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Research Article

Complete Moment Convergence of Weighted Sums for Arrays of Rowwise φ-Mixing Random Variables

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The complete moment convergence of weighted sums for arrays of rowwise φ-mixing random variables is investigated. By using moment inequality and truncation method, the sufficient conditions for complete moment convergence of weighted sums for arrays of rowwise φ-mixing random variables are obtained. The results of Ahmed et al. (2002) are complemented. As an application, the complete moment convergence of moving average processes based on a φ-mixing random sequence is obtained, which improves the result of Kim et al. (2008).

1. Introduction

Hsu and Robbins [1] introduced the concept of complete convergence of $\{X_n\}$. A sequence $\{X_n, n = 1, 2, \ldots\}$ is said to converge completely to a constant C if

$$\sum_{n=1}^{\infty} P(|X_n - C| > \epsilon) < \infty, \quad \forall \epsilon > 0. \tag{1.1}$$

Moreover, they proved that the sequence of arithmetic means of independent identically distributed (i.i.d.) random variables converge completely to the expected value if the variance of the summands is finite. The converse theorem was proved by Erdös [2]. This result has been generalized and extended in several directions, see Baum and Katz [3], Chow [4], Gut [5], Taylor et al. [6], and Cai and Xu [7]. In particular, Ahmed et al. [8] obtained the following result in Banach space.

Theorem A. *Let $\{X_{ni}; i \geq 1, n \geq 1\}$ be an array of rowwise independent random elements in a separable real Banach space $(B, \|\cdot\|)$. Let $P(\|X_{ni}\| > x) \leq CP(|X| > x)$ for some random variable X, constant C and all n, i and $x > 0$. Suppose that $\{a_{ni}, i \geq 1, n \geq 1\}$ is an array of constants such that*

$$\sup_{i\geq 1}|a_{ni}| = O(n^{-r}), \quad \text{for some } r > 0,$$
$$\sum_{i=1}^{\infty}|a_{ni}| = O(n^{\alpha}), \quad \text{for some } \alpha \in [0, r). \tag{1.2}$$

Let β be such that $\alpha + \beta \neq -1$ and fix $\delta > 0$ such that $1 + \alpha/r < \delta \leq 2$. Denote $s = \max(1 + (\alpha + \beta + 1)/r,\ \delta)$. If $E|X|^s < \infty$ and $S_n = \sum_{i=1}^{\infty} a_{ni}X_{ni} \to 0$ in probability, then $\sum_{n=1}^{\infty} n^{\beta}P(\|S_n\| > \epsilon) < \infty$ for all $\epsilon > 0$.

Chow [4] established the following refinement which is a complete moment convergence result for sums of (i.i.d.) random variables.

Theorem B. *Let $EX_1 = 0$, $1 \leq p < 2$ and $r \geq p$. Suppose that $E[|X_1|^r + |X_1|\log(1 + |X_1|)] < \infty$. Then*

$$\sum_{n=1}^{\infty} n^{(r/p)-2-(1/p)}E\left(\left|\sum_{i=1}^{n} X_i\right| - \epsilon n^{1/p}\right)^{+} < \infty, \quad \forall \epsilon > 0. \tag{1.3}$$

The main purpose of this paper is to discuss again the above results for arrays of rowwise φ-mixing random variables. The author takes the inspiration in [8] and discusses the complete moment convergence of weighted sums for arrays of rowwise φ-mixing random variables by applying truncation methods. The results of Ahmed et al. [8] are extended to φ-mixing case. As an application, the corresponding results of moving average processes based on a φ-mixing random sequence are obtained, which extend and improve the result of Kim and Ko [9].

For the proof of the main results, we need to restate a few definitions and lemmas for easy reference. Throughout this paper, C will represent positive constants, the value of which may change from one place to another. The symbol $I(A)$ denotes the indicator function of A; $[x]$ indicates the maximum integer not larger than x. For a finite set B, the symbol $\sharp B$ denotes the number of elements in the set B.

Definition 1.1. A sequence of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be a sequence of φ-mixing random variables, if

$$\varphi(m) = \sup_{k\geq 1}\left\{|P(B \mid A) - P(B)|\ ; A \in \mathfrak{I}_1^k, B \in \mathfrak{I}_{k+m}^{\infty}, P(A) > 0\right\} \longrightarrow 0, \quad \text{as } m \longrightarrow \infty, \tag{1.4}$$

where $\mathfrak{I}_j^k = \sigma\{X_i; j \leq i \leq k\}$, $1 \leq j \leq k \leq \infty$.

Definition 1.2. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X (write $\{X_i\} \prec X$) if there exists a constant C, such that $P\{|X_n| > x\} \leq CP\{|X| > x\}$ for all $x \geq 0$ and $n \geq 1$.

The following lemma is a well-known result.

Lemma 1.3. *Let the sequence* $\{X_n, n \geq 1\}$ *of random variables be stochastically dominated by a random variable* X*. Then for any* $p > 0, x > 0$

$$E|X_n|^p I(|X_n| \leq x) \leq C\left[E|X|^p I(|X| \leq x) + x^p P\{|X| > x\}\right], \tag{1.5}$$

$$E|X_n|^p I(|X_n| > x) \leq CE|X|^p I(|X| > x). \tag{1.6}$$

Definition 1.4. A real-valued function $l(x)$, positive and measurable on $[A, \infty)$ for some $A > 0$, is said to be slowly varying if $\lim_{x\to\infty} l(x\lambda)/l(x) = 1$ for each $\lambda > 0$.

By the properties of slowly varying function, we can easily prove the following lemma. Here we omit the details of the proof.

Lemma 1.5. *Let* $l(x) > 0$ *be a slowly varying function as* $x \to \infty$*, then there exists* C *(depends only on* r*) such that*

(i) $Ck^{r+1}l(k) \leq \sum_{n=1}^{k} n^r l(n) \leq Ck^{r+1}l(k)$ *for any* $r > -1$ *and positive integer* k*,*

(ii) $Ck^{r+1}l(k) \leq \sum_{n=k}^{\infty} n^r l(n) \leq Ck^{r+1}l(k)$ *for any* $r < -1$ *and positive integer* k*.*

The following lemma will play an important role in the proof of our main results. The proof is due to Shao [10].

Lemma 1.6. *Let* $\{X_i, 1 \leq i \leq n\}$ *be a sequence of* φ*-mixing random variables with mean zero. Suppose that there exists a sequence* $\{C_n\}$ *of positive numbers such that* $E(\sum_{i=k+1}^{k+m} X_i)^2 \leq C_n$ *for any* $k \geq 0, n \geq 1, m \leq n$*. Then for any* $q \geq 2$*, there exists* $C = C(q, \varphi(\cdot))$ *such that*

$$E\max_{1\leq j\leq n}\left|\sum_{i=k+1}^{k+j} X_i\right|^q \leq C\left[C_n^{q/2} + E\max_{k+1\leq i\leq k+n}|X_i|^q\right]. \tag{1.7}$$

Lemma 1.7. *Let* $\{X_i, 1 \leq i \leq n\}$ *be a sequence of* φ*-mixing random variables with* $\sum_{i=1}^{\infty} \varphi^{1/2}(i) < \infty$*, then there exists* C *such that for any* $k \geq 0$ *and* $n \geq 1$

$$E\left(\sum_{i=k+1}^{k+n} X_i\right)^2 \leq C\sum_{i=k+1}^{k+n} EX_i^2. \tag{1.8}$$

Proof. By Lemma 5.4.4 in [11] and Hölder's inequality, we have

$$
\begin{aligned}
E\left(\sum_{i=k+1}^{k+n} X_i\right)^2 &= \sum_{i=k+1}^{k+n} EX_i^2 + 2\sum_{k+1\le i<j\le k+n} EX_iX_j \\
&\le \sum_{i=k+1}^{k+n} EX_i^2 + 4\sum_{k+1\le i<j\le k+n} \varphi^{1/2}(j-i)\left(EX_i^2\right)^{1/2}\left(EX_j^2\right)^{1/2} \\
&\le \sum_{i=k+1}^{k+n} EX_i^2 + 2\sum_{i=k+1}^{k+n-1}\sum_{j=i+1}^{k+n} \varphi^{1/2}(j-i)\left(EX_i^2+EX_j^2\right) \\
&\le \left(1+4\sum_{i=1}^{n}\varphi^{1/2}(i)\right)\sum_{i=k+1}^{k+n} EX_i^2.
\end{aligned}
\tag{1.9}
$$

Therefore, (1.8) holds. □

2. Main Results

Now we state our main results. The proofs will be given in Section 3.

Theorem 2.1. *Let $\{X_{ni}, i \ge 1, n \ge 1\}$ be an array of rowwise φ-mixing random variables with $EX_{ni} = 0$, $\{X_{ni}\} \prec X$ and $\sum_{m=1}^{\infty}\varphi^{1/2}(m) < \infty$. Let $l(x) > 0$ be a slowing varying function, and $\{a_{ni}, i \ge 1, n \ge 1\}$ be an array of constants such that*

$$
\begin{aligned}
&\sup_{i\ge 1}|a_{ni}| = O\left(n^{-r}\right), \quad for\ some\ r > 0, \\
&\sum_{i=1}^{\infty}|a_{ni}| = O(n^{\alpha}), \quad for\ some\ \alpha \in [0, r).
\end{aligned}
\tag{2.1}
$$

(a) *If $\alpha + \beta + 1 > 0$ and there exists some $\delta > 0$ such that $(\alpha/r) + 1 < \delta \le 2$, and $s = \max(1 + ((\alpha+\beta+1)/r), \delta)$, then $E|X|^s l(|X|^{1/r}) < \infty$ implies*

$$
\sum_{n=1}^{\infty} n^{\beta} l(n) E\left[\sup_{k\ge 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right| - \epsilon\right]^{+} < \infty, \quad \forall\epsilon > 0. \tag{2.2}
$$

(b) *If $\beta = -1, \alpha > 0$, then $E|X|^{1+(\alpha/r)}(1 + l(|X|^{1/r})) < \infty$ implies*

$$
\sum_{n=1}^{\infty} n^{-1} l(n) E\left[\sup_{k\ge 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right| - \epsilon\right]^{+} < \infty, \quad \forall\epsilon > 0. \tag{2.3}
$$

Remark 2.2. If $\alpha+\beta+1<0$, then $E|X|<\infty$ implies that (2.2) holds. In fact,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\beta} l(n) E\left[\sup_{k\geq 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|-\epsilon\right]^{+} &\leq \sum_{n=1}^{\infty} n^{\beta} l(n)\sum_{i=1}^{\infty}|a_{ni}|E|X_{ni}|+\epsilon\sum_{n=1}^{\infty} n^{\beta} l(n)\\ &\leq C\sum_{n=1}^{\infty} n^{\beta+\alpha} l(n) E|X|+\epsilon\sum_{n=1}^{\infty} n^{\beta} l(n)<\infty.\end{aligned} \tag{2.4}$$

Remark 2.3. Note that

$$\begin{aligned}\infty>\sum_{n=1}^{\infty} n^{\beta} l(n) E\left[\sup_{k\geq 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|-\epsilon\right]^{+} &= \sum_{n=1}^{\infty} n^{\beta} l(n)\int_{0}^{\infty} P\left\{\sup_{k\geq 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|-\epsilon>x\right\}dx\\ &=\int_{0}^{\infty}\sum_{n=1}^{\infty} n^{\beta} l(n) P\left\{\sup_{k\geq 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|>x+\epsilon\right\}dx.\end{aligned} \tag{2.5}$$

Therefore, from (2.5), we obtain that the complete moment convergence implies the complete convergence, that is, under the conditions of Theorem 2.1, result (2.2) implies

$$\sum_{n=1}^{\infty} n^{\beta} l(n) P\left\{\sup_{k\geq 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|>\epsilon\right\}<\infty, \tag{2.6}$$

and (2.3) implies

$$\sum_{n=1}^{\infty} n^{-1} l(n) P\left\{\sup_{k\geq 1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|>\epsilon\right\}<\infty. \tag{2.7}$$

Corollary 2.4. *Under the conditions of Theorem 2.1,*

(1) *if* $\alpha+\beta+1>0$ *and there exists some* $\delta>0$ *such that* $(\alpha/r)+1<\delta\leq 2$, *and* $s=\max(1+((\alpha+\beta+1)/r),\delta)$, *then* $E|X|^{s} l(|X|^{1/r})<\infty$ *implies*

$$\sum_{n=1}^{\infty} n^{\beta} l(n) E\left[\left|\sum_{i=1}^{\infty} a_{ni}X_{ni}\right|-\epsilon\right]^{+}<\infty,\quad \forall\epsilon>0, \tag{2.8}$$

(2) *if* $\beta=-1,\alpha>0$, *then* $E|X|^{1+(\alpha/r)}(1+l(|X|^{1/r}))<\infty$ *implies*

$$\sum_{n=1}^{\infty} n^{-1} l(n) E\left[\left|\sum_{i=1}^{\infty} a_{ni}X_{ni}\right|-\epsilon\right]^{+}<\infty,\quad \forall\epsilon>0. \tag{2.9}$$

Corollary 2.5. *Let* $\{X_{ni}, i\geq 1, n\geq 1\}$ *be an array of rowwise* φ*-mixing random variables with* $EX_{ni}=0,\{X_{ni}\}\prec X$ *and* $\sum_{m=1}^{\infty}\varphi^{1/2}(m)<\infty$. *Suppose that* $l(x)>0$ *is a slowly varying function.*

(1) *Let* $p > 1$ *and* $1 \le t < 2$. *If* $E|X|^{pt} l(|X|^t) < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{p-2-(1/t)} l(n) E\left[\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_{ni}\right| - \epsilon n^{1/t}\right]^{+} < \infty, \quad \forall \epsilon > 0. \tag{2.10}$$

(2) *Let* $1 < t < 2$. *If* $E|X|^t[1 + l(|X|^t)] < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{-1-(1/t)} l(n) E\left[\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_{ni}\right| - \epsilon n^{1/t}\right]^{+} < \infty, \quad \forall \epsilon > 0. \tag{2.11}$$

Corollary 2.6. *Suppose that* $X_n = \sum_{i=-\infty}^{\infty} a_{i+n} Y_i$, $n \ge 1$, *where* $\{a_i, -\infty < i < \infty\}$ *is a sequence of real numbers with* $\sum_{-\infty}^{\infty} |a_i| < \infty$, *and* $\{Y_i, -\infty < i < \infty\}$ *is a sequence of* φ*-mixing random variables with* $EY_i = 0$, $\{Y_i\} \prec Y$ *and* $\sum_{m=1}^{\infty} \varphi^{1/2}(m) < \infty$. *Let* $l(x)$ *be a slowly varying function.*

(1) *Let* $1 \le t < 2, r \ge 1 + (t/2)$. *If* $E|Y|^r l(|Y|^t) < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{(r/t)-2-(1/t)} l(n) E\left[\left|\sum_{i=1}^{n} X_i\right| - \epsilon n^{1/t}\right]^{+} < \infty, \quad \forall \epsilon > 0. \tag{2.12}$$

(2) *Let* $1 < t < 2$. *If* $E|Y|^t[1 + l(|Y|^t)] < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{-1-(1/t)} l(n) E\left[\left|\sum_{i=1}^{n} X_i\right| - \epsilon n^{1/t}\right]^{+} < \infty, \quad \forall \epsilon > 0. \tag{2.13}$$

Remark 2.7. Corollary 2.6 obtains the result about the complete moment convergence of moving average processes based on a φ-mixing random sequence with different distributions. We extend the results of Chen et al. [12] from the complete convergence to the complete moment convergence. The result of Kim and Ko [9] is a special case of Corollary 2.6 (1). Moreover, our result covers the case of $r = t$, which was not considered by Kim and Ko.

3. Proofs of the Main Results

Proof of Theorem 2.1. Without loss of generality, we can assume

$$\sup_{i\ge 1} |a_{ni}| \le n^{-r}, \qquad \sum_{i=1}^{\infty} |a_{ni}| \le n^{\alpha}. \tag{3.1}$$

Let $S_{nk}(x) = \sum_{i=1}^{k} a_{ni}X_{ni}I(|a_{ni}X_{ni}| \le n^{-r}x)$ for any $k \ge 1$, $n \ge 1$, and $x \ge 0$. First note that $E|X|^{s}l(|X|^{1/r}) < \infty$ implies $E|X|^{t} < \infty$ for any $0 < t < s$. Therefore, for $x > n^{r}$,

$$\begin{aligned}
x^{-1}n^{r}\sup_{k\ge1}E|S_{nk}(x)| &= x^{-1}n^{r}\sup_{k\ge1}E\left|\sum_{i=1}^{k} a_{ni}X_{ni}I\left(|a_{ni}X_{ni}| > n^{-r}x\right)\right| \qquad (EX_{ni}=0)\\
&\le \sum_{i=1}^{\infty} E|a_{ni}X_{ni}|I\left(|a_{ni}X_{ni}| > n^{-r}x\right) \le \sum_{i=1}^{\infty} E|a_{ni}X|I\left(|a_{ni}X| > n^{-r}x\right)\\
&\le \sum_{i=1}^{\infty}|a_{ni}|E|X|I(|X|>x) \le n^{\alpha}E|X|I(|X|>x)\\
&\le x^{\alpha/r}E|X|I(|X|>x) \le E|X|^{1+(\alpha/r)}I(|X|>n^{r}) \longrightarrow 0 \quad \text{as } n\longrightarrow\infty.
\end{aligned} \tag{3.2}$$

Hence, for n large enough we have $\sup_{k\ge1}E|S_{nk}(x)| < (\epsilon/2)n^{-r}x$. Then

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta}l(n)E\left[\sup_{k\ge1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right| - \epsilon\right]^{+}\\
&\quad= \sum_{n=1}^{\infty} n^{\beta}l(n)\int_{\epsilon}^{\infty} P\left\{\sup_{k\ge1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right| \ge x\right\}dx\\
&\quad= \sum_{n=1}^{\infty} n^{\beta-r}l(n)\epsilon\int_{n^{r}}^{\infty} P\left\{\sup_{k\ge1}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right| \ge \epsilon n^{-r}x\right\}dx\\
&\quad\le C\sum_{n=1}^{\infty} n^{\beta-r}l(n)\int_{n^{r}}^{\infty} P\left\{\sup_{i}|a_{ni}X_{ni}| > n^{-r}x\right\}dx\\
&\qquad+ C\sum_{n=1}^{\infty} n^{\beta-r}l(n)\int_{n^{r}}^{\infty} P\left\{\sup_{k\ge1}|S_{nk}(x)-ES_{nk}(x)| \ge n^{-r}x\frac{\epsilon}{2}\right\}dx := I_1 + I_2.
\end{aligned} \tag{3.3}$$

Noting that $\alpha+\beta > -1$, by Lemma 1.5, Markov inequality, (1.6), and (3.1), we have

$$\begin{aligned}
I_1 &\le C\sum_{n=1}^{\infty} n^{\beta-r}l(n)\int_{n^{r}}^{\infty}\sum_{i=1}^{\infty} P\left\{|a_{ni}X_{ni}| > n^{-r}x\right\}dx\\
&\le C\sum_{n=1}^{\infty} n^{\beta-r}l(n)\int_{n^{r}}^{\infty} n^{r}x^{-1}\sum_{i=1}^{\infty} E|a_{ni}X_{ni}|I\left(|a_{ni}X_{ni}| > n^{-r}x\right)dx\\
&\le C\sum_{n=1}^{\infty} n^{\beta+\alpha}l(n)\int_{n^{r}}^{\infty} x^{-1}E|X|I(|X|>x)dx\\
&\le C\sum_{n=1}^{\infty} n^{\beta+\alpha}l(n)\sum_{k=n}^{\infty}\int_{k^{r}}^{k^{r+1}} x^{-1}E|X|I(|X|>x)dx
\end{aligned}$$

$$\leq C\sum_{n=1}^{\infty}n^{\beta+\alpha}l(n)\sum_{k=n}^{\infty}k^{-1}E|X|I(|X|>k^{r})\leq C\sum_{k=1}^{\infty}k^{-1}E|X|I(|X|>k^{r})\sum_{n=1}^{k}n^{\beta+\alpha}l(n)$$
$$\leq C\sum_{k=1}^{\infty}k^{\beta+\alpha}l(k)E|X|I(|X|>k^{r})\leq CE|X|^{1+((1+\alpha+\beta)/r)}l\left(|X|^{1/r}\right)<\infty. \tag{3.4}$$

Now we estimate I_2, noting that $\sum_{m=1}^{\infty}\varphi^{1/2}(m)<\infty$, by Lemma 1.7, we have

$$\begin{aligned}&\sup_{1\leq m<\infty}E\left(\sum_{i=1}^{m}a_{ni}X_{ni}I(|a_{ni}X_{ni}|\leq n^{-r}x)-E\sum_{i=1}^{m}a_{ni}X_{ni}I(|a_{ni}X_{ni}|\leq n^{-r}x)\right)^{2}\\&\quad\leq C\sum_{i=1}^{\infty}Ea_{ni}^{2}X_{ni}^{2}I(|a_{ni}X_{ni}|\leq n^{-r}x).\end{aligned}\tag{3.5}$$

By Lemma 1.6, Markov inequality, C_r inequality, and (1.5), for any $q\geq 2$, we have

$$\begin{aligned}&P\left\{\sup_{k\geq 1}|S_{nk}(x)-ES_{nk}(x)|\geq n^{-r}x\frac{\epsilon}{2}\right\}\leq Cn^{rq}x^{-q}E\sup_{k\geq 1}|S_{nk}(x)-ES_{nk}(x)|^{q}\\&\quad\leq Cn^{rq}x^{-q}\left[\left(\sum_{i=1}^{\infty}Ea_{ni}^{2}X_{ni}^{2}I(|a_{ni}X_{ni}|\leq n^{-r}x)\right)^{q/2}+\sum_{i=1}^{\infty}E|a_{ni}X_{ni}|^{q}I(|a_{ni}X_{ni}|\leq n^{-r}x)\right]\\&\quad\leq Cn^{rq}x^{-q}\left(\sum_{i=1}^{\infty}Ea_{ni}^{2}X^{2}I(|a_{ni}X|\leq n^{-r}x)\right)^{q/2}+Cn^{rq}x^{-q}\sum_{i=1}^{\infty}E|a_{ni}X|^{q}I(|a_{ni}X|\leq n^{-r}x)\\&\qquad+C\left(\sum_{i=1}^{\infty}P\{|a_{ni}X|>n^{-r}x\}\right)^{q/2}+C\sum_{i=1}^{\infty}P\{|a_{ni}X|>n^{-r}x\}\\&\quad:=J_1+J_2+J_3+J_4.\end{aligned}\tag{3.6}$$

So,

$$I_2\leq\sum_{n=1}^{\infty}n^{\beta-r}l(n)\int_{n^r}^{\infty}(J_1+J_2+J_3+J_4)dx. \tag{3.7}$$

From (3.4), we have $\sum_{n=1}^{\infty}n^{\beta-r}l(n)\int_{n^r}^{\infty}J_4dx<\infty$.

For J_1, we consider the following two cases.

If $s > 2$, then $EX^2 < \infty$. Taking $q \geq 2$ such that $\beta + (q(\alpha - r)/2) < -1$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta-r} l(n) \int_{n^r}^{\infty} J_1 dx \\
&\quad \leq C \sum_{n=1}^{\infty} n^{\beta-r+rq} l(n) \int_{n^r}^{\infty} x^{-q} \left(\sum_{i=1}^{\infty} a_{ni}^2 \right)^{q/2} dx \\
&\quad \leq C \sum_{n=1}^{\infty} n^{\beta-r+rq} l(n) n^{q(\alpha-r)/2} n^{r(-q+1)} \leq C \sum_{n=1}^{\infty} n^{\beta+(q(\alpha-r)/2)} l(n) < \infty.
\end{aligned} \tag{3.8}$$

If $s \leq 2$, we choose s' such that $1 + (\alpha/r) < s' < s$. Taking $q \geq 2$ such that $\beta + (qr/2)(1 + (\alpha/r) - s') < -1$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta-r} l(n) \int_{n^r}^{\infty} J_1 dx \\
&\quad \leq C \sum_{n=1}^{\infty} n^{\beta-r+rq} l(n) \int_{n^r}^{\infty} x^{-q} \left(\sum_{i=1}^{\infty} |a_{ni}| |a_{ni}|^{s'-1} E |a_{ni} X|^{2-s'} |X|^{s'} I(|a_{ni} X| \leq n^{-r} x) \right)^{q/2} dx \\
&\quad \leq C \sum_{n=1}^{\infty} n^{\beta-r+rq} l(n) n^{q\alpha/2} n^{-(qr/2)(s'-1)} \int_{n^r}^{\infty} x^{-q} (n^{-r} x)^{(q/2)(2-s')} dx \\
&\quad \leq C \sum_{n=1}^{\infty} n^{\beta+(qr/2)(1+(\alpha/r)-s')} l(n) < \infty.
\end{aligned} \tag{3.9}$$

So, $\sum_{n=1}^{\infty} n^{\beta-r} l(n) \int_{n^r}^{\infty} J_1 dx < \infty$.

Now, we estimate J_2. Set $I_{nj} = \{i \geq 1 \mid (n(j+1))^{-r} < |a_{ni}| \leq (nj)^{-r}\}$, $j = 1, 2, \ldots$. Then $\cup_{j \geq 1} I_{nj} = N$, where N is the set of positive integers. Note also that for all $k \geq 1, n \geq 1$,

$$\begin{aligned}
n^{\alpha} &\geq \sum_{i=1}^{\infty} |a_{ni}| = \sum_{j=1}^{\infty} \sum_{i \in I_{nj}} |a_{ni}| \\
&\geq \sum_{j=1}^{\infty} (\sharp I_{nj}) (n(j+1))^{-r} \geq n^{-r} \sum_{j=k}^{\infty} (\sharp I_{nj}) (j+1)^{-rq} (k+1)^{rq-r}.
\end{aligned} \tag{3.10}$$

Hence, we have

$$\sum_{j=k}^{\infty} (\sharp I_{nj}) j^{-rq} \leq C n^{\alpha+r} k^{r-rq}. \tag{3.11}$$

Note that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta-r} l(n) \int_{n^r}^{\infty} J_2 dx \\
&= C \sum_{n=1}^{\infty} n^{\beta-r+rq} l(n) \int_{n^r}^{\infty} x^{-q} \sum_{j=1}^{\infty} \sum_{i\in I_{nj}} E|a_{ni}X|^q I(|a_{ni}X| \le n^{-r}x) dx \\
&= C \sum_{n=1}^{\infty} n^{\beta-r+rq} l(n) \sum_{j=1}^{\infty} (\sharp I_{nj})(nj)^{-rq} \sum_{k=n}^{\infty} \int_{k^r}^{(k+1)^r} x^{-q} E|X|^q I(|X| \le x(j+1)^r) dx \\
&\le C \sum_{n=1}^{\infty} n^{\beta-r+rq} l(n) \sum_{j=1}^{\infty} (\sharp I_{nj})(nj)^{-rq} \sum_{k=n}^{\infty} k^{r(-q+1)-1} E|X|^q I(|X| \le (k+1)^r (j+1)^r) \\
&= C \sum_{n=1}^{\infty} n^{\beta-r} l(n) \sum_{k=n}^{\infty} k^{r(-q+1)-1} \sum_{j=1}^{\infty} (\sharp I_{nj}) j^{-rq} \sum_{i=0}^{(k+1)(j+1)-1} E|X|^q I(i^r < |X| \le (i+1)^r) \\
&\le C \sum_{n=1}^{\infty} n^{\beta-r} l(n) \sum_{k=n}^{\infty} k^{r(-q+1)-1} \sum_{j=1}^{\infty} (\sharp I_{nj}) j^{-rq} \sum_{i=0}^{2(k+1)-1} E|X|^q I(i^r < |X| \le (i+1)^r) \\
&\quad + C \sum_{n=1}^{\infty} n^{\beta-r} l(n) \sum_{k=n}^{\infty} k^{r(-q+1)-1} \sum_{j=1}^{\infty} (\sharp I_{nj}) j^{-rq} \sum_{i=2(k+1)}^{(k+1)(j+1)} E|X|^q I(i^r < |X| \le (i+1)^r) \\
&:= J_2' + J_2''.
\end{aligned} \tag{3.12}$$

Taking $q \ge 2$ large enough such that $\beta + \alpha - rq + r < -1$, for J_2', by Lemma 1.6 and (3.11), we get

$$\begin{aligned}
J_2' &\le C \sum_{n=1}^{\infty} n^{\beta-r} l(n) \sum_{k=n}^{\infty} k^{r(-q+1)-1} n^{\alpha+r} \sum_{i=0}^{2(k+1)-1} E|X|^q I(i^r < |X| \le (i+1)^r) \\
&= C \sum_{k=1}^{\infty} k^{r(-q+1)-1} \sum_{i=0}^{2(k+1)-1} E|X|^q I(i^r < |X| \le (i+1)^r) \sum_{n=1}^{k} n^{\beta+\alpha} l(n) \\
&\le C \sum_{k=1}^{\infty} k^{\beta+\alpha-rq+r} l(k) \sum_{i=0}^{2(k+1)-1} E|X|^q I(i^r < |X| \le (i+1)^r) \\
&\le C + C \sum_{i=3}^{\infty} E|X|^q I(i^r < |X| \le (i+1)^r) \sum_{k=[i/2]}^{\infty} k^{\beta+\alpha-rq+r} l(k) \\
&\le C + C \sum_{i=3}^{\infty} i^{\beta+\alpha-rq+r+1} l(i) E|X|^q I(i^r < |X| \le (i+1)^r) \le C + CE|X|^{1+((\beta+\alpha+1)/r)} l\left(|X|^{1/r}\right) < \infty.
\end{aligned} \tag{3.13}$$

For J_2'', we obtain

$$
\begin{aligned}
J_2'' &\le C\sum_{n=1}^{\infty} n^{\beta-r} l(n) \sum_{k=n}^{\infty} k^{r(-q+1)-1} \sum_{j=1}^{\infty} (\sharp I_{nj}) j^{-rq} \sum_{i=2(k+1)}^{(j+1)(k+1)} E|X|^q I\left(i^r < |X| \le (i+1)^r\right) \\
&\le C\sum_{n=1}^{\infty} n^{\beta-r} l(n) \sum_{k=n}^{\infty} k^{r(-q+1)-1} \sum_{i=2(k+1)}^{\infty} E|X|^q I\left(i^r < |X| \le (i+1)^r\right) \sum_{j=\left[i(k+1)^{-1}\right]-1}^{\infty} (\sharp I_{nj}) j^{-rq} \\
&\le C\sum_{n=1}^{\infty} n^{\beta-r} l(n) \sum_{k=n}^{\infty} k^{r(-q+1)-1} \sum_{i=2(k+1)}^{\infty} n^{r+\alpha} i^{r(1-q)} k^{-r(1-q)} E|X|^q I\left(i^r < |X| \le (i+1)^r\right) \\
&= C\sum_{k=1}^{\infty} k^{-1} \sum_{i=2(k+1)}^{\infty} i^{r(1-q)} E|X|^q I\left(i^r < |X| \le (i+1)^r\right) \sum_{n=1}^{k} n^{\beta+\alpha} l(n) \\
&\le C\sum_{k=1}^{\infty} k^{\beta+\alpha} l(k) \sum_{i=2(k+1)}^{\infty} i^{r(1-q)} E|X|^q I\left(i^r < |X| \le (i+1)^r\right) \\
&\le C\sum_{i=4}^{\infty} i^{\beta+\alpha+1+r-rq} E|X|^q I\left(i^r < |X| \le (i+1)^r\right) \le CE|X|^{1+((\beta+\alpha+1)/r)} l\left(|X|^{1/r}\right) < \infty.
\end{aligned}
\tag{3.14}
$$

So $\sum_{n=1}^{\infty} n^{\beta-r} l(n) \int_{n^r}^{\infty} J_2 dx < \infty$. Finally, we prove $\sum_{n=1}^{\infty} n^{\beta-r} l(n) \int_{n^r}^{\infty} J_3 dx < \infty$. In fact, noting $1 + (a/r) < s' < s$ and $\beta + (qr/2)(1 + (\alpha/r) - s') < -1$, using Markov inequality and (3.1), we get

$$
\begin{aligned}
\sum_{n=1}^{\infty} n^{\beta-r} l(n) \int_{n^r}^{\infty} J_3 dx &\le C\sum_{n=1}^{\infty} n^{\beta-r} l(n) \int_{n^r}^{\infty} \left(\sum_{i=1}^{\infty} n^{rs'} x^{-s'} E|a_{ni}X|^{s'}\right)^{q/2} dx \\
&\le C\sum_{n=1}^{\infty} n^{\beta-r} l(n) n^{qrs'/2} n^{-r(s'-1)(q/2)} n^{\alpha(q/2)} \int_{n^r}^{\infty} x^{-s'(q/2)} dx \\
&\le C\sum_{n=1}^{\infty} n^{\beta-r+r(q/2)+\alpha(q/2)} l(n) n^{r(-s'(q/2)+1)} \le C\sum_{n=1}^{\infty} n^{\beta+(qr/2)(1+(\alpha/2)-s')} l(n) < \infty.
\end{aligned}
\tag{3.15}
$$

Thus, we complete the proof in (a). Next, we prove (b). Note that $E|X|^{1+\alpha/r} < \infty$ implies that (3.2) holds. Therefore, from the proof in (a), to complete the proof of (b), we only need to prove

$$
I_2 = C\sum_{n=1}^{\infty} n^{-1-r} l(n) \int_{n^r}^{\infty} P\left\{\sup_{k\ge1} |S_{nk}(x) - ES_{nk}(x)| \ge n^{-r} x \frac{\epsilon}{2}\right\} dx < \infty. \tag{3.16}
$$

In fact, noting $\beta = -1$, $\alpha + \beta + 1 > 0$, $\alpha + \beta - r < -1$ and $E|X|^{1+\alpha/r} l(|X|^{1/r}) < \infty$. By taking $q = 2$ in the proof of (3.12), (3.13), and (3.14), we get

$$C \sum_{n=1}^{\infty} n^{-1+r} l(n) \int_{n^r}^{\infty} x^{-2} \sum_{i=1}^{\infty} E a_{ni}^2 X^2 I(|a_{ni} X| \leq n^{-r} x) dx \leq C + CE|X|^{1+(\alpha/r)} l\left(|X|^{1/r}\right) < \infty. \quad (3.17)$$

Then, by (3.17), we have

$$\begin{aligned}
I_2 &\leq C \sum_{n=1}^{\infty} n^{-1-r} l(n) \int_{n^r}^{\infty} n^{2r} x^{-2} E|S_{xn} - ES_{xn}|^2 dx \\
&\leq C \sum_{n=1}^{\infty} n^{-1+r} l(n) \int_{n^r}^{\infty} x^{-2} \sum_{i=1}^{\infty} E a_{ni}^2 X_{ni}^2 I(|a_{ni} X_{ni}| \leq n^{-r} x) dx \\
&\leq C \sum_{n=1}^{\infty} n^{-1+r} l(n) \int_{n^r}^{\infty} x^{-2} \sum_{i=1}^{\infty} E a_{ni}^2 X^2 I(|a_{ni} X| \leq n^{-r} x) dx \qquad (3.18) \\
&\quad + C \sum_{n=1}^{\infty} n^{-1-r} l(n) \int_{n^r}^{\infty} \sum_{i=1}^{\infty} P\{|a_{ni} X| > n^{-r} x\} dx \\
&\leq C \sum_{n=1}^{\infty} n^{-1+r} l(n) \int_{n^r}^{\infty} x^{-2} \sum_{i=1}^{\infty} E a_{ni}^2 X^2 I(|a_{ni} X| \leq n^{-r} x) dx + C < \infty.
\end{aligned}$$

The proof of Theorem 2.1 is completed. □

Proof of Corollary 2.4. Note that

$$\left[\left|\sum_{i=1}^{\infty} a_{ni} X_{ni}\right| - \epsilon\right]^{+} \leq \left[\sup_{k \geq 1} \left|\sum_{i=1}^{k} a_{ni} X_{ni}\right| - \epsilon\right]^{+}. \quad (3.19)$$

Therefore, (2.8) and (2.9) hold by Theorem 2.1. □

Proof of Corollary 2.5. By applying Theorem 2.1, taking $\beta = p - 2$, $a_{ni} = n^{-1/t}$ for $1 \leq i \leq n$, and $a_{ni} = 0$ for $i > n$, then we obtain (2.10). Similarly, taking $\beta = -1$, $a_{ni} = n^{-1/t}$ for $1 \leq i \leq n$, and $a_{ni} = 0$ for $i > n$, we obtain (2.11) by Theorem 2.1. □

Proof of Corollary 2.6. Let $X_{ni} = Y_i$ and $a_{ni} = n^{-1/t} \sum_{j=1}^{n} a_{i+j}$ for all $n \geq 1$, $-\infty < i < \infty$. Since $\sum_{-\infty}^{\infty} |a_i| < \infty$, we have $\sup_i |a_{ni}| = O(n^{-1/t})$ and $\sum_{i=-\infty}^{\infty} |a_{ni}| = O(n^{1-1/t})$. By applying Corollary 2.4, taking $\beta = (r/t) - 2$, $r = 1/t$, $\alpha = 1 - (1/t)$, we obtain

$$\sum_{n=1}^{\infty} n^{(r/t)-2-(1/t)} l(n) E\left[\left|\sum_{i=1}^{n} X_i\right| - \epsilon n^{1/t}\right]^{+} = \sum_{n=1}^{\infty} n^{\beta} l(n) E\left[\left|\sum_{i=-\infty}^{\infty} a_{ni} X_{ni}\right| - \epsilon\right]^{+} < \infty, \quad \forall \epsilon > 0. \quad (3.20)$$

Therefore, (2.12) and (2.13) hold. □

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