi \otimes \Phi)\Delta \\
&\quad = (m \otimes \mathrm{I}_{C\otimes C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_{C\otimes C})(\mathrm{I}_{B\otimes C} \otimes F \otimes \mathrm{I}_C)([(\mathrm{I}_B \otimes \Gamma)\Phi] \otimes \Phi)\Delta \\
&\quad = (m \otimes \mathrm{I}_{C\otimes C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_{C\otimes C})(\mathrm{I}_{B\otimes C} \otimes F \otimes \mathrm{I}_C)([(m \otimes \mathrm{I}_{C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_C)(\Phi \otimes \Phi)\Delta] \otimes \Phi)\Delta,
\end{aligned} \tag{3.5}$$

and thus if $W = (\mathrm{I}_B \otimes F \otimes \mathrm{I}_{C\otimes C})(\mathrm{I}_{B\otimes C} \otimes F \otimes \mathrm{I}_C)$, then

$$\begin{aligned}
&(\mathrm{I}_B \otimes \Gamma \otimes \mathrm{I}_C)(\mathrm{I}_B \otimes \Gamma)\Phi \\
&\quad = (m \otimes \mathrm{I}_{C\otimes C\otimes C})W(m \otimes \mathrm{I}_{C\otimes C\otimes B\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_{C\otimes B\otimes C})(\Phi \otimes \Phi \otimes \Phi)(\Delta \otimes \mathrm{I}_A)\Delta.
\end{aligned} \tag{3.6}$$

Thus, since $(\mathrm{I}_A \otimes \Delta)\Delta = (\Delta \otimes \mathrm{I}_A)\Delta$, to prove (3.2), it is enough to show that

$$\begin{aligned}&(m \otimes \mathrm{I}_{C\otimes C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_{C\otimes C})(\mathrm{I}_{B\otimes C} \otimes m \otimes \mathrm{I}_{C\otimes C})(\mathrm{I}_{B\otimes C\otimes B} \otimes F \otimes \mathrm{I}_C)\\ &\quad = (m \otimes \mathrm{I}_{C\otimes C\otimes C})W(m \otimes \mathrm{I}_{C\otimes C\otimes B\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_{C\otimes B\otimes C}).\end{aligned} \tag{3.7}$$

Let $b_1, b_2, b_3 \in B$ and $c_1, c_2, c_3 \in C$. Then for the left hand side of (3.7), we have

$$\begin{aligned}&(m \otimes \mathrm{I}_{C\otimes C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_{C\otimes C})(\mathrm{I}_{B\otimes C} \otimes m \otimes \mathrm{I}_{C\otimes C})(\mathrm{I}_{B\otimes C\otimes B} \otimes F \otimes \mathrm{I}_C)(b_1 \otimes c_1 \otimes b_2 \otimes c_2 \otimes b_3 \otimes c_3)\\ &\quad = (m \otimes \mathrm{I}_{C\otimes C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_{C\otimes C})(\mathrm{I}_{B\otimes C} \otimes m \otimes \mathrm{I}_{C\otimes C})(b_1 \otimes c_1 \otimes b_2 \otimes b_3 \otimes c_2 \otimes c_3)\\ &\quad = (m \otimes \mathrm{I}_{C\otimes C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_{C\otimes C})(b_1 \otimes c_1 \otimes (b_2 b_3) \otimes c_2 \otimes c_3)\\ &\quad = (m \otimes \mathrm{I}_{C\otimes C\otimes C})(b_1 \otimes (b_2 b_3) \otimes c_1 \otimes c_2 \otimes c_3)\\ &\quad = b_1(b_2 b_3) \otimes c_1 \otimes c_2 \otimes c_3\\ &\quad = (b_1 b_2 b_3) \otimes c_1 \otimes c_2 \otimes c_3,\end{aligned} \tag{3.8}$$

and for the right hand side of (3.7),

$$\begin{aligned}&(m \otimes \mathrm{I}_{C\otimes C\otimes C})W(m \otimes \mathrm{I}_{C\otimes C\otimes B\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_{C\otimes B\otimes C})(b_1 \otimes c_1 \otimes b_2 \otimes c_2 \otimes b_3 \otimes c_3)\\ &\quad = (m \otimes \mathrm{I}_{C\otimes C\otimes C})W(m \otimes \mathrm{I}_{C\otimes C\otimes B\otimes C})(b_1 \otimes b_2 \otimes c_1 \otimes c_2 \otimes b_3 \otimes c_3)\\ &\quad = (m \otimes \mathrm{I}_{C\otimes C\otimes C})W(b_1 b_2 \otimes c_1 \otimes c_2 \otimes b_3 \otimes c_3)\\ &\quad = (m \otimes \mathrm{I}_{C\otimes C\otimes C})(b_1 b_2 \otimes b_3 \otimes c_1 \otimes c_2 \otimes c_3)\\ &\quad = (b_1 b_2 b_3) \otimes c_1 \otimes c_2 \otimes c_3.\end{aligned} \tag{3.9}$$

Therefore, (3.7) is satisfied and the proof is complete. □

Theorem 3.2. *Let (A, Δ) be a compact quantum semigroup with a left counit. Suppose that B, C, Φ, and Γ are as in Theorem 3.1. Then the compact quantum semigroup (C, Γ) has a left counit.*

Proof. Let $\epsilon : A \to \mathbb{C}$ be a left counit for (A, Δ). Define the unital *-algebra homomorphism $\omega : A \to B \otimes \mathbb{C} = B$ by $\omega(a) = 1_B \otimes \epsilon(a) = \epsilon(a)1_B$ $(a \in A)$. Then the universal property of (C, Φ) shows that there is a character $\widehat{\epsilon} : C \to \mathbb{C}$ such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\quad\Phi\quad} & B \otimes C \\ \Big\| & & \Big\downarrow {\scriptstyle I_B \otimes \widehat{\epsilon}} \\ A & \xrightarrow{\quad\omega\quad} & B \otimes \mathbb{C} \end{array} \tag{3.10}$$

We show that $(\widehat{\epsilon} \otimes \mathrm{I}_C)\Gamma = \mathrm{I}_C$, and thus $\widehat{\epsilon}$ is a counit for (C, Γ). By the universal property of (C, Φ), it is enough to show that

$$(\mathrm{I}_B \otimes [(\widehat{\epsilon} \otimes \mathrm{I}_C)\Gamma])\Phi = \Phi. \tag{3.11}$$

We have

$$\begin{aligned}
(\mathrm{I}_B \otimes [(\widehat{\epsilon} \otimes \mathrm{I}_C)\Gamma])\Phi &= (\mathrm{I}_B \otimes \widehat{\epsilon} \otimes \mathrm{I}_C)(\mathrm{I}_B \otimes \Gamma)\Phi \\
&= (\mathrm{I}_B \otimes \widehat{\epsilon} \otimes \mathrm{I}_C)(m \otimes \mathrm{I}_{C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_C)(\Phi \otimes \Phi)\Delta \\
&= (m \otimes \widehat{\epsilon} \otimes \mathrm{I}_C)(\mathrm{I}_B \otimes F \otimes \mathrm{I}_C)(\Phi \otimes \Phi)\Delta \\
&= (m \otimes \mathrm{I}_C)(\mathrm{I}_B \otimes \widehat{\epsilon} \otimes \mathrm{I}_B \otimes \mathrm{I}_C)(\Phi \otimes \Phi)\Delta \\
&= (m \otimes \mathrm{I}_C)([(\mathrm{I}_B \otimes \widehat{\epsilon})\Phi] \otimes \Phi)\Delta \\
&= (m \otimes \mathrm{I}_C)(\omega \otimes \Phi)\Delta \\
&= (m \otimes \mathrm{I}_C)(\mathrm{I}_B \otimes \Phi)(\omega \otimes \mathrm{I}_A)\Delta.
\end{aligned} \tag{3.12}$$

Since ϵ is a left counit for (A, Δ), we have

$$(\omega \otimes \mathrm{I}_A)\Delta(a) = 1_B \otimes a, \tag{3.13}$$

for every $a \in A$. This implies that

$$\begin{aligned}
(m \otimes \mathrm{I}_C)(\mathrm{I}_B \otimes \Phi)(\omega \otimes \mathrm{I}_A)\Delta(a) &= (m \otimes \mathrm{I}_C)(\mathrm{I}_B \otimes \Phi)(1_B \otimes a) \\
&= \Phi(a),
\end{aligned} \tag{3.14}$$

for every a in A. This completes the proof. □

Analogous of Theorem 3.2 is satisfied for quantum groups that have right and (two-sided) counits. Some natural questions about the structure of the compact quantum semigroup (C, Γ) arise.

Question 1. Let (A, Δ) and (C, Γ) be as in Theorem 3.1.

(i) Suppose that (A, Δ) has one of the left or weak left cancellation properties. Does this hold for (C, Γ)? In particular, suppose the following.

(ii) Suppose that (A, Δ) is a compact quantum group. Is (C, Γ) a compact quantum group?

(iii) Are the converses of (i) and (ii) satisfied?

We consider some parts of these questions for a simple example in the next section.

4. Some Examples

In this section, we consider a class of examples. Let $A = \mathbb{C}^n$ be the C*-algebra of functions on the commutative finite space $\{1,\ldots,n\}$, and let (C,Φ) be the quantum family of all maps from $\mathfrak{Q}A$ to $\mathfrak{Q}A$. A direct computation shows that C is the universal C*-algebra generated by n^2 elements $\{c_{ij} : 1 \le i,j \le n\}$ that satisfy the relations

(1) $c_{ij}^2 = c_{ij} = c_{ij}^*$ for every $i,j = 1,\ldots,n$,

(2) $\sum_{j=1}^n c_{ij} = 1$ for every $i = 1,\ldots,n$, and

(3) $c_{ij}c_{ik} = 0$ for every $i,k,j = 1,\ldots,n$.

Also, $\Phi : A \to A \otimes C$ is defined by $\Phi(e_k) = \sum_{i=1}^n e_i \otimes c_{ik}$, where $e_1,\ldots,e_n$ is the standard basis for A. Suppose that

$$\xi : \{1,\ldots,n\} \times \{1,\ldots,n\} \longrightarrow \{1,\ldots,n\} \tag{4.1}$$

is a semigroup multiplication. Then ξ induces a comultiplication $\Delta : A \to A \otimes A$:

$$\Delta(e_k) = \sum_{r,s=1}^n \Delta_k^{rs} e_r \otimes e_s, \tag{4.2}$$

defined by $\Delta_k^{rs} = \delta_{k\xi(r,s)}$, where δ is the Kronecker delta. We compute the comultiplication $\Gamma : C \to C \otimes C$, induced by Δ as in Theorem 3.1. We have

$$\begin{aligned}(\Phi\otimes\Phi)\Delta(e_k) &= (\Phi\otimes\Phi)\left(\sum_{r,s=1}^n \Delta_k^{rs} e_r \otimes e_s\right) = \sum_{r,s=1}^n \Delta_k^{rs}\Phi(e_r)\otimes\Phi(e_s) \\ &= \sum_{r,s=1}^n\sum_{j=1}^n\sum_{i=1}^n \Delta_k^{rs} e_j \otimes c_{jr} \otimes e_i \otimes c_{is},\end{aligned} \tag{4.3}$$

and therefore

$$\begin{aligned}&(m\otimes \mathrm{I}_{C\otimes C})(\mathrm{I}_B \otimes F \otimes \mathrm{I}_C)(\Phi\otimes\Phi)\Delta(e_k) \\ &\quad= (m\otimes \mathrm{I}_{C\otimes C})\left(\sum_{j=1}^n\sum_{i=1}^n\sum_{r,s=1}^n \Delta_k^{rs} e_j \otimes e_i \otimes c_{jr} \otimes c_{is}\right) \\ &\quad= \sum_{l=1}^n\sum_{r,s=1}^n \Delta_k^{rs} e_l \otimes c_{lr} \otimes c_{ls} \\ &\quad= \sum_{l=1}^n e_l \otimes \left(\sum_{r,s=1}^n \Delta_k^{rs} c_{lr}\otimes c_{ls}\right).\end{aligned} \tag{4.4}$$

This equals to $(\mathrm{I}_A \otimes \Gamma)\Phi(e_k) = (\mathrm{I}_A \otimes \Gamma)\sum_{i=1}^{n} e_i \otimes c_{ik} = \sum_{l=1}^{n} e_l \otimes \Gamma(c_{lk})$. Thus Γ is defined by

$$\Gamma(c_{lk}) = \sum_{r,s=1}^{n} \Delta_k^{rs} c_{lr} \otimes c_{ls}. \tag{4.5}$$

We now consider the special case $n = 2$, in more details. There are only four semigroup structures (up to isomorphism and anti-isomorphism) on the set $\{1,2\}$:

$\xi_1 : 11 = 1,\ 12 = 2,\ 21 = 2,\ 22 = 1.$
$\xi_2 : 11 = 1,\ 12 = 2,\ 21 = 2,\ 22 = 2.$
$\xi_3 : 11 = 1,\ 12 = 1,\ 21 = 1,\ 22 = 1.$
$\xi_4 : 11 = 1,\ 12 = 1,\ 21 = 2,\ 22 = 2.$

The semigroup structure ξ_1 is a group structure and ξ_4 has right cancellation property. In Semigroup Theory, ξ_2, ξ_3, and ξ_4, are called semilattice, null, and left-zero band structures, respectively. For semigroup $(\{1,2\},\xi_i)$, let $(\mathbb{C}^2,\Delta_i)$ and (C,Γ_i) be the corresponding quantum semigroups, as above. A simple computation shows that:

$$\begin{aligned}
&\Gamma_1(c_{11}) = c_{11} \otimes c_{11} + c_{12} \otimes c_{12}, \qquad \Gamma_1(c_{12}) = c_{11} \otimes c_{12} + c_{12} \otimes c_{11},\\
&\Gamma_1(c_{21}) = c_{21} \otimes c_{21} + c_{22} \otimes c_{22}, \qquad \Gamma_1(c_{22}) = c_{21} \otimes c_{22} + c_{22} \otimes c_{21},\\
&\Gamma_2(c_{11}) = c_{11} \otimes c_{11}, \qquad \Gamma_2(c_{12}) = c_{12} \otimes c_{12} + c_{11} \otimes c_{12} + c_{12} \otimes c_{11},\\
&\Gamma_2(c_{21}) = c_{21} \otimes c_{21}, \qquad \Gamma_2(c_{22}) = c_{22} \otimes c_{22} + c_{21} \otimes c_{22} + c_{22} \otimes c_{21},\\
&\Gamma_3(c_{11}) = 1, \qquad \Gamma_3(c_{12}) = 0,\\
&\Gamma_3(c_{21}) = 1, \qquad \Gamma_3(c_{22}) = 0,\\
&\Gamma_4(c) = c \otimes 1 \quad (\forall c \in C).
\end{aligned} \tag{4.6}$$

As we have explained in Section 2, $(\mathbb{C}^2,\Delta_1)$ is a compact quantum group and $(\mathbb{C}^2,\Delta_4)$ is a compact quantum semigroup with right cancellation property. From the above computations, it is clear that the compact quantum semigroup (C,Γ_4) has right cancellation property. Now, we show that (C,Γ_1) is also a compact quantum group: the unital C*-algebra C is generated by the two unitary elements $x = c_{11} - c_{12}$ and $y = c_{21} - c_{22}$ (see the following remark for more details). A simple computation shows that

$$\Gamma_1(x) = x \otimes x, \qquad \Gamma_1(y) = y \otimes y. \tag{4.7}$$

This easily implies that (C,Γ_1) has left and right cancellation properties, and therefore (C,Γ_1) is a compact quantum group.

Remark 4.1. (1) The algebra $A = \mathbb{C}^2$ is the universal C*-algebra generated by a unitary self-adjoint element, say $(1,-1)$. It follows from the proof of Theorem 3.3 of [2], that C becomes the universal C*-algebra generated by two unitary self-adjoint elements. A model for C is the

C^*-algebra of all continuous maps from closed unit interval to 2×2 matrix algebra, which take diagonal matrices at the endpoints of the interval, equivalently

$$C = \left\{ \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} : f_{ij} \in \mathcal{C}[0,1], f_{12}(0) = f_{12}(1) = f_{21}(0) = f_{21}(1) = 0 \right\}, \tag{4.8}$$

with unitary self-adjoint generators

$$x = \begin{pmatrix} \cos(\pi t) & \sin(\pi t) \\ \sin(\pi t) & -\cos(\pi t) \end{pmatrix}, \qquad y = \begin{pmatrix} -\cos(\pi t) & \sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{pmatrix}. \tag{4.9}$$

In this representation of C, the generators c_{ij}'s become: $c_{11} = (1+x)/2$, $c_{12} = (1-x)/2$, $c_{21} = (1+y)/2$ and $c_{22} = (1-y)/2$. Also, the homomorphism $\Phi : \mathbb{C}^2 \to \mathbb{C}^2 \otimes C = C \oplus C$ is defined by $\Phi(1,-1) = (x,y)$. This representation of the C^*-algebra C is one of the elementary examples of noncommutative spaces; see Section II.2.β of [9].

(2) There is another quantum semigroup structure on quantum families of all maps from any finite quantum space to itself introduced by Sołtan [2].

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