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Research Article

Examples of Rational Toral Rank Complex

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There is a CW complex $\mathcal{T}(X)$, which gives a rational homotopical classification of almost free toral actions on spaces in the rational homotopy type of X associated with rational toral ranks and also presents certain relations in them. We call it the *rational toral rank complex* of X. It represents a variety of toral actions. In this note, we will give effective 2-dimensional examples of it when X is a finite product of odd spheres. This is a combinatorial approach in rational homotopy theory.

1. Introduction

Let X be a simply connected CW complex with $\dim H^*(X;\mathbb{Q}) < \infty$ and $r_0(X)$ be the *rational toral rank* of X, which is the largest integer r such that an r-torus $T^r = S^1 \times \cdots \times S^1$ (r-factors) can act continuously on a CW-complex Y in the rational homotopy type of X with all its isotropy subgroups finite (such an action is called *almost free*) [1]. It is a very interesting rational invariant. For example, the inequality

$$r_0(X) = r_0(X) + r_0\left(S^{2n}\right) < r_0\left(X \times S^{2n}\right) \qquad (*)$$

can hold for a formal space X and an integer $n > 1$ [2]. It must appear as one phenomenon in a variety of almost free toral actions. The example $(*)$ is given due to Halperin by using *Sullivan minimal model* [3].

Put the Sullivan minimal model $M(X) = (\Lambda V, d)$ of X. If an r-torus T^r acts on X by $\mu : T^r \times X \to X$, there is a minimal KS extension with $|t_i| = 2$ for $i = 1, \ldots, r$

$$(\mathbb{Q}[t_1, \ldots, t_r], 0) \longrightarrow (\mathbb{Q}[t_1, \ldots, t_r] \otimes \wedge V, D) \longrightarrow (\wedge V, d) \qquad (1.1)$$

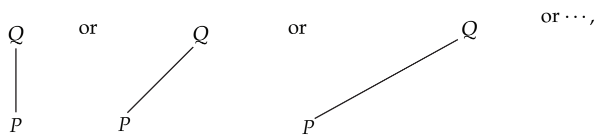
with $Dt_i = 0$ and $Dv \equiv dv$ modulo the ideal $(t_1, \dots, t_r)$ for $v \in V$ which is induced from the Borel fibration [4]

$$X \longrightarrow ET^r \times^{\mu}_{T^r} X \longrightarrow BT^r. \tag{1.2}$$

According to [1, Proposition 4.2], $r_0(X) \geq r$ if and only if there is a KS extension of above satisfying $\dim H^*(\mathbb{Q}[t_1, \dots, t_r] \otimes \wedge V, D) < \infty$. Moreover, then T^r acts freely on a finite complex that has the same rational homotopy type as X. So we will discuss this note by Sullivan models.

We want to give a classification of rationally almost free toral actions on X associated with rational toral ranks and also present certain relations in them. Recall a finite-based CW complex $\mathcal{T}(X)$ in [5, Section 5]. Put $\mathcal{X}_r = \{(\mathbb{Q}[t_1, \dots, t_r] \otimes \wedge V, D)\}$ the set of isomorphism classes of KS extensions of $M(X) = (\Lambda V, d)$ such that $\dim H^*(\mathbb{Q}[t_1, \dots, t_r] \otimes \wedge V, D) < \infty$. First, the set of 0-cells $\mathcal{T}_0(X)$ is the finite sets $\{(s, r) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}\}$ where the point $P_{s,r}$ of the coordinate (s, r) exists if there is a model $(\Lambda W, d_W) \in \mathcal{X}_r$ and $r_0(\Lambda W, d_W) = r_0(X) - s - r$. Of course, the model may not be uniquely determined. Note that the base point $P_{0,0} = (0, 0)$ always exists by X itself.

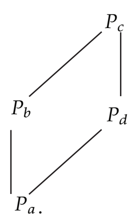
Next, 1-skeltons (vertexes) of the 1-skelton $\mathcal{T}_1(X)$ are represented by a KS-extension $(\mathbb{Q}[t], 0) \to (\mathbb{Q}[t] \otimes \Lambda W, D) \to (\Lambda W, d_W)$ with $\dim H^*(\mathbb{Q}[t] \otimes \Lambda W, D) < \infty$ for $(\Lambda W, d_W) \in \mathcal{X}_r$, where $W = \mathbb{Q}(t_1, \dots, t_r) \oplus V$ and $d_W|_V = d$. It is given as



where P exists by $(\Lambda W, d_W)$, and Q exists by $(\mathbb{Q}[t] \otimes \Lambda W, D)$. The 2 cell is given if there is a (homotopy) commutative diagram of restrictions

$$\begin{array}{ccc}
(\Lambda W, d_W) & \longleftarrow & (\mathbb{Q}[t_{r+2}] \otimes \Lambda W, D_{r+2}) \\
\uparrow & & \uparrow \\
(\mathbb{Q}[t_{r+1}] \otimes \Lambda W, D_{r+1}) & \longleftarrow & (\mathbb{Q}[t_{r+1}, t_{r+2}] \otimes \Lambda W, D)
\end{array}$$

which represents (a horizontal deformation of)



Here P_a exists by $(\Lambda W, d_W)$, P_b(or P_d) by $(\mathbb{Q}[t_{r+1}] \otimes \Lambda W, D_{r+1})$, P_c by $(\mathbb{Q}[t_{r+1}, t_{r+2}] \otimes \Lambda W, D)$, and P_d(or P_b) by $(\mathbb{Q}[t_{r+2}] \otimes \Lambda W, D_{r+2})$. Then we say that a 2 cell attaches to (the tetragon) $P_aP_bP_cP_d$. Thus, we can construct the 2-skelton $\mathcal{T}_2(X)$.

Generally, an n-cell is given by an n-cube where a vertex of $(\mathbb{Q}[t_{r+1},\ldots,t_{r+n}]\otimes\Lambda W, D)$ of height $r+n$, n-vertexes $\{(\mathbb{Q}[t_{r+1},\ldots,\overset{\vee}{t_{r+i}},\ldots,t_{r+n}]\otimes\Lambda W, D_{(i)})\}_{1\le i\le n}$ of height $r+n-1$, …, a vertex $(\Lambda W, d_W)$ of height r. Here $\vee$ is the symbol which removes the below element, and the differential $D_{(i)}$ is the restriction of D.

We will call this connected regular complex $\mathcal{T}(X)=\cup_{n\ge 0}\mathcal{T}_n(X)$ the *rational toral rank complex* (r.t.r.c.) of X. Since $r_0(X)<\infty$ in our case, it is a finite complex. For example, when $X=S^3\times S^3$ and $Y=S^5$, we have

$$\mathcal{T}(X)\vee\mathcal{T}(Y)=\mathcal{T}_1(X)\vee\mathcal{T}_1(Y)=\mathcal{T}_1(X\times Y)=\mathcal{T}(X\times Y), \tag{1.3}$$

which is an unusual case. Then, of course, $r_0(X)+r_0(Y)=r_0(X\times Y)$. Recall that $r_0(S^3\times S^3)+r_0(S^7)=r_0(S^3\times S^3\times S^7)$ but $\mathcal{T}_1(S^3\times S^3)\vee\mathcal{T}_1(S^7)\subsetneq\mathcal{T}_1(S^3\times S^3\times S^7)$ [5, Example 3.5]. In Section 2, we see that r.t.r.c. is not complicated as a CW complex but delicate. We see in Theorems 2.2 and 2.3 that the differences between $X=Z\times S^7$ and $Y=Z\times S^9$ for some products Z of odd spheres make certain different homotopy types of r.t.r.c., respectively. Remark that the above inequality $(*)$ is a property on $\mathcal{T}_0(X)$ or $\mathcal{T}_1(X)$ as the example of Theorem 2.4(1). We see in Theorem 2.4(2) an example that $\mathcal{T}_1(X)=\mathcal{T}_1(X\times\mathbb{C}P^n)$ but $\mathcal{T}_2(X)\subsetneq\mathcal{T}_2(X\times\mathbb{C}P^n)$, which is a higher-dimensional phenomenon of $(*)$.

2. Examples

In this section, the symbol $P_iP_jP_kP_l$ means the tetragon, which is the cycle with vertexes P_i, P_j, P_k, P_l, and edges P_iP_j, P_jP_k, P_kP_l, P_lP_i.

In general, it is difficult to show that a point of $\mathcal{T}_0(X)$ does not exist on a certain coordinate. So the following lemma is useful for our purpose.

Lemma 2.1. *If X has the rational homotopy type of the product of finite odd spheres and finite complex projective spaces, then $(1,r)\notin\mathcal{T}_0(X)$ for any r.*

Proof. Suppose that X has the rational homotopy type of the product of n odd spheres and m complex projective spaces. Put a minimal model $A=(\mathbb{Q}[t_1,\ldots,t_{n-1},x_1,\ldots,x_m]\otimes\Lambda(v_1,\ldots,v_n,y_1,\ldots,y_m),D)$ with $|t_1|=\cdots=|t_{n-1}|=|x_1|=\cdots=|x_m|=2$ and $|v_i|,|y_i|$ odd. If $\dim H^*(A)<\infty$, then A is pure; that is, $Dv_i, Dy_i\in\mathbb{Q}[t_1,\ldots,t_{n-1},x_1,\ldots,x_m]$ for all i. Therefore, from [2, Lemma 2.12], $r_0(A)=1$. Thus, we have $(1,r_0(X)-1)=(1,n-1)\notin\mathcal{T}_0(X)$. □

Theorem 2.2. *Put $X=S^3\times S^3\times S^3\times S^7\times S^7$ and $Y=S^3\times S^3\times S^3\times S^7\times S^9$. Then $\mathcal{T}_1(X)=\mathcal{T}_1(Y)$. But $\mathcal{T}(X)$ is contractible and $\mathcal{T}(Y)\simeq S^2$.*

Proof. Let $M(X)=(\Lambda V,0)=(\Lambda(v_1,v_2,v_3,v_4,v_5),0)$ with $|v_1|=|v_2|=|v_3|=3$ and $|v_4|=|v_5|=7$. Then

$$\mathcal{T}_0(X)=\{P_{0,0},P_{0,1},P_{0,2},P_{0,3},P_{0,4},P_{0,5},P_{2,1},P_{2,2},P_{2,3},P_{3,1},P_{3,2}\}. \tag{2.1}$$

For example, they are given as follows.

(0) $P_{0,0}$ is given by $(\Lambda V,0)$.

(1) $P_{0,1}$ is given by $(\mathbb{Q}[t_1]\otimes\Lambda V,D)$ with $Dv_1=t_1^2$ and $Dv_2=Dv_3=Dv_4=Dv_5=0$.

(2) $P_{0,2}$ is given by $(\mathbb{Q}[t_1,t_2] \otimes \Lambda V, D)$ with $Dv_1 = t_1^2$, $Dv_2 = t_2^2$, and $Dv_3 = Dv_4 = Dv_5 = 0$.

(3) $P_{0,3}$ is given by $(\mathbb{Q}[t_1,t_2,t_3] \otimes \Lambda V, D)$ with $Dv_1 = t_1^2$, $Dv_2 = t_2^2$, $Dv_3 = t_3^2$, and $Dv_4 = Dv_5 = 0$.

(4) $P_{0,4}$ is given by $(\mathbb{Q}[t_1,t_2,t_3,t_4] \otimes \Lambda V, D)$ with $Dv_1 = t_1^2$, $Dv_2 = t_2^2$, $Dv_3 = t_3^2$, $Dv_4 = t_4^4$, and $Dv_5 = 0$.

(5) $P_{0,5}$ is given by $(\mathbb{Q}[t_1,t_2,t_3,t_4,t_5] \otimes \Lambda V, D)$ with $Dv_1 = t_1^2$, $Dv_2 = t_2^2$, $Dv_3 = t_3^2$, $Dv_4 = t_4^4$, and $Dv_5 = t_5^4$.

(6) $P_{2,1}$ is given by $(\mathbb{Q}[t_1] \otimes \Lambda V, D)$ with $Dv_1 = Dv_2 = Dv_3 = Dv_5 = 0$ and $Dv_4 = v_1v_2t_1 + t_1^4$

(7) $P_{2,2}$ is given by $(\mathbb{Q}[t_1,t_2] \otimes \Lambda V, D)$ with $Dv_1 = Dv_2 = 0$, $Dv_3 = t_2^2$, $Dv_4 = v_1v_2t_1 + t_1^2$, and $Dv_5 = 0$.

(8) $P_{2,3}$ is given by $(\mathbb{Q}[t_1,t_2,t_3] \otimes \Lambda V, D)$ with $Dv_1 = Dv_2 = 0$, $Dv_3 = t_2^2$, $Dv_4 = t_1^2 + v_1v_2t_1$, and $Dv_5 = t_3^4$.

(9) $P_{3,1}$ is given by $(\mathbb{Q}[t_1] \otimes \Lambda V, D)$ with $Dv_1 = Dv_2 = Dv_3 = 0$, $Dv_4 = v_1v_2t_1 + t_1^4$, and $Dv_5 = v_1v_3t_1$.

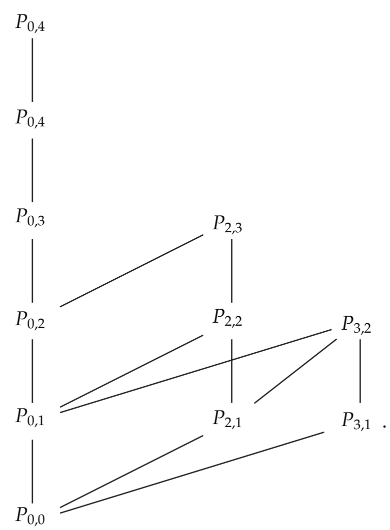
(10) $P_{3,2}$ is given by $(\mathbb{Q}[t_1,t_2] \otimes \Lambda V, D)$ with $Dv_4 = v_1v_2t_1 + t_1^4$ and $Dv_5 = v_1v_3t_1 + t_2^4$.

(11) $P_{4,1}$, that is, a point of the coordinate $(4,1)$ does not exist. Indeed, if it exists, it must be given by a model $(\mathbb{Q}[t_1] \otimes \Lambda V, D)$ whose differential is $Dv_1 = Dv_2 = Dv_3 = 0$ and $Dv_4, Dv_5 \in \mathbb{Q}[t_1] \otimes \Lambda(v_1,v_2,v_3)$ by degree reason. But, for any D satisfying such conditions, we have $\dim H^*(\mathbb{Q}[t_1,t_2] \otimes \Lambda V, \widetilde{D}) < \infty$ for a KS extension

$$(\mathbb{Q}[t_2],0) \longrightarrow \left(\mathbb{Q}[t_1,t_2] \otimes \Lambda V, \widetilde{D}\right) \longrightarrow (\mathbb{Q}[t_1] \otimes \Lambda V, D), \tag{2.2}$$

that is, $r_0(\mathbb{Q}[t_1] \otimes \Lambda V, D) > 0$. It contradicts the definition of $P_{4,1}$.

$\mathcal{T}_1(X)$ is given as



For example, the edges (1 simplexes)

$$\{P_{0,0}P_{0,1},\ P_{0,1}P_{0,2},\ P_{0,2}P_{0,3},\ P_{0,3}P_{0,4},\dots,\ P_{0,0}P_{3,1},\ P_{3,1}P_{3,2}\} \tag{2.3}$$

are given as follows.

(1) $P_{0,1}P_{3,2}$ is given by the projection $(\mathbb{Q}[t_1,t_2]\otimes\Lambda V,D)\to(\mathbb{Q}[t_1]\otimes\Lambda V,D_1)$ where $Dv_1=Dv_2=Dv_3=0$, $Dv_4=v_1v_2t_2+t_1^4$, $Dv_5=v_1v_3t_2+t_2^4$, and $D_1v_1=D_1v_2=D_1v_3=D_1v_5=0$ and $D_1v_4=t_1^4$.

(2) $P_{2,1}P_{3,2}$ is given by $Dv_1=Dv_2=Dv_3=0$, $Dv_4=v_1v_2t_1+t_1^4$, and $Dv_5=v_1v_3t_2+t_2^4$.

(3) $P_{3,1}P_{3,2}$ is given by $Dv_1=Dv_2=Dv_3=0$, $Dv_4=v_1v_2t_1+t_1^4$, and $Dv_5=v_1v_3t_1+t_2^4$.

$\mathcal{T}_2(X)$ is given as follows.

(1) $P_{0,0}P_{2,1}P_{3,2}P_{3,1}$ is attached by a 2 cell from $Dv_1=Dv_2=Dv_3=0$, $Dv_4=v_1v_2(t_1+t_2)+t_1^4$ and $Dv_5=v_1v_3t_2+t_2^4$. (Then $P_{2,1}$ is given by $D_1v_4=v_1v_2t_1+t_1^4$, $D_1v_5=0$, and $P_{3,1}$ is given by $D_2v_4=v_1v_2t_2$, $D_2v_5=v_1v_3t_2+t_2^4$.)

(2) $P_{0,0}P_{0,1}P_{3,2}P_{3,1}$ is attached by a 2 cell from $Dv_1=Dv_2=Dv_3=0$, $Dv_4=v_1v_2t_2+t_1^4$, and $Dv_5=v_1v_3t_2+t_2^4$.

(3) $P_{0,0}P_{0,1}P_{2,2}P_{2,1}$ is attached by a 2 cell from $Dv_1=Dv_2=Dv_3=0$, $Dv_4=v_1v_2t_2+t_2^4$, and $Dv_5=t_1^4$.

(4) $P_{0,1}P_{0,2}P_{2,3}P_{2,2}$ is attached by a 2 cell from $Dv_1=Dv_2=0$, $Dv_3=t_3^2$, $Dv_4=v_1v_2t_2+t_2^4$, and $Dv_5=t_1^4$.

(5) $P_{0,0}P_{0,1}P_{3,2}P_{2,1}$ is *not* attached by a 2 cell. Indeed, assume that a 2 cell attaches on it. Notice that $P_{3,2}$ is given by $(\mathbb{Q}[t_1,t_2]\otimes\Lambda V,D)$ with $Dv_1=Dv_2=Dv_3=0$ and

$$Dv_4=\alpha(v_1,v_2,v_3)+f,\qquad Dv_5=\beta(v_1,v_2,v_3)+g, \tag{2.4}$$

where $\alpha,\beta\in(v_1,v_2,v_3)$ and $\{f,g\}$ is a regular sequence in $\mathbb{Q}[t_1,t_2]$. Since $P_{0,1}P_{3,2}\in\mathcal{T}_1(X)$, both α and β must be contained in the ideal (t_i) for some i. Also they are not in (t_1t_2) by degree reason. Furthermore, since $P_{2,1}P_{3,2}\in\mathcal{T}_1(X)$, we can put that both α and β are contained in the monogenetic ideal (v_iv_j) for some $1\le i<j\le 3$ without losing generality. Then, $\dim H^*(\mathbb{Q}[t_1,t_2,t_3]\otimes\Lambda V,\widetilde{D})<\infty$ for a KS extension

$$(\mathbb{Q}[t_3],0)\longrightarrow\left(\mathbb{Q}[t_1,t_2,t_3]\otimes\Lambda V,\widetilde{D}\right)\longrightarrow(\mathbb{Q}[t_1,t_2]\otimes\Lambda V,D), \tag{2.5}$$

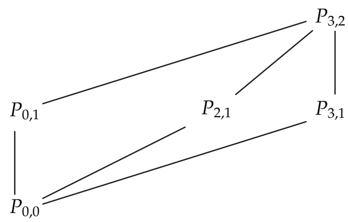
by putting $\widetilde{D}v_k=t_3^2$ for $k\in\{1,2,3\}$ with $k\neq i,j$ and $\widetilde{D}v_n=Dv_n$ for $n\neq k$. Thus, we have $r_0(\mathbb{Q}[t_1,t_2]\otimes\Lambda V,D)>0$. It contradicts to the definition of $P_{3,2}$.

Notice there is no 3 cell since it must attach to a 3 cube (in graphs) in general. Thus, we see that $\mathcal{T}(X)=\mathcal{T}_2(X)$ is contractible.

On the other hand, let $M(Y)=(\Lambda W,0)=(\Lambda(w_1,w_2,w_3,w_4,w_5),0)$ with $|w_1|=|w_2|=|w_3|=3$, $|w_4|=7$ and $|w_5|=9$. Then we see that $\mathcal{T}_1(X)=\mathcal{T}_1(Y)$ from same arguments. But, in $\mathcal{T}_2(Y)$, $P_{0,0}P_{0,1}P_{3,2}P_{2,1}$ is attached by a 2 cell since we can put $Dw_1=Dw_2=Dw_3=0$ and

$$Dw_4=w_1w_2t_2+t_2^4,\qquad Dw_5=w_1w_3t_1t_2+t_1^5, \tag{2.6}$$

by degree reason. Here $P_{0,1}$ is given by $D_1w_4 = 0$, $D_1w_5 = t_1^5$, and $P_{2,1}$ is given by $D_2w_4 = w_1w_2t_2 + t_2^4$, $D_2w_5 = 0$. Others are same as $\mathfrak{T}_2(X)$. Then three 2 cells on $P_{0,0}P_{0,1}P_{3,2}P_{2,1}$, $P_{0,0}P_{2,1}P_{3,2}P_{3,1}$, and $P_{0,0}P_{0,1}P_{3,2}P_{3,1}$ in $\mathfrak{T}_2(Y)$ make the following:



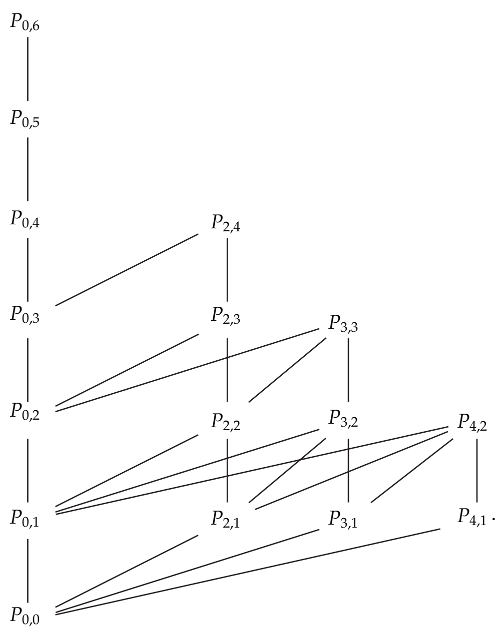
to be homeomorphic to S^2. Thus $\mathfrak{T}(Y) = \mathfrak{T}_2(Y) \simeq S^2$. □

Theorem 2.3. *Put* $X = S^3 \times S^3 \times S^3 \times S^3 \times S^7 \times S^7$ *and* $Y = S^3 \times S^3 \times S^3 \times S^3 \times S^7 \times S^9$. *Then* $\mathfrak{T}_1(X) = \mathfrak{T}_1(Y)$. *But* $\mathfrak{T}(X) \simeq S^2$ *and* $\mathfrak{T}(Y) \simeq \vee_{i=1}^6 S_i^2$.

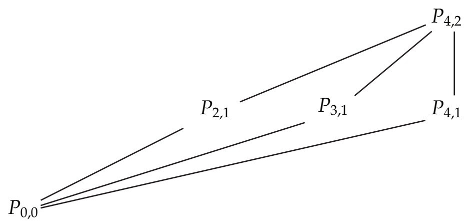
Proof. We see as the proof of Theorem 2.2 that

$$\mathfrak{T}_0(X) = \{P_{0,0}, P_{0,1}, P_{0,2}, P_{0,3}, P_{0,4}, P_{0,5}, P_{0,6}, P_{2,1}, P_{2,2}, P_{2,3}, P_{2,4}, P_{3,1}, P_{3,2}, P_{3,3}, P_{4,1}, P_{4,2}\} \quad (2.7)$$

and both $\mathfrak{T}_1(X)$ and $\mathfrak{T}_1(Y)$ are given as



For all tetragons in $\mathfrak{T}_1(X)$ except the following 4 tetragons:
(1) $P_{0,0}P_{0,1}P_{3,2}P_{2,1}$, (2) $P_{0,1}P_{0,2}P_{3,3}P_{2,2}$, (3) $P_{0,0}P_{0,1}P_{4,2}P_{2,1}$, and (4) $P_{0,0}P_{0,1}P_{4,2}P_{3,1}$, 2 cells attach in $\mathfrak{T}_2(X)$. The proof is similar to it of Theorem 2.2. Thus we see that $\mathfrak{T}_2(X)$ is homotopy equivalent to



which is homeomorphic to S^2. For example, when $M(X) = (\Lambda V, 0) = (\Lambda(v_1, v_2, v_3, v_4, v_5, v_6), 0)$ with $|v_1| = |v_2| = |v_3| = |v_4| = 3$ and $|v_5| = |v_6| = 7$, 2 cells attach $P_{0,0}P_{2,1}P_{4,2}P_{3,1}$, $P_{0,0}P_{3,1}P_{4,2}P_{4,1}$ and $P_{0,0}P_{2,1}P_{4,2}P_{4,1}$ from $Dv_1 = \cdots = Dv_4 = 0$,

$$\begin{aligned} &Dv_5 = v_1v_2t_1 + t_1^4, \qquad Dv_6 = v_1v_3t_1 + v_2v_4t_2 + t_2^4, \\ &Dv_5 = v_1v_2t_1 + t_1^4, \qquad Dv_6 = v_1v_3(t_1 + t_2) + v_2v_4t_2 + t_2^4, \\ &Dv_5 = v_1v_2t_1 + t_1^4, \qquad Dv_6 = v_1v_3t_2 + v_2v_4t_2 + t_2^4, \end{aligned} \tag{2.8}$$

respectively.

In $\mathcal{T}_2(Y)$, 2 cells attach all tetragons in $\mathcal{T}_1(Y)$ by degree reason. For example, when $M(Y) = (\Lambda W, 0) = (\Lambda(w_1, w_2, w_3, w_4, w_5, w_6), 0)$ with $|w_1| = |w_2| = |w_3| = |w_4| = 3$, $|w_5| = 7$ and $|w_6| = 9$, put $Dw_1 = Dw_2 = Dw_3 = 0$ and

(1) $Dw_4 = 0,\ Dw_5 = w_1w_3t_2 + t_2^4, Dw_6 = w_2w_3t_1t_2 + t_1^5$,

(2) $Dw_4 = t_3^2,\ Dw_5 = w_1w_3t_2 + t_2^4, Dw_6 = w_2w_3t_1t_2 + t_1^5$,

(3) $Dw_4 = 0,\ Dw_5 = w_1w_2t_2 + t_2^4,\ Dw_6 = w_3w_4t_1t_2 + t_1^5$,

(4) $Dw_4 = 0,\ Dw_5 = w_1w_3t_2 + t_2^4, Dw_6 = w_1w_4t_2^2 + w_2w_3t_1t_2 + t_1^5$,

for (1)~(4) of above. Then we can check that $\mathcal{T}(Y) \simeq \vee_{i=1}^6 S_i^2$ ($\mathcal{T}(Y)$ cannot be embedded in $\mathbb{R}^3$). □

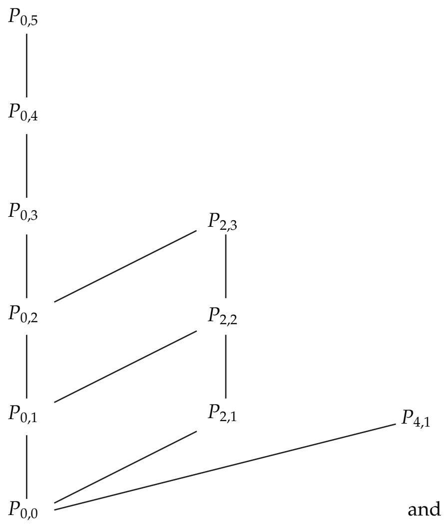
Theorem 2.4. *Even when $r_0(X) = r_0(X \times \mathbb{C}P^n)$ for the n-dimensional complex projective space $\mathbb{C}P^n$, it does not fold that $\mathcal{T}(X) = \mathcal{T}(X \times \mathbb{C}P^n)$ in general. For example,*

(1) *When $X = S^3 \times S^3 \times S^3 \times S^3 \times S^7$ and $n = 4$, then $\mathcal{T}_1(X) \subsetneq \mathcal{T}_1(X \times \mathbb{C}P^4)$.*

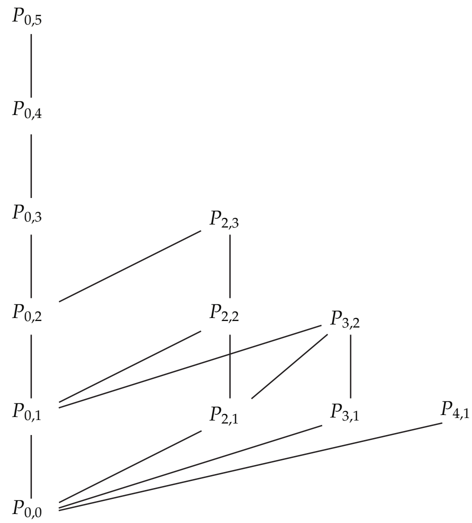
(2) *When $X = S^3 \times S^3 \times S^3 \times S^7 \times S^7$ and $n = 4$, then $\mathcal{T}_1(X) = \mathcal{T}_1(X \times \mathbb{C}P^4)$ but $\mathcal{T}_2(X) \subsetneq \mathcal{T}_2(X \times \mathbb{C}P^4)$.*

Proof. Put $M(\mathbb{C}P^n) = (\Lambda(x, y), d)$ with $dx = 0$ and $dy = x^{n+1}$ for $|x| = 2$ and $|y| = 2n + 1$. Put $(\mathbb{Q}[t_1, \ldots, t_r] \otimes \Lambda V \otimes \Lambda(x, y), D)$ the model of a Borel space $ET^r \times_{T^r} (X \times \mathbb{C}P^n)$ of $X \times \mathbb{C}P^n$.

(1) $\mathcal{T}_1(X)$ and $\mathcal{T}_1(X \times \mathbb{C}P^4)$ are given as



and



respectively. For $M(X) = (\Lambda V, 0) = (\Lambda(v_1, v_2, v_3, v_4, v_5), 0)$ with $|v_1| = |v_2| = |v_3| = |v_4| = 3$ and $|v_5| = 7$. Here $P_{4,1}$ is given by $Dv_i = 0$ for $i = 1, 2, 3, 4$ and $Dv_5 = v_1v_2t_1 + v_3v_4t_1 + t_1^4$. It is contained in both $\mathcal{T}_0(X)$ and $\mathcal{T}_0(X \times \mathbb{C}P^4)$. On the other hand, $P_{3,2}$ is given by $Dv_i = 0$ for $i = 1, 2, 3$, $Dv_4 = t_2^2$, $Dv_5 = v_1v_2t_1 + t_1^4$, $Dx = 0$, and $Dy = x^5 + v_1v_3t_1^2$. Then $P_{3,1}$ is given by $Dv_i = 0$ for $i = 1, 2, 3, 4$, $Dv_5 = v_1v_2t_1 + t_1^4$, $Dx = 0$, and $Dy = x^5 + v_1v_3t_1^2$. They are contained only in $\mathcal{T}_0(X \times \mathbb{C}P^4)$.

(2) Both $\mathcal{T}_1(X)$ and $\mathcal{T}_1(X \times \mathbb{C}P^4)$ are same as one in Theorem 2.2. Notice that $P_{0,0}P_{0,1}P_{3,2}P_{2,1}$ is attached by a 2 cell in $\mathcal{T}_2(X \times \mathbb{C}P^4)$ from $Dv_i = 0$ for $i = 1, 2, 3$, $Dv_4 = v_1v_2t_1 + t_1^4$, $Dv_5 = t_2^4$, $Dx = 0$, and $Dy = x^5 + v_1v_3t_1t_2$. So $\mathcal{T}(X \times \mathbb{C}P^4) = \mathcal{T}(Y)$ for $Y = S^3 \times S^3 \times S^3 \times S^7 \times S^9$. □

Remark 2.5. The author must mention about the spaces X_1 and X_2 in [5, Examples 3.8 and 3.9] such that $\mathcal{T}_1(X_1) = \mathcal{T}_1(X_2)$. We can check that 2 cells attach on both $P_0P_5P_9P_8$ of them (compare [5, page 506]).

Remark 2.6. In [5, Question 1.6], a rigidity problem is proposed. It says that does $\mathcal{T}_0(X)$ with coordinates determine $\mathcal{T}_1(X)$? For $\mathcal{T}(X)$, it is false as we see in above examples. But it seems that there are certain restrictions. For example, is $\mathcal{T}_2(X)$ simply connected?

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