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Research Article

On Generalized Approximative Properties of Systems

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Generalized concepts of b-completeness, b-independence, b-minimality, and b-basicity are introduced. Corresponding concept of a space of coefficients is defined, and some of its properties are stated.

1. Introduction

Theory of classical basis (including Schauder basis) is sufficiently well developed, and quite a good number of monographs such as Day [1], Singer [2, 3], Young [4], Bilalov, Veliyev [5], Ch. Heil [6], and so forth have been dedicated to it so far. There are different versions and generalizations of the concept of classical basis (for more details see [2, 3]). One of such generalizations is proposed in [5, 7]. In these works, the concept of b_Y-invariance is first introduced and then used to obtain main results. It should be noted that the condition of b_Y-invariance implies the fact that the corresponding mapping is tensorial.

In our work, we neglect the b_Y-invariance condition. We give more detailed consideration to the space of coefficients. We also state a criterion of basicity.

2. Needful Notations and Concepts

Let X, Y, Z be some Banach spaces, and let $\|\cdot\|_X$, $\|\cdot\|_Y$, $\|\cdot\|_Z$ be the corresponding norms. Assume that we are given some bounded bilinear mapping $b : X \times Y \to Z$, that is, $\|b(x;y)\|_Z \le c\,\|x\|_X\|y\|_Y$, for all $x \in X$, for all $y \in Y$, where c is an absolute constant. For simplicity, we denote $xy \equiv b(x;y)$. Let $M \subset Y$ be some set. By $L^b[M]$ we denote the *b-span* of M. So, by definition, $L^b[M] \equiv \{z \in Z : \exists\{x_k\}_1^n \subset X, \exists\{y_k\}_1^n \subset M, z = \sum_{k=1}^n x_k y_k\}$. By $\overline{(\cdot)}$ we denote the closure in the corresponding space.

System $\{y_n\}_{n\in\mathbb{N}} \subset Y$ is called *b-linearly* independent if $\sum_{n=1}^{\infty} x_n y_n = 0$ *in* Z implies that $x_n = 0$, for all $n \in \mathbb{N}$.

In the context of the above mentioned, the concept of usual completeness is stated as follows.

Definition 2.1. System $\{y_n\}_{n\in\mathbb{N}} \subset Y$ is called *b-complete* in Z if $\overline{L^b[\{y_n\}_{n\in\mathbb{N}}]} \equiv Z$. We will also need the concepts of *b-biorthogonal* system and *b-basis*.

Definition 2.2. System $\{y_n^*\}_{n\in\mathbb{N}} \subset L(Z;X)$ is called *b-biorthogonal* to the system $\{y_n\}_{n\in\mathbb{N}} \subset Y$ if $y_n^*(xy_k) = \delta_{nk}x$, for all $n,k \in \mathbb{N}$, for all $x \in X$, where δ_{nk} is the Kronecker symbol.

Definition 2.3. System $\{y_n\}_{n\in\mathbb{N}} \subset Y$ is called *b-basis* in Z if for for all $z \in Z$, $\exists!\{x_n\}_{n\in\mathbb{N}} \subset X$: $z = \sum_{n=1}^{\infty} x_n y_n$.

To obtain our main results we will use the following concept.

Definition 2.4. System $\{y_n\}_{n\in\mathbb{N}} \subset Y$ is called nondegenerate if $\exists c_n > 0 :\ \|x\|_X \le c_n \|xy_n\|_Z$, for all $x \in X$, for all $n \in \mathbb{N}$.

3. Main Results

3.1. Space of Coefficients

Let $\{y_n\}_{n\in\mathbb{N}} \subset Y$ be some system. Assume that

$$\mathcal{K}_{\overline{y}} \equiv \left\{ \{x_n\}_{n\in N} \subset X : \text{the series } \sum_{n=1}^{\infty} x_n y_n \text{ converges in } Z \right\}. \tag{3.1}$$

With regard to the ordinary operations of addition and multiplication by a complex number, $\mathcal{K}_{\overline{y}}$ is a linear space. We introduce a norm $\|\cdot\|_{\mathcal{K}_{\overline{y}}}$ in $\mathcal{K}_{\overline{y}}$ as follows:

$$\|\overline{x}\|_{\mathcal{K}_{\overline{y}}} = \sup_m \left\| \sum_{n=1}^{m} x_n y_n \right\|_Z, \tag{3.2}$$

where $\overline{x} \equiv \{x_n\}_{n\in\mathbb{N}} \in \mathcal{K}_{\overline{y}}$. In fact, it is clear that

$$\begin{gathered} \|\lambda\ \overline{x}\|_{\mathcal{K}_{\overline{y}}} = |\lambda|\ \|\overline{x}\|_{\mathcal{K}_{\overline{y}}}, \quad \forall \lambda \in \mathbb{C}; \\ \|\overline{x}_1 + \overline{x}_2\|_{\mathcal{K}_{\overline{y}}} \le \|\overline{x}_1\|_{\mathcal{K}_{\overline{y}}} + \|\overline{x}_2\|_{\mathcal{K}_{\overline{y}}}, \quad \forall \overline{x}_k \in \mathcal{K}_{\overline{y}},\ k = 1,2. \end{gathered} \tag{3.3}$$

Assume that $\|\overline{x}\|_{\mathcal{K}_{\overline{y}}} = 0$ for some $\overline{x} \equiv \{x_n\}_{n\in\mathbb{N}} \in \mathcal{K}_{\overline{y}}$. Let $n_0 = \inf\{n : x_k = 0,\ \text{for all } k \le n-1\}$. In what follows we will suppose that the system $\{y_n\}_{n\in\mathbb{N}}$ is nondegenerate. Let $n_0 < +\infty$. We have

$$\sup_m \left\| \sum_{n=1}^{m} x_n y_n \right\|_Z \ge \left\| \sum_{n=1}^{n_0} x_n y_n \right\|_Z = \|x_{n_0} y_{n_0}\|_Z \ge \frac{1}{c_n} \|x_{n_0}\|_X > 0. \tag{3.4}$$

But this is contrary to $\|\overline{x}\|_{\mathcal{K}_{\overline{y}}} = 0$. Consequently, $n_0 = +\infty$, that is, $\overline{x} = 0$. Thus, $(\mathcal{K}_{\overline{y}};\ \|\cdot\|_{\mathcal{K}_{\overline{y}}})$ is a normed space. Let us show that it is complete. Let $\{\overline{x}_n\}_{n\in\mathbb{N}} \subset \mathcal{K}_{\overline{y}}$ be some fundamental sequence with $\overline{x}_n \equiv \{x_k^{(n)}\}_{k\in\mathbb{N}} \subset X$. For arbitrary fixed number $k \in \mathbb{N}$. We have

$$\begin{aligned}\left\|x_k^{(n)} - x_k^{(n+p)}\right\|_X &\leq c_k \left\|\left(x_k^{(n)} - x_k^{(n+p)}\right)y_k\right\|_Z \\ &= c_k \left\|\sum_{i=1}^{k}\left(x_i^{(n)} - x_i^{(n+p)}\right)y_i - \sum_{i=1}^{k-1}\left(x_i^{(n)} - x_i^{(n+p)}\right)\ y_i\right\|_Z \\ &\leq 2c_k \sup_m \left\|\sum_{i=1}^{m}\left(x_i^{(n)} - x_i^{(n+p)}\right)y_i\right\| \\ &= 2c_k \left\|\overline{x}_n - \overline{x}_{n+p}\right\|_{\mathcal{K}_{\overline{y}}} \longrightarrow 0, \quad \text{as } n, p \longrightarrow \infty.\end{aligned} \tag{3.5}$$

As a result, for all fixed $k \in \mathbb{N}$ the sequence $\{x_k^{(n)}\}_{n\in\mathbb{N}}$ is fundamental in X. Let $x_k^{(n)} \to x_k$, $n \to \infty$. Let us take arbitrary positive ε. Then $\exists n_0$: for all $n \geq n_0$, for all $p \in \mathbb{N}$, we have $\|\overline{x}_n - \overline{x}_{n+p}\|_{\mathcal{K}_{\overline{y}}} < \varepsilon$. Thus

$$\left\|\sum_{k=1}^{m}\left(x_k^{(n)} - x_k^{(n+p)}\right)y_k\right\|_Z < \varepsilon, \quad \forall n \geq n_0,\ \forall p, m \in \mathbb{N}. \tag{3.6}$$

Passing to the limit as $p \to \infty$ yields

$$\left\|\sum_{k=1}^{m}\left(x_k^{(n)} - x_k\right)y_k\right\|_Z \leq \varepsilon, \quad \forall n \geq n_0,\ \forall m \in \mathbb{N}. \tag{3.7}$$

It is easy to see that

$$\left\|\sum_{k=m}^{m+p}\left(x_k^{(n)} - x_k\right)\ y_k\right\|_Z \leq 2\varepsilon, \quad \forall n \geq n_0,\ \forall m, p \in \mathbb{N}. \tag{3.8}$$

Since the series $\sum_{k=1}^{\infty} x_k^{(n)} y_k$ converges in Z, it is clear that $\exists m_0^{(n)}$: for all $m \geq m_0^{(n)}$, for all $p \in \mathbb{N}$, we have

$$\left\|\sum_{k=m}^{m+p} x_k^{(n)} y_k\right\|_Z < \varepsilon. \tag{3.9}$$

Then it follows from the previous inequality that

$$\begin{gathered}\left\|\sum_{k=m}^{m+p} x_k y_k\right\|_Z \leq \left\|\sum_{k=m}^{m+p}\left(x_k^{(n)} - x_k\right)y_k\right\|_Z + \left\|\sum_{k=m}^{m+p} x_k^{(n)} y_k\right\|_Z \leq 3\varepsilon, \\ \forall m \geq m_0^{(n)}, \quad \forall p \in \mathbb{N}.\end{gathered} \tag{3.10}$$

Consequently, the series $\sum_{k=1}^{\infty} x_k y_k$ converges in Z, and therefore $\overline{x} \equiv \{x_k\}_{k\in\mathbb{N}} \in \mathcal{K}_{\overline{y}}$. From (3.7) it follows directly that $\|\overline{x}_n - \overline{x}\|_{\mathcal{K}_{\overline{y}}} \to 0,\ n \to \infty$. Thus, $\mathcal{K}_{\overline{y}}$ is a Banach space.

Let us consider the operator $K : \mathcal{K}_{\overline{y}} \to Z$, defined by the expression

$$K\overline{x} = \sum_{n=1}^{\infty} x_n y_n, \overline{x} \equiv \{x_n\}_{n\in\mathbb{N}}. \tag{3.11}$$

It is obvious that K is a linear operator. Let $z = K\overline{x}$. We have

$$\|K\ \overline{x}\|_Z = \|z\|_Z = \left\|\sum_{n=1}^{\infty} x_n\ y_n\right\|_Z \le \sup_m \left\|\sum_{n=1}^{m} x_n\ y_n\right\|_Z = \|\overline{x}\|_{\mathcal{K}_{\overline{y}}}. \tag{3.12}$$

It follows that $K \in L(\mathcal{K}_{\overline{y}}; Z)$ and $\|K\| \le 1$. Let $\overline{x}_0 \equiv \{x;\ 0;\cdots\},\ \ x \in X$. It is clear that $\|K\overline{x}_0\|_Z = \|\overline{x}_0\|_{\mathcal{K}_{\overline{y}}}$. Consequently, $\|K\| = 1$. It is absolutely obvious that if the system $\{y_n\}_{n\in\mathbb{N}} \subset Y$ is *b-linearly independent*, then KerK $= \{0\}$. In this case, $\exists K^{-1} : Z \to \mathcal{K}_{\overline{y}}$. If $\operatorname{Im} K$ is closed, then, by the Banach's theorem on the inverse operator, we obtain that $K^{-1} \in L(\operatorname{Im} K; \mathcal{K}_{\overline{y}})$. The same considerations are valid in the case when the system $\{y_n\}_{n\in\mathbb{N}}$ has a *b-biorthogonal* system. We will call operator K a coefficient operator. Thus, we have proved the following.

Theorem 3.1. *Every nondegenerate system $S_{\overline{y}} \equiv \{y_n\}_{n\in\mathbb{N}} \subset Y$ is corresponded by a Banach space of coefficients $\mathcal{K}_{\overline{y}}$ and coefficient operator $K \in L(\mathcal{K}_{\overline{y}}; Z)$, $\|K\| = 1$. If the system $S_{\overline{y}}$ is b-linearly independent or has a b-biorthogonal system, then $\exists K^{-1}$. Moreover, if* $\operatorname{Im} K$ *is closed, then* $K^{-1} \in L(\operatorname{Im} K; \mathcal{K}_{\overline{y}})$.

In what follows, we will need the concept of *b-basis* in the space of coefficients $\mathcal{K}_{\overline{y}}$.

Definition 3.2. System $\{T_n\}_{n\in\mathbb{N}} \subset L(X; \mathcal{K}_{\overline{y}})$ is called *b-basis* in $\mathcal{K}_{\overline{y}}$ if for for all $\overline{x} \in \mathcal{K}_{\overline{y}}$, $\exists! \{x_n\}_{n\in\mathbb{N}} \subset X : \overline{x} = \sum_{n=1}^{\infty} T_n\, x_n$ (convergence in $\mathcal{K}_{\overline{y}}$).

Consider the operators $E_n : X \longrightarrow \mathcal{K}_{\overline{y}} : E_n x = \{\delta_{nk} x\}_{k\in\mathbb{N}},\ n \in \mathbb{N}$. We have

$$\|E_n x\|_{\mathcal{K}_{\overline{y}}} = \|x y_n\|_Z \le \|y_n\|_Y \|x\|_X, \quad \forall x \in X. \tag{3.13}$$

Thus, $E_n \in L(X; \mathcal{K}_{\overline{y}})$, for all $n \in \mathbb{N}$. Take $\overline{x} \equiv \{x_n\}_{n\in\mathbb{N}} \in \mathcal{K}_{\overline{y}}$. Then

$$\left\|\overline{x} - \sum_{n=1}^{m} E_n x_n\right\|_{\mathcal{K}_{\overline{y}}} = \left\|\left\{\underbrace{0;\cdots;0}_{m};\ x_{m+1}; x_{m+2}; \cdots\right\}\right\|_{\mathcal{K}_{\overline{y}}} = \sup_p \left\|\sum_{n=m+1}^{m+p} x_n y_n\right\|_z \longrightarrow 0, \tag{3.14}$$

as $m \to \infty$, since the series $\sum_{n=1}^{\infty} x_n y_n$ converges in Z. As a result, we obtain that $\overline{x} = \sum_{n=1}^{\infty} E_n x_n$. Consider the operators $P_n : \mathcal{K}_{\overline{y}} \to X : P_n\ \overline{x} = x_n,\ n \in \mathbb{N}$. We have

$$\|P_n \overline{x}\|_X = \|x_n\|_X \le c_n \|x_n y_n\|_Z \le c_n \sup_m \left\|\sum_{k=1}^{m} x_k y_k\right\|_Z = c_n \|\overline{x}\|_{\mathcal{K}_{\overline{y}}}. \tag{3.15}$$

Consequently, $P_n \in L(\mathcal{K}_{\overline{y}}; X)$, $n \in \mathbb{N}$. Let us show that the expansion $\overline{x} = \sum_{n=1}^{\infty} E_n x_n$ is unique. Let $\sum_{n=1}^{\infty} E_n x_n = 0$. We have $0 = P_k(\sum_{n=1}^{\infty} E_n x_n) = \sum_{n=1}^{\infty} P_k(E_n x_n) = x_k$, for all $k \in \mathbb{N}$. As a result, we obtain that the system $\{E_n\}_{n\in\mathbb{N}}$ forms a *b-basis* for $\mathcal{K}_{\overline{y}}$. We will call this system a *canonical system*. So we have proved the following.

Theorem 3.3. *Let $\mathcal{K}_{\overline{y}}$ be a space of coefficients of nondegenerate system $\{y_n\}_{n\in\mathbb{N}} \subset Y$. Then the canonical system $\{E_n\}_{n\in\mathbb{N}}$ forms a b-basis for $\mathcal{K}_{\overline{y}}$.*

Let the nondegenerate system $\{y_n\}_{n\in\mathbb{N}} \subset Y$ form a *b-basis for Z*. Consider the coefficient operator $K : \mathcal{K}_{\overline{y}} \to Z$. By definition of *b-basis*, the equation $K\,\overline{x} = z$ is solvable with regard to $\overline{x} \in \mathcal{K}_{\overline{y}}$ for for all $z \in Z$. It is absolutely clear that $K\mathrm{er}\ K = \{0\}$. Then it follows from Theorem 3.1 and Banach theorem that $K^{-1} \in L(Z; \mathcal{K}_{\overline{y}})$. Consequently, operator K performs isomorphism between $\mathcal{K}_{\overline{y}}$ and Z.

Vice versa, let $\{y_n\}_{n\in\mathbb{N}} \subset Y$ be a nondegenerate system and let $\mathcal{K}_{\overline{y}}$ be a corresponding space of coefficients. Assume that a coefficient operator $K \in L(\mathcal{K}_{\overline{y}}; Z)$ is an isomorphism. Take for all $z \in Z$. It is clear that $\exists \overline{x} \equiv \{x_n\}_{n\in\mathbb{N}} \in \mathcal{K}_{\overline{y}} : K\overline{x} = z$, that is $z = \sum_{n=1}^{\infty} x_n y_n$ in Z. Consequently, z can be expanded in a series with respect to the system $\{y_n\}_{n\in\mathbb{N}}$. Let us show that such an expansion is unique. Let $\sum_{n=1}^{\infty} x_n^0 y_n = 0$ for some $\overline{x}^0 \equiv \{x_n^0\}_{n\in\mathbb{N}} \in \mathcal{K}_{\overline{y}}$. This means that $K\overline{x}^0 = 0 \Rightarrow \overline{x}^0 = 0 \Rightarrow \overline{x}_n^0 = 0$, for for all $n \in \mathbb{N}$. Thus, the system $\{y_n\}_{n\in\mathbb{N}}$ forms a *b-basis* for Z. So the following theorem is proved.

Theorem 3.4. *Let $\mathcal{K}_{\overline{y}}$ be a space of coefficients of nondegenerate system $\{y_n\}_{n\in\mathbb{N}}$ and let K be a corresponding coefficient operator. Then this system forms a b-basis for Z if and only if K is an isomorphism in $L(\mathcal{K}_{\overline{y}};\ Z)$.*

3.2. Criterion of Basicity

Let the systems $\{y_n\}_{n\in\mathbb{N}} \subset Y$ and $\{y_n^*\}_{n\in\mathbb{N}} \subset L(Z;\ X)$ be *b-biorthogonal*. Take for all $z \in Z$ and consider the partial sums

$$S_n z = \sum_{k=1}^{n} y_k^*(z) y_k, \quad n \in \mathbb{N}. \tag{3.16}$$

We have

$$\begin{aligned} S_n(S_m z) &= \sum_{k=1}^{n} y_k^*(S_m z) y_k = \sum_{k=1}^{n} y_k^* \left[\sum_{i=1}^{m} y_i^*(z) y_i \right] y_k \\ &= \sum_{k=1}^{n} \sum_{i=1}^{m} \delta_{ki} y_i^*(z) y_k = \sum_{k=1}^{\min\{n;m\}} y_k^*(z) y_k = S_{\min\{n;m\}} z, \quad \forall n, m \in \mathbb{N}. \end{aligned} \tag{3.17}$$

Hence, $S_n^2 = S_n$, for for all $n \in \mathbb{N}$, that is, S_n is a projector in Z. It follows directly from the estimate

$$\left\| y_k^*(z) y_k \right\|_Z \le c \left\| y_k^*(z) \right\|_X \left\| y_k \right\|_Y \le c \left\| y_k^* \right\| \ \left\| y_k \right\|_Y \|z\|_Z, \quad \forall z \in Z, \tag{3.18}$$

that S_n is a continuous projector. Suppose that the nondegenerate system $\{y_n\}_{n\in\mathbb{N}} \subset Y$ forms a *b-basis* for Z. Then for for all $z \in Z$ has a unique expansion $z = \sum_{n=1}^{\infty} x_n y_n$ in Z. We denote the correspondence $z \to x_n$ by $y_n^* : y_n^*(z) = x_n$, for all $n \in \mathbb{N}$. It is obvious that $y_n^* : Z \to X$ is a linear operator. Let $\mathcal{K}_{\overline{y}}$ be *a space of coefficients* of basis $\{y_n\}_{n\in\mathbb{N}}$, and let $K : \mathcal{K}_{\overline{y}} \to Z$ be the corresponding coefficient operator. By Theorem 3.4, K is an isomorphism. We have

$$\begin{aligned} \|y_n^*(z)\|_X &= \|x_n\|_X \le c_n \|x_n y_n\| \le c_n \sup_m \left\| \sum_{k=1}^{m} x_k y_k \right\|_Z = c_n \|\overline{x}\|_{\mathcal{K}_{\overline{y}}} \\ &= c_n \left\| K^{-1} z \right\|_{\mathcal{K}_{\overline{y}}} \le c_n \left\| K^{-1} \right\| \|z\|_Z, \end{aligned} \tag{3.19}$$

where $\overline{x} \equiv \{x_n\}_{n\in\mathbb{N}}$. Consequently, $\{y_n^*\}_{n\in\mathbb{N}} \subset L(Z; X)$. It follows directly from the uniqueness of the expansion that $y_n^*(x y_k) = \delta_{nk} x$; for all $n, k \in \mathbb{N}$, for for all $x \in X$. As a result, we obtain that the system $\{y_n^*\}_{n\in\mathbb{N}}$ is *b-biorthogonal* to $\{y_n\}_{n\in\mathbb{N}}$. Let us consider the projectors $S_m \in L(Z)$ for for all $z \in Z$:

$$S_m z = \sum_{n=1}^{m} y_n^*(z) y_n, \quad m \in \mathbb{N}. \tag{3.20}$$

As the series (3.20) converges for for all $z \in Z$, it follows from Banach-Steinhaus theorem that

$$M = \sup_m \|S_m\| < +\infty. \tag{3.21}$$

It is absolutely obvious that the system $\{y_n\}_{n\in\mathbb{N}}$ is *b-complete* in Z. Thus, if the system $\{y_n\}_{n\in\mathbb{N}}$ forms a *b-basis* for Z, then (1) it is *b-complete* in Z; (2) it has a *b-biorthogonal* system; (3) the corresponding family of projectors is uniformly bounded.

Vice versa, let the system $\{y_n\}_{n\in\mathbb{N}}$ be *b-complete in* Z and have a *b-biorthogonal* system $\{y_n^*\}_{n\in\mathbb{N}}$. Assume that the corresponding family of projectors $\{S_m\}_{m\in\mathbb{N}}$ is uniformly bounded, that is, relation (3.21) holds. Let $z \in Z$ be an arbitrary element. Let us take arbitrary positive ε. It is clear that $\exists \{x_n\}_{n=1}^{m_0} \subset X : \ \|z - \sum_{n=1}^{m_0} x_n y_n\|_Z < \varepsilon$. Let $z_0 = \sum_{n=1}^{m_0} x_n y_n$. We have

$$y_n^*(z_0) = \sum_{k=1}^{m_0} y_n^*(x_k y_k) = x_n, \quad n = \overline{1, m_0};\ y_n^*(z_0) = 0,\ \forall n > m_0. \tag{3.22}$$

As a result, we obtain for $m \ge m_0$ that

$$\|z - S_m z\|_Z \le \|z - z_0\|_Z + \|z_0 - S_m z\|_Z \le \varepsilon + \left\| \sum_{n=1}^{m} y_n^*(z - z_0) y_n \right\|_Z \le (M+1)\ \varepsilon. \tag{3.23}$$

From the arbitrariness of ε we get $\lim_{m\to\infty} S_m z = z$. Consequently, for for all $z \in Z$ can be expanded in a series with respect to the system $\{y_n\}_{n\in\mathbb{N}}$. The existence of *b-biorthogonal* system implies the uniqueness of the expansion. Thus, the following theorem is valid.

Theorem 3.5. *Nondegenerate system* $\{y_n\}_{n\in\mathbb{N}} \subset Y$ *forms a* b*-basis for* Z *if and only if the following conditions are satisfied:*

(1) *the system is* b*-complete in* Z*;*

(2) *it has a* b*-biorthogonal system* $\{y_n^*\}_{n\in\mathbb{N}} \subset L(Z; X)$*;*

(3) *the family of projectors* (3.21) *is uniformly bounded.*

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