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Research Article

Pullback Attractors for Nonclassical Diffusion Equations in Noncylindrical Domains

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The existence and uniqueness of a variational solution are proved for the following nonautonomous nonclassical diffusion equation $u_t - \varepsilon\Delta u_t - \Delta u + f(u) = g(x,t)$, $\varepsilon \in (0,1]$, in a noncylindrical domain with homogeneous Dirichlet boundary conditions, under the assumption that the spatial domains are bounded and increase with time. Moreover, the nonautonomous dynamical system generated by this class of solutions is shown to have a pullback attractor $\widehat{\mathcal{A}}_\varepsilon$, which is upper semicontinuous at $\varepsilon = 0$.

1. Introduction

In recent years, the evolution equations on noncylindrical domains, that is, spatial domains which vary in time so their Cartesian products with the time variable are noncylindrical sets, have been investigated extensively (see, e.g., [1–3]). Much of the progress has been made for nested spatial domains which expand in time. However, the results focus mainly on formulation of the problems and existence and uniqueness theory, while the existence of attractors of such systems has been less considered, except some recent works for the reaction-diffusion equation (or the heat equation) [4, 5]. This is not really surprising since such systems are intrinsically nonautonomous even if the equations themselves contain no time-dependent terms and require the concept of a nonautonomous attractor, which has only been introduced in recent years.

In this paper, we consider a class of nonautonomous nonclassical diffusion equations on bounded spatial domains which are expanding in time. First, we show how the first initial boundary value problem for these equations can be formulated as a variational problem with appropriate function spaces, and then we establish the existence and uniqueness over a finite time interval of variational solutions. Next, we show that the process of two parameter generated by such solutions has a nonautonomous pullback attractor. Finally, we study the upper semicontinuity of the obtained pullback attractor.

Let $\{\Omega_t\}_{t\in\mathbb{R}}$ be a family of nonempty bounded open subsets of $\mathbb{R}^N$ such that

$$s < t \Longrightarrow \Omega_s \subset \Omega_t. \tag{1.1}$$

From now on, we will frequently use the following notations:

$$\begin{gathered} Q_{\tau,T} := \bigcup_{t\in(\tau,T)} \Omega_t \times \{t\}, \\ Q_\tau := \bigcup_{t\in(\tau,\infty)} \Omega_t \times \{t\}, \quad \forall\tau \in \mathbb{R}, \\ \sum_{\tau,T} := \bigcup_{t\in(\tau,T)} \partial\Omega_t \times \{t\}, \\ \sum_{\tau} := \bigcup_{t\in(\tau,\infty)} \partial\Omega_t \times \{t\}. \end{gathered} \tag{1.2}$$

In this paper we consider the following nonautonomous equation:

$$(P_\epsilon) \begin{cases} u_t - \varepsilon\Delta u_t - \Delta u + f(u) = g(x,t) & \text{in } Q_\tau, \\ u = 0 & \text{on } \sum_\tau, \\ u|_{t=\tau} = u_\tau(x), & x \in \Omega_\tau, \end{cases} \tag{1.3}$$

where $\varepsilon \in [0,1]$, the nonlinear term f and the external force g satisfy some conditions specified later on. This equation is called the nonclassical diffusion equation when $\varepsilon > 0$, and when $\varepsilon = 0$, it turns to be the classical reaction-diffusion equation.

Nonclassical diffusion equation arises as a model to describe physical phenomena, such as non-Newtonian flows, soil mechanics, and heat conduction (see, e.g., [6–9]). In the last few years, the existence and long-time behavior of solutions to nonclassical diffusion equations has attracted the attention of many mathematicians. However, to the best of our knowledge, all existing results are devoted to the study of the equation in cylindrical domains. For example, under a Sobolev growth rate of the nonlinearity f, problem (1.3) in cylindrical domains has been studied [10–13] for the autonomous case, that is the case g not depending on time t and in [14, 15] for the nonautonomous case. In this paper, we will study the existence and long-time behavior of solutions to problem (1.3) in the case of noncylindrical domains, the nonlinearity f of polynomial type satisfying some dissipativity condition, and the external force g depending on time t. It is noticed that this question for problem (1.3) in the case $\varepsilon = 0$, that is, for the reaction-diffusion equation, has only been studied recently in [4, 5].

In order to study problem (1.3), we make the following assumptions.

(H1) The function $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies that

$$-\beta + \alpha_1 |s|^p \leq f(s)s \leq \beta + \alpha_2 |s|^p, \tag{1.4}$$

$$f'(s) \geq -\ell, \tag{1.5}$$

for some $p \geq 2$, where α_1, α_2, β, ℓ are nonnegative constants. By (1.4), there exist nonnegative constants $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$, $\widetilde{\beta}$ such that

$$-\widetilde{\beta} + \widetilde{\alpha}_1 |s|^p \leq F(s) \leq \widetilde{\beta} + \widetilde{\alpha}_2 |s|^p, \tag{1.6}$$

where $F(u) = \int_0^u f(s)ds$ is the primitive of f.

(H2) The external force $g \in L^2_{\text{loc}}(\mathbb{R}^{N+1})$.

(H3) The initial datum $u_\tau \in H_0^1(\Omega_\tau) \cap L^p(\Omega_\tau)$ is given.

Since the open set Ω_t changes with time t, problem (1.3) is nonautonomous even when the external force g is independent of time. Thus, in order to study the long-time behavior of solutions to (1.3), we use the theory of pullback attractors. This theory has been developed for both the nonautonomous and random dynamical systems and has shown to be very useful in the understanding of the dynamics of these dynamical systems (see [16] and references therein). The existence of a pullback attractor for problem (1.3) in the case $\varepsilon = 0$, that is, for the classical reaction-diffusion equation, has been derived recently in [4]. In the case $\varepsilon > 0$, since (1.3) contains the term $-\varepsilon\Delta u_t$, this is essentially different from the classical reaction-diffusion equation. For example, the reaction-diffusion equation has some kind of "regularity"; for example, although the initial datum only belongs to a weaker topology space, the solution will belong to a stronger topology space with higher regularity, and hence we can use the compact Sobolev embeddings to obtain the existence of attractors easily. However, for problem (1.3) when $\varepsilon > 0$, because of $-\Delta u_t$, if the initial datum u_τ belongs to $H_0^1(\Omega_\tau) \cap L^p(\Omega_\tau)$, the solution $u(t)$ with the initial condition $u(\tau) = u_\tau$ is always in $H_0^1(\Omega_t) \cap L^p(\Omega_t)$ and has no higher regularity, which is similar to hyperbolic equations. This brings some difficulty in establishing the existence of attractors for the nonclassical diffusion equations. Other difficulty arises since the considered domain is not cylindrical, so the standard techniques used for studying evolution equations in cylindrical domains cannot be used directly. Therefore, up to now, although there are many results on attractors for evolution equations in cylindrical domains (see, e.g., [17, 18]), little seems to be known for the equations in noncylindrical domains.

In this paper, we first exploit the penalty method to prove the existence and uniqueness of a variational solution satisfying the energy equality to problem (1.3). Next, we prove the existence of a pullback attractor $\widehat{\mathcal{A}}_\varepsilon$ for the process associated to problem (1.3). Finally, we study the continuous dependence on ε of the solutions to problem (1.3), in particular we show that the solutions of the nonclassical diffusion equations converge to the solution of the classical reaction-diffusion equation as $\varepsilon \to 0$. Hence using an abstract result derived recently by Carvalho et al. [19] and techniques similar to the ones used in [14], we prove the upper semicontinuity of pullback attractors $\widehat{\mathcal{A}}_\varepsilon$ in $L^2(\Omega_t)$ at $\varepsilon = 0$. The last result means that the pullback attractors $\widehat{\mathcal{A}}_\varepsilon$ of the nonclassical diffusion equations converge to the pullback attractor $\widehat{\mathcal{A}}_0$ of the classical reaction-diffusion equations as $\varepsilon \to 0$, in the sense of the Hausdorff semidistance.

The paper is organized as follows. In Section 2, for the convenience of readers, we recall some results on the penalty method and the theory of pullback attractors. After some preliminary results in Section 2, we proceed by a penalty method to solve approximated problem, and then we also prove the existence and uniqueness of the solution to problem (1.3) in Section 3. In Section 4, a uniform estimate for the solutions is then obtained under an additional assumption of the external force g, and this will lead to the proof of existence of a pullback attractor $\widehat{\mathcal{A}}_\varepsilon$ in an appropriate framework. The upper semicontinuity of pullback attractors $\widehat{\mathcal{A}}_\varepsilon$ at $\varepsilon = 0$ is investigated in Section 5. In the last section, we give some discussions and related open problems.

Notations. In what follows, we will introduce some notations which are frequently used in the paper. Denote $H_r := L^2(\Omega_r)$ and $V_r := H_0^1(\Omega_r)$ for each $r \in \mathbb{R}$, and denote by $(\cdot,\cdot)_r$ and $|\cdot|_r$ the usual inner product and associated norm in H_r and by $((\cdot,\cdot))$ and $\|\cdot\|_r$ the usual gradient inner product and associated norm in V_r. For each $s < t$, consider V_s as a closed subspace of V_t with the functions belonging to V_s being trivially extended by zero. It follows from (1.1) that $\{V_t\}_{t\in[\tau,T]}$ can be considered as a family of closed subspaces of V_T for each $T > \tau$ with

$$s < t \Longrightarrow V_s \subset V_t. \tag{1.7}$$

In addition, H_r will be identified with its topological dual H_r^* by means of the Riesz theorem and V_r will be considered as a subspace of H_r^* with $v \in V_r$ identified with the element $f_v \in H_r^*$ defined by

$$f_v(h) = (v,h)_r, \quad h \in H_r. \tag{1.8}$$

The duality product between V_r^* and V_r will be denoted by $\langle\cdot,\cdot\rangle$.

2. Preliminaries

2.1. Penalty Method

To study problem (1.3), for each $T > \tau$, we consider the following auxiliary problem:

$$\begin{gathered} u_t - \varepsilon\Delta u_t - \Delta u + f(u) = g(x,t) \quad \text{in } Q_{\tau,T}, \\ u = 0 \quad \text{on } \sum_{\tau,T}, \\ u|_{t=\tau} = u_\tau(x), \quad x \in \Omega_\tau, \end{gathered} \tag{2.1}$$

where $\tau \in \mathbb{R}$, $u_\tau : \Omega_\tau \to \mathbb{R}$ and $g : Q_\tau \to \mathbb{R}$ are given functions.

The method of penalization due to Lions (see [20]) will be used to prove the existence and uniqueness of a solution to problem (2.1) satisfying an energy equality a.e. in (τ,T) and, as a consequence, the existence and uniqueness of a solution to problem (1.3) satisfying the energy equality a.e. in (τ, ∞). To begin, fix $T > \tau$ and for each $t \in [\tau, T]$ denote by

$$V_t^{\perp} := \{v \in V_T : ((v, \omega))_T = 0, \forall \omega \in V_t\} \tag{2.2}$$

the orthogonal subspace of V_t with respect the inner product in V_T and by $P(t) \in \mathcal{L}(V_T)$ the orthogonal projection operator from V_T onto $V_T^{\perp}$, which is defined as

$$P(t)v \in V_t^{\perp}, \qquad v - P(t)v \in V_t, \tag{2.3}$$

for each $v \in V_T$. Finally, define $P(t) = P(T)$ for all $t > T$ and observe that $P(T)$ is the zero of $\mathcal{L}(V_T)$.

We will now approximate $P(t)$ by operators which are more regular in time. Consider the family $p(t; \cdot, \cdot)$ of symmetric bilinear forms on V_T defined by

$$p(t; v, \omega) := ((P(t)v, \omega))_T, \quad \forall v, \omega \in V_T, \ \forall t \geq \tau. \tag{2.4}$$

It can be proved that the mapping $[\tau, \infty) \ni t \mapsto p(t; v, \omega) \in \mathbb{R}$ is measurable for all v, $\omega \in V_T$. Moreover, $|p(t; v, \omega)| \leq \|v\|_T \|\omega\|_T$. For each integer $k \geq 1$ and each $t \geq \tau$, we define

$$p_k(t; v, \omega) := k \int_0^{1/k} p(t + r; v, \omega) dr, \quad \forall v, \omega \in V_T, \ \forall t \geq \tau, \tag{2.5}$$

and denote by $P_k(t) \in \mathcal{L}(V_T)$ the associated operator defined by

$$((P_k(t)v, \omega))_T := p_k(t; v, \omega), \quad \forall v, \omega \in V_T, \ \forall t \geq \tau. \tag{2.6}$$

Lemma 2.1 (see [2, 4]). *For any integer $1 \leq h \leq k$, any $t \geq \tau$ and every $v, \omega \in V_T$,*

$$\begin{gathered} p_k(t; v, \omega) = p_k(t; \omega, v), \\ 0 \leq p_h(t; v, v) \leq p_k(t; v, v) \leq p(t; v, v) = \|P(t)v\|_T^2 \leq \|v\|_T^2, \\ p_k'(t; v, v) := \frac{d}{dt} p_k(t; v, v) = k\left(p\left(t + \frac{1}{k}; v, v\right) - p(t; v, v)\right) \leq 0, \\ ((P_k(t)v, z))_T = 0, \quad \forall z \in V_t. \end{gathered} \tag{2.7}$$

Moreover, for every sequence $\{v_k\} \subset L^2(\tau, T; V_T)$ weak convergent to v in $L^2(\tau, T; V_T)$,

$$\liminf_{k \to +\infty} \int_\tau^T p_k(t; v_k(t), v_k(t)) dt \geq \int_\tau^T p(t; v(t), v(t)) dt. \tag{2.8}$$

Let $J : V_T \to V_T^*$ be the Riesz isomorphism defined by

$$\langle Jv, \omega \rangle_T := ((v, \omega))_T, \quad \forall v, \omega \in V_T, \tag{2.9}$$

and for each integer $k \geq 1$ and each $t \in [\tau, T]$, we denote

$$A_k(t) := -\Delta + kJP_k(t). \tag{2.10}$$

Obviously, $A_k(t) \in \mathcal{L}(V_T, V_T^*)$, $t \in [\tau, T]$, is a family of symmetric linear operators such that the mapping $t \in [\tau, T] \mapsto A_k(t) \in \mathcal{L}(V_T, V_T^*)$ is measurable, bounded, and satisfies

$$\langle A_k(t)v, v\rangle_T \geq \|v\|_T^2, \quad \forall v \in V_T,\ \forall t \in [\tau, T]. \tag{2.11}$$

Let $u_\tau \in V_T$ be given and for each $k \geq 1$ consider the following problem:

$$\begin{gathered}\left(u_k'(t), v\right)_T + \langle A_k(t)u_k(t), v\rangle_T \\ + \varepsilon\left\langle A_k(t)u_k'(t), v\right\rangle_T + \left(f(u_k(t)), v\right)_T = \left(g(t), v\right)_T, \quad \forall v \in V_T, \\ ((u_k(\tau), v))_T = ((u_\tau, v))_T.\end{gathered} \tag{2.12}$$

The idea of the penalty method is as follows: for each $k \geq 1$ we first prove the existence of a solution u_k to problem (2.12) (a problem in a cylindrical domain) using standard methods such as the Galerkin method, and then show that u_k converges to a solution to problem (2.1) (a problem in a noncylindrical domain) in some suitable sense, and as a consequence, the existence of a solution to problem (1.3) (see Section 3 for details).

2.2. Pullback Attractors

Since the open set Ω_t changes with time t, problem (1.3) is nonautonomous even when the external force g is independent of time. Thus, in order to study the long-time behavior of solutions to (1.3), we use the theory of pullback $\mathfrak{D}$-attractors which is a modification of the theory in [16].

Consider a process $U(\cdot,\cdot)$ on a family of metric spaces $\{(X_t, d_t); t \in \mathbb{R}\}$, that is, a family $\{U(t,\tau); -\infty < \tau \leq t < +\infty\}$ of mappings $U(t,\tau) : X_\tau \to X_t$ such that $U(\tau,\tau)x = x$ for all $x \in X_\tau$ and

$$U(t,\tau) = U(t,r)U(r,\tau) \quad \forall \tau \leq r \leq t. \tag{2.13}$$

In addition, suppose $\mathfrak{D}$ is a nonempty class of parameterized sets of the form $\widehat{\mathfrak{D}} = \{D(t); D(t) \subset X_t, D(t) \neq \emptyset, t \in \mathbb{R}\}$.

Definition 2.2 (see [4]). The process $U(\cdot,\cdot)$ is said to be pullback $\mathfrak{D}$-asymptotically compact if the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X_t for any $t \in \mathbb{R}$, any $\widehat{\mathfrak{D}} \in \mathfrak{D}$, and any sequences $\{\tau_n\}$ and $\{x_n\}$ with $\tau_n \to -\infty$ and $x_n \in D(\tau_n)$.

Definition 2.3 (see [4]). A family $\widehat{\mathcal{B}} \in \mathfrak{D}$ is said to be pullback $\mathfrak{D}$-absorbing for the process $U(\cdot,\cdot)$ if for any $t \in \mathbb{R}$ and any $\widehat{\mathfrak{D}} \in \mathfrak{D}$, there exists $\tau_0(t, \widehat{\mathfrak{D}}) \leq t$ such that

$$U(t,\tau)\mathfrak{D}(\tau) \subset \mathcal{B}(t), \tag{2.14}$$

for all $\tau \leq \tau_0(t, \widehat{\mathfrak{D}})$.

Remark 2.4. Note that if $\hat{\mathcal{B}} \in \mathfrak{D}$ is pullback $\mathfrak{D}$-absorbing for the process $U(\cdot,\cdot)$ and $B(t)$ is a compact subset of X_t for any $t \in \mathbb{R}$, then the process $U(\cdot,\cdot)$ is pullback $\mathfrak{D}$-asymptotically compact.

For each $t \in \mathbb{R}$, let $\operatorname{dist}_t(D_1, D_2)$ be the Hausdorff semi-distance between nonempty subsets D_1 and D_2 of X_t, which is defined as

$$\operatorname{dist}_t(D_1, D_2) = \sup_{x \in D_1} \inf_{y \in D_2} d_{X_t}(x, y) \quad \text{for } D_1, D_2 \subset X_t. \tag{2.15}$$

Definition 2.5 (see [4]). The family $\hat{\mathcal{A}} = \{A(t); A(t) \subset X_t, A(t) \neq \emptyset, t \in \mathbb{R}\}$ is said to be a pullback $\mathfrak{D}$-attractor for $U(\cdot,\cdot)$ if

(1) $a(t)$ is a compact set of X_t for all $t \in \mathbb{R}$,

(2) $\hat{\mathcal{A}}$ is pullback $\mathfrak{D}$-attracting, that is,

$$\lim_{\tau \to -\infty} \operatorname{dist}_t(U(t,\tau)D(\tau), A(t)) = 0 \quad \forall \hat{\mathfrak{D}} \in \mathfrak{D}, \forall t \in \mathbb{R}, \tag{2.16}$$

(3) $\hat{\mathcal{A}}$ is invariant, that is,

$$U(t,\tau)A(\tau) = A(t) \quad \text{for} - \infty < \tau \le t < +\infty. \tag{2.17}$$

Theorem 2.6 (see [4]). *Suppose that the process $U(\cdot,\cdot)$ is pullback $\mathfrak{D}$-asymptotically compact and that $\hat{B} \in \mathfrak{D}$ is a family of pullback $\mathfrak{D}$-absorbing sets for $U(\cdot,\cdot)$. Then, the family $\hat{\mathcal{A}} = \{A(t); t \in \mathbb{R}\}$ defined by $A(t) := \Lambda(\hat{B}, t)$, $t \in \mathbb{R}$, where for each $\hat{D} \in \mathfrak{D}$ and $t \in \mathbb{R}$,*

$$\Lambda\left(\hat{D}, t\right) := \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} U(t,\tau)D(\tau)}^{X_t} \ (\textit{closure in } X_t) \tag{2.18}$$

is a pullback $\mathfrak{D}$-attractor for $U(\cdot,\cdot)$, which in addition satisfies

$$A(t) = \overline{\bigcup_{\hat{D} \in \mathfrak{D}} \Lambda\left(\hat{D}, t\right)}^{X_t}. \tag{2.19}$$

Furthermore, $\hat{\mathcal{A}}$ is minimal in the sense that if $\hat{C} = \{C(t); t \in \mathbb{R}\}$ is a family of nonempty sets such that $C(t)$ is a closed subset of X_t and

$$\lim_{\tau \to -\infty} \operatorname{dist}_t(U(t,\tau)B(\tau), C(t)) = 0 \quad \text{for any } \ t \in \mathbb{R}, \ \textit{then } A(t) \subset C(t) \ \forall t \in \mathbb{R}. \tag{2.20}$$

2.3. The Upper Semicontinuity of Pullback Attractors

We now state some results on upper semicontinuity of pullback attractors, which are slight modifications of those in [19]. Because the proof is very similar to the one in [19], so we omit it here.

Definition 2.7. Let $\{U_\varepsilon(\cdot,\cdot) : \varepsilon \in [0,1]\}$ be a family of evolution processes in a family of Banach spaces $\{X_t\}$ with corresponding pullback $\mathfrak{D}$-attractors $\{A_\varepsilon(t) : \varepsilon \in [0,1],\ t \in \mathbb{R}\}$. For any bounded interval $I \in \mathbb{R}$, we say that $\{A_\varepsilon(\cdot)\}$ is upper semicontinuous at $\varepsilon = 0$ for $t \in I$ if

$$\limsup_{\varepsilon \to 0}\sup_{t\in I} \mathrm{dist}_t(A_\varepsilon(t), A_0(t)) = 0. \tag{2.21}$$

Theorem 2.8. *Let $\{U_\varepsilon(\cdot,\cdot) : \varepsilon \in [0,1]\}$ be a family of processes with corresponding pullback $\mathfrak{D}$-attractors $\{A_\varepsilon(\cdot) : \varepsilon \in [0,1]\}$. Then, for any bounded interval $I \subset \mathbb{R}$, $\{A_\varepsilon(\cdot) : \varepsilon \in [0,1]\}$ is upper semicontinuous at 0 for $t \in I$ if for each $t \in \mathbb{R}$, for each $T > 0$, and for each compact subset K of $X_{t-\tau}$, the following conditions hold:*

(i) $\sup_{\tau\in[0,T]}\sup_{x\in K} \mathrm{dist}_t(U_\varepsilon(t, t-\tau)x, U_0(t, t-\tau)x) \to 0$ *as* $\varepsilon \to 0$,

(ii) $\bigcup_{\varepsilon\in[0,1]} \bigcup_{t\le t_0} A_\varepsilon(t)$ *is bounded for given* t_0,

(iii) $\overline{\bigcup_{0<\varepsilon\le 1} A_\varepsilon(t)}$ *is compact for each* $t \in \mathbb{R}$.

3. Existence and Uniqueness of Variational Solutions

For each $T > \tau$, denote

$$\begin{aligned} \widetilde{Q}_{\tau,T} &:= \Omega_T \times (\tau, T), \\ U_{\tau,T} &:= \Big\{\Phi \in L^\infty(\tau, T; V_T) \cap L^p\Big(\widetilde{Q}_{\tau,T}\Big), \Phi' \in L^2(\tau, T; V_T), \\ &\qquad \Phi(\tau) = \Phi(T) = 0, \Phi(t) \in V_t \text{ a.e. in } (\tau, T)\Big\}. \end{aligned} \tag{3.1}$$

Definition 3.1. A variational solution of (2.1) is a function u such that

(C1) $u \in L^\infty(\tau, T, V_T) \cap L^p(\widetilde{Q}_{\tau,T})$, $u' \in L^2(\tau, T; V_T)$,

(C2) for all $\Phi \in U_{\tau,T}$,

$$\begin{aligned} &\int_\tau^T \left[-\left(u(t), \Phi'(t)\right)_T + ((u(t), \Phi(t)))_T + \varepsilon\left(\left(u'(t), \Phi(t)\right)\right)_T + \left(f(u), \Phi(t)\right)_T\right] dt \\ &\quad = \int_\tau^T \left(g(t), \Phi(t)\right)_T dt, \end{aligned} \tag{3.2}$$

(C3) $u(t) \in V_t$ a.e. in (τ, T),

(C4) $\lim_{t\downarrow\tau}(t-\tau)^{-1} \int_\tau^t |u(r) - u_\tau|_T^2\, dr = 0$.

Remark 3.2. If $T_2 > T_1 > \tau$ and u is a variational solution of (2.1) with $T = T_2$, then the restriction of u to $\widetilde{Q}_{\tau,T_1}$ is a variational solution of (2.1) with $T = T_1$.

Denote $\widetilde{Q}_\tau := \mathrm{U}_{T>\tau}\widetilde{Q}_{(\tau,T)}$.

Definition 3.3. A variational solution of (1.3) is a function $u : \widetilde{Q}_\tau \to \mathbb{R}$ such that for each $T > \tau$, its restriction to $\widetilde{Q}_{\tau,T}$ is a variational solution of (2.1).

To prove the uniqueness of variational solutions to problem (2.1), we need the following lemmas.

Lemma 3.4 (see [4]). *Assume that* $v \in L^2(\tau, T, V_T) \cap L^p(\widetilde{Q}_{\tau,T})$ *and there exist* $\xi \in L^2(\tau, T, V_T^*)$ *and* $\eta \in L^{p/p-1}(\widetilde{Q}_{\tau,T})$ *such that*

$$\int_\tau^T \left(v(t), \Phi'(t)\right)_T dt = -\int_\tau^T \langle \xi(t), \Phi(t) \rangle_T dt - \int_\tau^T \left(\eta(t), \Phi(t)\right)_T dt, \tag{3.3}$$

for every function $\Phi \in U_{\tau,T}$.

For each $0 < h < T - \tau$, *define* v_h *by*

$$v_h := \begin{cases} h^{-1}(v(t+h) - v(t)) & \text{a.e. in } (\tau, T-h), \\ 0 & \text{a.e. in } (T-h, T). \end{cases} \tag{3.4}$$

Then

$$\lim_{h \downarrow 0} \int_\tau^T (v_h(t), \omega(t))_T dt = \int_\tau^T \langle \xi(t), \omega(t) \rangle_T dt + \int_\tau^T \left(\eta(t), \omega(t)\right)_T dt, \tag{3.5}$$

for every function $\omega \in L^2(\tau, T; V_T) \cap L^p(\widetilde{Q}_{\tau,T})$ *such that* $\omega(t) \in V_t$ *a.e. in* (τ, T).

Remark 3.5. If $\tau < T' < T$ and $\Phi \in L^2(\tau, T'; V_T) \cap L^p(\Omega_T \times (\tau, T'))$, with $\Phi' \in L^2(\tau, T'; H_T)$ satisfies $\Phi(\tau) = \Phi(T') = 0$ and $\Phi(t) \in V_t$ a.e. in (τ, T'), then the trivial extension $\widetilde{\Phi}$ of Φ satisfies $\widetilde{\Phi} \in U_{\tau,T}$, with $(\widetilde{\Phi})' = \widetilde{\Phi}'$. Using the open sets $\widehat{\Omega}_t := \Omega_{t+T-T'}, \tau \leq t \leq T'$, it is easy to see that under the conditions of (3.5), one also has

$$\lim_{h \downarrow 0} \int_\tau^{T'-h} (v_h(t), \omega(t))_T dt = \int_\tau^{T'} \langle \xi(t), \omega(t) \rangle_T dt + \int_\tau^{T'} \left(\eta(t), \omega(t)\right)_T dt, \tag{3.6}$$

for every $\tau \leq T' \leq T$ and every function $\omega \in L^2(\tau, T; V_T) \cap L^p(\widetilde{Q}_{\tau,T})$ such that $\omega(t) \in V_t$ a.e. in (τ, T).

Lemma 3.6 (see [4]). *Let* $v_i \in L^2(\tau, T; V_T) \cap L^p(\widetilde{Q}_{\tau,T})$, $i = 1, 2$, *be two functions such that* $v_i(t) \in V_t$ *a.e. in* (τ, T) *for* $i = 1, 2$. *Assume that there exist* $\xi_i \in L^2(\tau, T; V_T^*), \eta_i \in L^{p/p-1}(\widetilde{Q}_{\tau,T})$, $i = 1, 2$ *such that*

$$\int_\tau^T \left(v_i(t), \Phi'(t)\right)_T dt = -\int_\tau^T \langle \xi_i(t), \Phi(t) \rangle_T dt - \int_\tau^T \left(\eta_i(t), \Phi(t)\right)_T dt, \quad i = 1, 2, \tag{3.7}$$

for every function $\Phi \in U_{\tau,T}$. *Then, for every pair* $\tau \leq s < t \leq T$ *of Lebesgue points of the inner product function* $(v_1, v_2)_T$ *it holds*

$$\begin{aligned}(v_1(t), v_2(t))_T - (v_1(s), v_2(s))_T = &\int_s^t \langle \xi_1(r), v_2(r)\rangle_T dr + \int_s^t \langle \xi_2(r), v_1(r)\rangle_T dr \\ &+ \int_s^t (\eta_1(r), v_2(r))_T dr + \int_s^t (\eta_2(r), v_1(r))_T dr \\ &+ \lim_{h\downarrow 0} h^{-1} \int_s^{t-h} (v_1(r+h) - v_1(r), v_2(r+h) - v_2(r))_T dr.\end{aligned} \tag{3.8}$$

If u is a variational solution of problem (2.1), then τ is the Lebesgue point of $|u|_T^2$ since the condition (C4) is satisfied. The next corollary gives an obvious consequence of (3.8).

Corollary 3.7. *If u is a variational solution of* (2.1), *then for every Lebesgue point $t \in (\tau, T)$ of $|u|_T^2$ it holds*

$$\begin{aligned}&|u(t)|_T^2 + \varepsilon\|u(t)\|_T^2 + 2\int_\tau^t \|u(r)\|_T^2 dr + 2\int_\tau^t (f(u(r)), u(r))_T dr \\ &\quad = |u_\tau|_T^2 + \varepsilon\|u_\tau\|_T^2 + 2\int_\tau^t (g(r), u(r))_T dr + \lim_{h\downarrow 0} h^{-1}\int_\tau^{t-h} |u(r+h) - u(r)|_T^2 dr.\end{aligned} \tag{3.9}$$

Proof. If u is a variational solution of (2.1), then we have

$$\begin{aligned}&\int_\tau^T \left[-(u(t), \Phi'(t))_T + \varepsilon\left(\left(\frac{\partial u}{\partial t}, \Phi(t)\right)\right)_T + ((u(t), \Phi(t)))_T + (f(u(t)), \Phi(t))_T\right] dt \\ &\quad = \int_\tau^T (g(t), \Phi(t))_T dt.\end{aligned} \tag{3.10}$$

Applying Lemma 3.6 with $v_1 = v_2 = u$, we get

$$\begin{aligned}|u(t)|_T^2 - |u_\tau|_T^2 = &-\int_\tau^t \left(\left(\varepsilon\frac{\partial u}{\partial t} + u(r), u(r)\right)\right)_T dr - \int_\tau^t \left(\left(\varepsilon\frac{\partial u}{\partial t} + u(r), u(r)\right)\right)_T dr \\ &- \int_\tau^t (f(u(r)) - g(r), u(r))_T dr - \int_\tau^t (f(u(r)) - g(r), u(r))_T dr \\ &+ \lim_{h\to 0} h^{-1}\int_\tau^{t-h} |u(r+h) - u(r)|_T^2 dr \\ = &-2\int_\tau^t \|u(r)\|_T^2 dr - 2\int_\tau^t (f(u(r)), u(r))_T dr + 2\int_\tau^t (g(r), u(r))_T dr \\ &- \varepsilon\|u(t)\|_T^2 + \varepsilon\|u_\tau\|_T^2 + \lim_{h\to 0} h^{-1}\int_\tau^{t-h} |u(r+h) - u(r)|_T^2 dr.\end{aligned} \tag{3.11}$$

Hence, it implies the desired result. □

The aim of this section is to obtain a variational solution of (2.1) such that

$$|u(t)|_T^2 + \varepsilon\|u(t)\|_T^2 + 2\int_\tau^t \|u(r)\|_T^2 dr + 2\int_\tau^t \left(f(u(r)), u(r)\right)_T dr = |u_\tau|_T^2 + \varepsilon\|u_\tau\|_T^2 + 2\int_\tau^t \left(g(r), u(r)\right)_T dr. \tag{3.12}$$

We will say that u satisfies *the energy equality* in (τ, T) if (3.12) is satisfied a.e. in (τ, T). Analogously, if u is a variational solution of (1.3), we will say that u satisfies the energy equality a.e. in $(\tau, +\infty)$ if for each $T > \tau$ the restriction of u to $\widetilde{Q}_{\tau,T}$ satisfies the energy equality (3.12) a.e. in (τ, T).

For any function $v \in L^2(\tau, T; H_T)$ and any $t \in (\tau, T]$, we put

$$\eta_{v,T}(t) := \limsup_{h\downarrow 0} h^{-1} \int_\tau^{t-h} |v(r+h) - v(r)|_T^2 dr. \tag{3.13}$$

Then $\eta_{v,T}$ is a nondecreasing function. By Corollary 3.7, a variational solution u of (1.3) satisfies the energy equality a.e. in (τ, T) if and only if $\eta_{u,T}(t) = 0$ for a.e. $t \in (\tau, T)$. In fact, using the continuity of the following mapping:

$$t \in [\tau, T] \mapsto |u_\tau|_T^2 + \varepsilon\|u_\tau\|_T^2 + 2\int_\tau^t \left[\left(g(r), u(r)\right)_T - \|u(r)\|_T^2 - \left(f(u(r)), u(r)\right)_T\right] dr, \tag{3.14}$$

one can see that a variational solution u of (1.3) satisfies the energy equality a.e. in (τ, T) if and only if $\eta_{u,T}(T) = 0$.

The next lemma provides a sufficient condition for u to satisfy the energy equality a.e. in (τ, T).

Lemma 3.8. *Let u be a variational solution of* (2.1) *and suppose that there exists a sequence $\{t_n\} \subset (\tau, T)$ of Lebesgue points of $|u|_T^2$ such that $t_n \to T$ and*

$$\limsup_{n\longrightarrow+\infty} \left(|u(t_n)|_T^2 + \varepsilon\|u(t_n)\|_T^2\right) \le |u_\tau|_T^2 + \varepsilon\|u_\tau\|_T^2 + 2\int_\tau^T \left[\left(g(r), u(r)\right)_T - \|u(r)\|_T^2 - \left(f(u(r)), u(r)\right)_T\right] dr. \tag{3.15}$$

Then, u satisfies the energy equality a.e. in (τ, T).

Proof. It is sufficient to prove that $\eta_{u,T}(T) = 0$. Since $t_n \to T$ and $\eta_{u,T}$ is nondecreasing, by Corollary 3.7, we have

$$\begin{aligned}
\eta_{u,T}(T) \le \limsup_{n\uparrow+\infty} \eta_{u,T}(t_n) = \limsup_{n\to+\infty}\Bigg(&|u(t_n)|_T^2 + \varepsilon\|u(t_n)\|_T^2 - |u_\tau|_T^2 - \varepsilon\|u_\tau\|_T^2 \\
&-2\int_\tau^{t_n}\Big[\big(g(r),u(r)\big)_T - \|u(r)\|_T^2 - \big(f(u(r)),u(r)\big)_T\Big]dr\Bigg) \\
\le \limsup_{n\to+\infty}\Big(|u(t_n)|_T^2 + \varepsilon\|u(t_n)\|_T^2\Big) &- |u_\tau|_T^2 - \varepsilon\|u_\tau\|_T^2 \\
-2\int_\tau^{T}\Big[\big(g(r),u(r)\big)_T - \|u(r)\|_T^2 &- \big(f(u(r)),u(r)\big)_T\Big]dr \\
\le 0.
\end{aligned} \tag{3.16}$$

This completes the proof. □

Proposition 3.9. *Let $u,\overline{u}$ be two variational solutions of* (2.1) *corresponding to the initial data $u_\tau, \overline{u}_\tau \in V_\tau \cap L^p(\Omega_\tau)$, respectively, which satisfy the energy equality a.e. in (τ,T). Then,*

$$\begin{aligned}
|u(t)-\overline{u}(t)|_T^2 + \varepsilon\|u(t)-\overline{u}(t)\|_T^2 + 2\int_\tau^t \|u(r)-\overline{u}(r)\|_T^2 dr \\
\le e^{2\ell(t-\tau)}\Big(|u_\tau-\overline{u}_\tau|_T^2 + \varepsilon\|u_\tau-\overline{u}_\tau\|_T^2\Big) \quad a.e.\ t\in(\tau,T).
\end{aligned} \tag{3.17}$$

Hence, it implies the uniqueness of variational solutions to (2.1) *satisfying the energy equality in (τ,T).*

Proof. Since $u,\overline{u}$ satisfy the energy equation, $\eta_{u,T}(t) = \eta_{\overline{u},T}(t) = 0$ for all $t\in(\tau,T)$ and

$$\begin{aligned}
&|u(t)-\overline{u}(t)|_T^2 + \varepsilon\|u(t)-\overline{u}(t)\|_T^2 + 2\int_\tau^t \|u(r)-\overline{u}(r)\|_T^2 dr \\
&\quad + 2\int_\tau^t \big(f(u(r)) - f(\overline{u}(r)), u(r)-\overline{u}(r)\big)_T dr \\
&\le |u_\tau-\overline{u}_\tau|_T^2 + \varepsilon\|u_\tau-\overline{u}_\tau\|_T^2 - 2\lim_{h\downarrow 0} h^{-1}\int_\tau^{t-h} \big(u(r+h)-u(r), \overline{u}(r+h)-\overline{u}(r)\big)_T dr.
\end{aligned} \tag{3.18}$$

On the other hand,

$$\left|h^{-1}\int_{\tau}^{t-h}(u(r+h)-u(r),\overline{u}(r+h)-\overline{u}(r))_T dr\right|^2 \leq \left(h^{-1}\int_{\tau}^{t-h}|u(r+h)-u(r)|^2 dr\right)\left(h^{-1}\int_{\tau}^{t-h}|\overline{u}(r+h)-\overline{u}(r)|^2 dr\right), \tag{3.19}$$

so

$$\lim_{h\downarrow 0} h^{-1}\int_{\tau}^{t-h}(u(r+h)-u(r),\overline{u}(r+h)-\overline{u}(r))_T dr = 0. \tag{3.20}$$

Using this and (1.5) in (3.12), one can conclude

$$\begin{aligned}
&|u(t)-\overline{u}(t)|_T^2+\varepsilon\|u(t)-\overline{u}(t)\|_T^2+2\int_{\tau}^{t}\|u(r)-\overline{u}(r)\|_T^2 dr \\
&\quad\leq |u_\tau-\overline{u}_\tau|_T^2+\varepsilon\|u_\tau-\overline{u}_\tau\|_T^2-2\int_{\tau}^{t}\left(f(u(r))-f(\overline{u}(r)),u(r)-\overline{u}(r)\right)_T dr \\
&\quad\leq |u_\tau-\overline{u}_\tau|_T^2+\varepsilon\|u_\tau-\overline{u}_\tau\|_T^2+2\ell\int_{\tau}^{t}|u(r)-\overline{u}(r)|_T^2 dr.
\end{aligned} \tag{3.21}$$

By an application of Gronwall's inequality, we get (3.17). □

The method of penalization will now be used to prove the existence and uniqueness of a variational solution to problem (2.1) satisfying an energy equality a.e. in (τ,T) and, as a consequence, the existence and uniqueness of a variational solution to problem (1.3) satisfying the energy equality a.e. in (τ,∞).

Theorem 3.10. *Let $g\in L^2(\tau,T;H_T)$, $u_\tau\in V_\tau\cap L^p(\Omega_\tau)$ be given. Then problem* (2.1) *has a unique variational solution satisfying the energy equality a.e. in* (τ,T).

Proof. We divide the proof into two steps.

Step 1. Existence of a weak solution to problem (2.12).

We will use the Galerkin method (see [20]). Take an orthonormal Hilbert basis $\{e_j\}$ of H_T formed by elements of $V_T\cap L^p(\Omega_T)$ such that the vector space generated by $\{e_j\}$ is dense in V_T and $L^p(\Omega_T)$. Then, one takes a sequence $\{u_{\tau_m}\}$ converging to u_τ in V_T, with $\{u_{\tau_m}\}$ in the vector space spanned by the m first $\{e_j\}$. For each integer $m\geq 1$, one considers the approximation $u_{k_m}(t)=\sum_{j=1}^{m}\gamma_{k_m,j}(t)e_j$, defined as a solution of

$$\begin{gathered}
\left(u'_{k_m}(t),e_j\right)_T+\langle A_k(t)u_{k_m}(t),e_j\rangle_T+\varepsilon\left\langle A_k(t)u'_{k_m}(t),e_j\right\rangle_T+(f(u_{k_m}(t)),e_j)_T=(g(t),e_j)_T, \\
((u_{k_m}(\tau),e_j))_T=((u_{\tau_m},e_j))_T.
\end{gathered} \tag{3.22}$$

Multiplying (3.22) by $\gamma'_{k_m,j}(t)$ and summing from $j = 1$ to m, we obtain

$$\left(u'_{k_m}(t), u'_{k_m}(t)\right)_T + \left\langle A_k(t)u_{k_m}(t), u'_{k_m}(t)\right\rangle + \varepsilon\left\langle A_k(t)u'_{k_m}(t), u'_{k_m}(t)\right\rangle_T + \left(f(u_{k_m}(t)), u'_{k_m}(t)\right)_T$$
$$= \left(g(t), u'_{k_m}(t)\right)_T, \tag{3.23}$$

or

$$\begin{aligned}&\left|u'_{k_m}(t)\right|_T^2 + \frac{1}{2}\frac{d}{dt}\|u_{k_m}(t)\|_T^2 + \varepsilon\left\|u'_{k_m}(t)\right\|_T^2 + k\left(\left(P_k(t)u_{k_m}(t), u'_{k_m}(t)\right)\right)_T \\ &\quad + \varepsilon k\left(\left(P_k(t)u'_{k_m}(t), u'_{k_m}(t)\right)\right)_T + \left(f(u_{k_m}(t)), u'_{k_m}(t)\right)_T = \left(g(t), u'_{k_m}(t)\right)_T.\end{aligned} \tag{3.24}$$

Thus,

$$\begin{aligned}\left(g(t), u'_{k_m}(t)\right)_T &\geq \left|u'_{k_m}(t)\right|_T^2 + \varepsilon\left\|u'_{k_m}(t)\right\|_T^2 + \frac{1}{2}\frac{d}{dt}\left[\|u_{k_m}(t)\|_T^2 + k((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T\right] \\ &\quad + \varepsilon k\left(\left(P_k(t)u'_{k_m}(t), u'_{k_m}(t)\right)\right)_T + \left(f(u_{k_m}(t)), u'_{k_m}(t)\right)_T \\ &= \left|u'_{k_m}(t)\right|_T^2 + \varepsilon\left\|u'_{k_m}(t)\right\|_T^2 + \varepsilon k\left(\left(P_k(t)u'_{k_m}(t), u'_{k_m}(t)\right)\right)_T \\ &\quad + \frac{1}{2}\frac{d}{dt}\left[\|u_{k_m}(t)\|_T^2 + k((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T + 2\int_{\Omega_T} F(u_{k_m}(x,t))dx\right].\end{aligned} \tag{3.25}$$

We have

$$\left(g(t), u'_{k_m}(t)\right)_T \leq \frac{1}{2}\left(|g(t)|_T^2 + \left|u'_{k_m}(t)\right|_T^2\right), \tag{3.26}$$

so

$$\begin{aligned}|g(t)|_T^2 &\geq 2\varepsilon\left\|u'_{k_m}(t)\right\|_T^2 + \frac{d}{dt}\left[\|u_{k_m}(t)\|_T^2 + k((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T + 2\int_{\Omega_T} F(u_{k_m}(x,t))dx\right] \\ &\quad + 2\varepsilon k\left(\left(P_k(t)u'_{k_m}(t), u'_{k_m}(t)\right)\right)_T + \left|u'_{k_m}(t)\right|_T^2.\end{aligned} \tag{3.27}$$

Integrating (3.27) on $[\tau, t], \tau \leq t \leq T$, we obtain

$$\begin{aligned} &2\varepsilon \int_\tau^t \left\| u'_{k_m}(r) \right\|_T^2 dr + 2\varepsilon k \int_\tau^t \left(\left(P_k(r) u'_{k_m}(r), u'_{k_m}(r) \right) \right)_T dr + \int_\tau^t \left| u'_{k_m}(r) \right|_T^2 dr \\ &\quad + \|u_{k_m}(t)\|_T^2 + k((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T + 2\int_{\Omega_T} F(u_{k_m}(x,t))dx \\ &\leq \int_\tau^t |g(r)|_T^2 dr + \|u_{k_m}(\tau)\|_T^2 + k((P_k(\tau)u_{k_m}(\tau), u_{k_m}(\tau)))_T + 2\int_{\Omega_T} F(u_{k_m}(x,\tau))dx. \end{aligned} \tag{3.28}$$

Since

$$\begin{aligned} \int_{\Omega_T} F(u_{k_m}(x,t))dx &\geq -\widetilde{\beta}|\Omega_T| + \widetilde{\alpha}_1 \|u_{k_m}(t)\|_{L^p(\Omega_T)}^p, \\ \int_{\Omega_T} F(u_{k_m}(x,\tau))dx &\leq \widetilde{\beta}|\Omega_T| + \widetilde{\alpha}_2 \|u_{\tau_m}\|_{L^p(\Omega_T)}^p, \end{aligned} \tag{3.29}$$

we have

$$\begin{aligned} &2\varepsilon \int_\tau^t \left\| u'_{k_m}(r) \right\|_T^2 dr + 2\varepsilon k \int_\tau^t \left(\left(P_k(r) u'_{k_m}(r), u'_{k_m}(r) \right) \right)_T dr + \int_\tau^t \left| u'_{k_m}(r) \right|_T^2 dr \\ &\quad + \|u_{k_m}(t)\|_T^2 + k((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T + 2\widetilde{\alpha}_1 \|u_{k_m}(t)\|_{L^p(\Omega_T)}^p \\ &\leq \int_\tau^t |g(r)|_T^2 dr + \|u_{k_m}(\tau)\|_T^2 + k((P_k(\tau)u_{k_m}(\tau), u_{k_m}(\tau)))_T + 4\widetilde{\beta}|\Omega_T| + 2\widetilde{\alpha}_2 \|u_{\tau_m}\|_{L^p(\Omega_T)}^p. \end{aligned} \tag{3.30}$$

From (3.30), we deduce that

$\{u_{k_m}\}$ is bounded in $L^\infty(\tau, T; V_T) \cap L^p(\widetilde{Q}_{\tau,T})$,

$\{u_{k_m}\} \rightharpoonup u_k$ weakly in $L^\infty(\tau, T; V_T) \cap L^p(\widetilde{Q}_{\tau,T})$,

$\{u'_{k_m}\}$ is bounded in $L^2(\tau, T; V_T)$,

$\{u'_{k_m}\} \rightharpoonup u'_k$ weakly in $L^2(\tau, T; V_T)$.

Since $\{u_{k_m}\}$ is bounded in $L^\infty(\tau, T; V_T) \cap L^p(\widetilde{Q}_{\tau,T})$, one can check that $\{f(u_{k_m})\}$ is bounded in $L^q(\tau, T; L^q(\Omega_T))$ with $q = p/(p-1)$, hence $f(u_{k_m}) \rightharpoonup \eta$ in $L^q(\tau, T; L^q(\Omega_T))$. We now prove that $\eta = f(u_k)$.

Indeed, we have

$V_T \subset\subset H_T \subset V_T^*$,

$\{u_{k_m}\}$ is bounded in $L^\infty(\tau, T; V_T)$,

$\{u'_{k_m}\}$ is bounded in $L^2(\tau, T; V_T^*)$.

By the Aubin-Lions lemma [20, Chapter 1], $\{u_{k_m}\}$ is relatively compact in $L^2(\tau,T;H_T)$. Therefore, one can assume that $u_{k_m} \to u_k$ strongly in $L^2(\tau,T;H_T)$, so $u_{k_m} \to u_k$ a.e. in $\widetilde{Q}_{\tau,T}$. Since f is continuous, $f(u_{k_m}) \to f(u_k)$ a.e. in $\widetilde{Q}_{\tau,T}$. Applying Lemma 1.3 in [20], we have

$$f(u_{k_m}) \rightharpoonup f(u_k) \quad \text{weakly in } L^q(\tau,T;L^q(\Omega_T)). \tag{3.31}$$

This implies that u_k is a weak solution of problem (2.12).

Step 2. Existence of a variational solution to (2.1) satisfying the energy equality. □

From (3.30), we have

$$\begin{aligned} k\int_\tau^T ((P_k(r)u_{k_m}(r),u_{k_m}(r)))_T dr \\ \le (T-\tau)\bigg(\int_\tau^t |g(r)|_T^2 dr + \|u_{k_m}(\tau)\|_T^2 + k((P_k(\tau)u_{k_m}(\tau),u_{k_m}(\tau)))_T \\ + 4\widetilde{\beta}|\Omega_T| + 2\widetilde{\alpha}_2\|u_{\tau_m}\|^p_{L^p(\Omega_T)}\bigg). \end{aligned} \tag{3.32}$$

Consider the function $\Phi : L^2(\tau,T;V_T) \to \mathbb{R}$ defined by

$$\Phi(v) = \int_\tau^T ((P_k(t)v(t),v(t)))_T dt, \quad v \in L^2(\tau,T;V_T). \tag{3.33}$$

It is easy to see that Φ is a continuous and convex function. It follows that $\int_\tau^T ((P_k(t)u_{k_m}(t),u_{k_m}(t)))_T dt$ is weakly lower semicontinuous in $L^2(\tau,T;V_T)$. Moreover, $\{u_{k_m}\} \rightharpoonup u_k$ weakly in $L^2(\tau,T;V_T)$, hence

$$\begin{aligned} k\int_\tau^T ((P_k(t)u_k(t),u_k(t)))_T dt \\ \le k\liminf_{m\to\infty}\int_\tau^T ((P_k(t)u_{k_m}(t),u_{k_m}(t)))_T dt \\ \le (T-\tau)\bigg(\int_\tau^t |g(r)|_T^2 dr + \|u_\tau\|_T^2 + k((P_k(\tau)u_\tau,u_\tau))_T + 4\widetilde{\beta}|\Omega_T| + 2\widetilde{\alpha}_2\|u_\tau\|^p_{L^p(\Omega_T)}\bigg). \end{aligned} \tag{3.34}$$

Since $\{u'_{k_m}\} \rightharpoonup u'_k$ weakly in $L^2(\tau,T;V_T)$, then, reasoning as above,

$$\begin{aligned}
2\varepsilon k\int_\tau^T & ((P_k(t)u'_k(t),u'_k(t)))_T dt \\
&\le 2\epsilon k\liminf_{m\to\infty}\int_\tau^T ((P_k(t)u_{k_m}(t),u_{k_m}(t)))_T dt \\
&\le \left(\int_\tau^t |g(r)|_T^2 dr + \|u_\tau\|_T^2 + k((P_k(\tau)u_\tau,u_\tau))_T + 4\widetilde{\beta}|\Omega_T| + 2\widetilde{\alpha}_2\|u_\tau\|^p_{L^p(\Omega_T)}\right).
\end{aligned} \tag{3.35}$$

From the facts that $u_{k_m} \rightharpoonup u_k$ weakly in $L^\infty(\tau,T;V_T)$, $u'_{k_m} \rightharpoonup u'_k$ weakly in $L^2(\tau,T;V_T)$ and the weak lower semicontinuity of the norm, we deduce that

$$\begin{aligned}
2\varepsilon\int_\tau^t & \|u'_k(r)\|_T^2 dr + 2\varepsilon k\int_\tau^t ((P_k(r)u'_k(r),u'_k(r)))_T dr + \int_\tau^t |u'_k(r)|_T^2 dr \\
&+ \|u_k(t)\|_T^2 + k\int_\tau^T ((P_k(t)u_k(t),u_k(t)))_T + 2\widetilde{\alpha}_1\|u_k(t)\|^p_{L^p(\Omega_T)} \\
&\le (5+T-\tau)\left(\int_\tau^t |g(r)|_T^2 dr + \|u_\tau\|_T^2 + k((P_k(\tau)u_\tau,u_\tau))_T + 4\widetilde{\beta}|\Omega_T| + 2\widetilde{\alpha}_2\|u_\tau\|^p_{L^p(\Omega_T)}\right) = C.
\end{aligned} \tag{3.36}$$

Since $u_\tau \in V_\tau \cap L^p(\Omega_\tau)$, $((P_k(\tau)u_\tau,u_\tau))_T = 0$ for all $k \ge 1$, (3.36) gives

$\{u_k\}$ is bounded in $L^\infty(\tau,T;V_T) \cap L^p(\tau,T;L^p(\Omega_T))$,
$\{u'_k\}$ is bounded in $L^2(\tau,T;V_T)$,
$\{u_k\} \rightharpoonup u$ weakly in $L^\infty(\tau,T;V_T) \cap L^p(\tau,T;L^p(\Omega_T))$,
$\{u'_k\} \rightharpoonup u'$ weakly in $L^2(\tau,T;V_T)$.

From Lemma 2.1, we have

$$\begin{aligned}
&\int_\tau^T \|P(t)u(t)\|_T^2 dt \le \liminf_{k\to\infty}\int_\tau^T ((P_k(t)u_k(t),u_k(t)))_T dt \le \liminf_{k\to\infty}\frac{C}{k} = 0, \\
&\text{that is, } P(t)u(t) = 0 \text{ a.e. in } (\tau,T) \text{ or } u(t) \in V_t \text{ a.e. in } (\tau,T), \\
&\int_\tau^T \|P(t)u'(t)\|_T^2 dt \le \liminf_{k\to\infty}\int_\tau^T ((P_k(t)u'_k(t),u'_k(t)))_T dt \le \liminf_{k\to\infty}\frac{C}{k} = 0, \\
&\text{that is, } P(t)u'(t) = 0 \text{ a.e. in } (\tau,T).
\end{aligned} \tag{3.37}$$

Moreover, (3.36) and the equality

$$u_k(t) - u_k(s) = \int_s^t u'_k(r)dr, \quad \forall s,\ t \in [\tau,T],\ \forall k \ge 1, \tag{3.38}$$

give

$$|u_k(t) - u_k(s)|_T \leq C^{1/2}|t-s|^{1/2} \quad \forall s,t \in [\tau, T], \ \forall k \geq 1. \tag{3.39}$$

It follows from (3.36) that $\|u_k(t)\|_T \leq C$ for all $t \in [\tau, T]$ and each $k \geq 1$. Since the injection of V_T into H_T is compact, the set $\{v \in V_T : \|v\|_T^2 \leq C\}$ is compact in H_T. By (3.39) and the Arzela-Ascoli theorem, there exists a subsequence, that will be still denoted by $\{u_k\}$, such that

$$u_k \to u \quad \text{in } C([\tau, T]; H_T) \text{ as } k \longrightarrow +\infty. \tag{3.40}$$

So, the condition (C4) is satisfied.

On the other hand, $\{u_k\}$ is bounded in $L^\infty(\tau, T; V_T)$ and $\{u_k'\}$is bounded in $L^2(\tau, T; V_T)$, applying the Aubin-Lions lemma and Lemma 1.3 in [20, Chapter 1], one has

$$f(u_k) \rightharpoonup f(u) \quad \text{weakly in } L^q(\tau, T, L^q(\Omega_T)). \tag{3.41}$$

Since u_k is the weak solution of the problem

$$\begin{gathered}
\left(u_k'(t), v\right)_T + \langle A_k(t)u_k(t), v\rangle_T \\
+ \varepsilon\left\langle A_k(t)u_k'(t), v\right\rangle_T + \left(f(u_k(t)), v\right)_T = \left(g(t), v\right)_T, \quad \forall v \in V_T, \\
((u_k(\tau), v))_T = ((u_\tau, v))_T,
\end{gathered} \tag{3.42}$$

taking to the limit as $k \to +\infty$ and using the fact that $P(t)u(t) = 0, P(t)u'(t) = 0$ a.e. in (τ, T), we can conclude that u is the solution of (2.1).

Now, we will show that u satisfies the energy equality in (τ, T). Multiplying (3.22) by $\gamma_{k_m,j}$ and summing from $j = 1$ to m, we obtain

$$\begin{gathered}
\left(u_{k_m}'(t), u_{k_m}(t)\right)_T + \langle A_k(t)u_{k_m}(t), u_{k_m}(t)\rangle_T \\
+ \varepsilon\left\langle A_k(t)u_{k_m}'(t), u_{k_m}(t)\right\rangle_T + \left(f(u_{k_m}(t)), u_{k_m}(t)\right)_T = \left(g(t), u_{k_m}(t)\right)_T.
\end{gathered} \tag{3.43}$$

Hence, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_{k_m}(t)|_T^2 + \|u_{k_m}(t)\|_T^2 + k((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T \\
&\quad + \frac{\epsilon}{2}\frac{d}{dt}\|u_{k_m}(t)\|_T^2 + \epsilon k\left(\left(P_k(t)u_{k_m}'(t), u_{k_m}(t)\right)\right)_T + \left(f(u_{k_m}(t)), u_{k_m}(t)\right)_T = \left(g(t), u_{k_m}(t)\right)_T,
\end{aligned} \tag{3.44}$$

or

$$
\begin{aligned}
&|u_{k_m}(t)|_T^2 + 2\int_\tau^t \|u_{k_m}(r)\|_T^2 dr + 2k\int_\tau^T ((P_k(r)u_{k_m}(r), u_{k_m}(r)))_T dr \\
&\quad + \epsilon\|u_{k_m}(t)\|_T^2 + 2k\epsilon\int_\tau^t \left(\left(P_k(r)u'_{k_m}(r), u_{k_m}(r)\right)\right)_T dr + 2\int_\tau^t \left(f(u_{k_m}(r)), u_{k_m}(r)\right)_T dr \\
&= 2\int_\tau^t \left(g(r), u_{k_m}(r)\right)_T dr + |u_{\tau_m}|_T^2 + \varepsilon\|u_{\tau_m}\|_T^2.
\end{aligned}
\tag{3.45}
$$

Since

$$
\begin{gathered}
\left(\left(P_k(t)u'_{k_m}(t), u_{k_m}(t)\right)\right)_T \geq \frac{1}{2}\frac{d}{dt}((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T, \\
((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T \geq 0, \\
|u_{k_m}(t)|_T^2 + 2\int_\tau^t \|u_{k_m}(r)\|_T^2 dr + \epsilon\|u_{k_m}(t)\|_T^2 + 2\int_\tau^t \left(f(u_{k_m}(r)), u_{k_m}(r)\right)_T dr \\
\leq 2\int_\tau^t \left(g(r), u_{k_m}(r)\right)_T dr + |u_{\tau_m}|_T^2 + \varepsilon\|u_{\tau_m}\|_T^2 + \varepsilon k((P_k(t)u_{\tau_m}, u_{\tau_m}))_T,
\end{gathered}
\tag{3.46}
$$

letting $m \to \infty$, we obtain

$$
\begin{aligned}
&|u_k(T)|_T^2 + 2\int_\tau^T \|u_k(r)\|_T^2 dr + \epsilon\|u_k(T)\|_T^2 + 2\int_\tau^T \left(f(u_k(r)), u_k(r)\right)_T dr \\
&\quad \leq 2\int_\tau^T \left(g(r), u_k(r)\right)_T dr + |u_\tau|_T^2 + \varepsilon\|u_\tau\|_T^2.
\end{aligned}
\tag{3.47}
$$

Now

$$
\begin{aligned}
&\int_\tau^T \left(f(u_k(r)), u_k(r)\right)_T dr \\
&\quad = \int_\tau^T \left(f(u_k(r)) - f(u(r)), u_k(r) - u(r)\right)_T dr + \int_\tau^T \left(f(u_k(r)), u(r)\right)_T dr \\
&\qquad + \int_\tau^T \left(f(u(r)), u_k(r)\right)_T dr - \int_\tau^T \left(f(u(r)), u(r)\right)_T dr \\
&\quad \geq -\ell\int_\tau^T |u_k(r) - u(r)|_T^2 dr + \int_\tau^T \left(f(u_k(r)), u(r)\right)_T dr \\
&\qquad + \int_\tau^T \left(f(u(r)), u_k(r)\right)_T dr - \int_\tau^T \left(f(u(r)), u(r)\right)_T dr.
\end{aligned}
\tag{3.48}
$$

This inequality and (3.47) give

$$\begin{aligned}|u_k(T)|_T^2 &+ 2\int_\tau^T \|u_k(r)\|_T^2 dr + \varepsilon\|u_k(T)\|_T^2 \\ &\le 2\int_\tau^T (g(r), u_k(r))_T dr + |u_\tau|_T^2 + \varepsilon\|u_\tau\|_T^2 + 2l\int_\tau^T |u_k(r) - u(r)|_T^2 dr \\ &\quad - \int_\tau^T (f(u_k(r)), u(r))_T dr - \int_\tau^T (f(u(r)), u_k(r))_T dr + \int_\tau^T (f(u(r)), u(r))_T dr.\end{aligned} \tag{3.49}$$

Since $u_k \rightharpoonup u$ weakly in $L^2(\tau, T; V_T)$, we get

$$\begin{aligned}|u(T)|_T^2 + \varepsilon\|u(T)\|_T^2 \le |u_\tau|_T^2 + \varepsilon\|u_\tau\|_T^2 &- 2\int_\tau^T (f(u(r)), u(r))_T dr \\ &- 2\int_\tau^T \|u(r)\|_T^2 dr + 2\int_\tau^T (g(r), u(r))_T dr.\end{aligned} \tag{3.50}$$

Applying Lemma 3.8 with $t_n = T$ for all n, one concludes that u satisfies the energy equality on (τ, T), and the desired uniqueness follows from Proposition 3.9.

4. Existence of Pullback $\mathcal{D}$-Attractors

The aim of this section is to establish the existence of a pullback attractor for problem (1.3).

Suppose that $g \in L^2_{\text{loc}}(\mathbb{R}^{N+1})$. Then, according to Theorem 3.10, for each $\tau \in \mathbb{R}$ and each $u_\tau \in V_\tau \cap L^p(\Omega_\tau)$ given, there exists a unique variational solution $u(\cdot; \tau, u_\tau)$ of problem (1.3) satisfying the energy equality a.e. in (τ, T) for all $T > \tau$.

Define

$$U(t,\tau)u_\tau := u(t;\tau,u_\tau), \quad -\infty < \tau \le t < +\infty, u_\tau \in V_\tau \cap L^p(\Omega_\tau). \tag{4.1}$$

It is easy to check that the family of mappings $\{U(t,\tau); -\infty < \tau \le t < +\infty\}$ is a process $U(\cdot,\cdot)$.

A uniform estimate in $V_T \cap L^p(\Omega_T)$ will be obtained now for the variational solutions of (1.3) satisfying the energy equality, and since the compactness of the embedding $V_T \cap L^p(\Omega_T) \hookrightarrow H_T$, this will immediately imply the existence of a pullback attractor for the process $U(\cdot,\cdot)$. The proof requires the following lemma.

Lemma 4.1 (see [21]). *Let $X \subset Y$ be Banach spaces such that X is reflexive and the injection of X in Y is compact. Suppose that $\{v_n\}$ is a bounded sequence in $L^\infty(t_0, T; X)$ such that $v_n \rightharpoonup v$ weakly in $L^p(t_0, T; X)$ for some $p \in [1, +\infty)$ and $v \in C^0([t_0, T]; Y)$. Then, $v(t) \in X$ for all $t \in [t_0, T]$ and*

$$\|v(t)\|_X \le \liminf_{n\to+\infty} \|v_n\|_{L^\infty(t_0,T;X)}, \quad \forall t \in [t_0, T]. \tag{4.2}$$

Proposition 4.2. *Suppose that the assumptions in Theorem 3.10 hold and $g \in L^2_{\mathrm{loc}}(\mathbb{R}^{N+1})$ satisfies*

$$C_{g,T} = \sup_{t \le T} \int_{t-1}^{t} |g(r)|_T^2 dr < +\infty. \tag{4.3}$$

Then, for any $u_\tau \in V_\tau \cap L^p(\Omega_\tau)$ given, the corresponding variational solution u of (1.3) satisfying the energy equality in (τ, T) also satisfies

$$\begin{aligned} &\|u(t)\|^p_{L^p(\Omega_T)} + \|u(t)\|_T^2 \\ &\quad \le C\left(e^{-\sigma_T(t-\tau)}\left(\|u_\tau\|_\tau^2 + \|u_\tau\|^p_{L^p(\Omega_\tau)}\right) + 1 + \frac{1}{1-e^{-\sigma_T}} C_{g,T}\right), \quad \forall t \in [\tau+1, T], \end{aligned} \tag{4.4}$$

where $\sigma_T = \min\{\lambda_{1,T}/2, 1/(\varepsilon+1), \alpha_1/\widetilde{\alpha}_2\}$, $\lambda_{1,T} > 0$ is the first eigenvalue of the operator $-\Delta$ in Ω_T with the homogeneous Dirichlet condition, $\alpha_1, \widetilde{\alpha}_2$ are the constants in $(H1)$, and the constant C is independent of t, τ, ε.

Proof. Assume that u_{k_m} are the Galerkin approximations of u_k defined by (3.22). From (3.44) and (3.24), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(|u_{k_m}(t)|_T^2 + (\varepsilon+1)\|u_{k_m}(t)\|_T^2\right) + \left|u'_{k_m}(t)\right|_T^2 + \|u_{k_m}(t)\|_T^2 \\ &\quad + \varepsilon\left\|u'_{k_m}(t)\right\|_T^2 + (\varepsilon+1)k\left(\left(P_k(t)u'_{k_m}(t), u_{k_m}(t)\right)\right)_T + k((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T \\ &\quad + \varepsilon k\left(\left(P_k(t)u'_{k_m}(t), u'_{k_m}(t)\right)\right)_T + \left(f(u_{k_m}(t)), u'_{k_m}(t) + u_{k_m}(t)\right)_T \\ &= (g(t), u_{k_m}(t))_T + \left(g(t), u'_{k_m}(t)\right)_T. \end{aligned} \tag{4.5}$$

Moreover,

$$\begin{aligned} \left(\left(P_k(t)u'_{k_m}(t), u_{k_m}(t)\right)\right)_T &\ge \frac{1}{2}\frac{d}{dt}((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T, \\ \left(f(u_{k_m}(t)), u'_{k_m}(t)\right)_T &= \frac{d}{dt}\int_{\Omega_T} F(u_{k_m}(x,t))dx, \\ (g(t), u_{k_m}(t))_T &\le \frac{1}{4\eta_1}|g(t)|_T^2 + \eta_1|u_{k_m}(t)|_T^2, \quad \forall \eta_1 > 0, \\ \left(g(t), u'_{k_m}(t)\right)_T &\le \frac{1}{4\eta_2}|g(t)|_T^2 + \eta_2\left|u'_{k_m}(t)\right|_T^2, \quad \forall \eta_2 > 0, \end{aligned} \tag{4.6}$$

so

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Bigg(|u_{k_m}(t)|_T^2 + (\varepsilon+1)\|u_{k_m}(t)\|_T^2 + (\varepsilon+1)k((P_k(t)u_{k_m}(t),u_{k_m}(t)))_T \\
&\qquad +2\int_{\Omega_T} F(u_{k_m}(x,t))dx\Bigg) + \left|u'_{k_m}(t)\right|_T^2 + \|u_{k_m}(t)\|_T^2 + \varepsilon\left\|u'_{k_m}(t)\right\|_T^2 \\
&\qquad + \varepsilon k\left(\left(P_k(t)u'_{k_m}(t),u'_{k_m}(t)\right)\right)_T + k((P_k(t)u_{k_m}(t),u_{k_m}(t)))_T + \left(f(u_{k_m}(t)),u_{k_m}(t)\right)_T \\
&\quad \leq \frac{1}{4\eta_1}\left|g(t)\right|_T^2 + \eta_1|u_{k_m}(t)|_T^2 + \frac{1}{4\eta_2}\left|g(t)\right|_T^2 + \eta_2\left|u'_{k_m}(t)\right|_T^2, \quad \forall \eta_1,\eta_2>0.
\end{aligned} \tag{4.7}$$

Since

$$\begin{aligned}
\left(f(u_{k_m}(t)),u_{k_m}(t)\right)_T &= \int_{\Omega_T} f(u_{k_m}(t))u_{k_m}(t)dx \\
&\geq \int_{\Omega_T}\left(-\beta + \alpha_1|u_{k_m}(t)|^p dx\right) = -\beta|\Omega_T| + \alpha_1\|u_{k_m}(t)\|_{L^p(\Omega_T)}^p,
\end{aligned} \tag{4.8}$$

we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Bigg(|u_{k_m}(t)|_T^2 + (\varepsilon+1)\|u_{k_m}(t)\|_T^2 + (\varepsilon+1)k((P_k(t)u_{k_m}(t),u_{k_m}(t)))_T \\
&\qquad + 2\int_{\Omega_T} F(u_{k_m}(x,t))dx\Bigg) + (1-\eta_2)\left|u'_{k_m}(t)\right|_T^2 + \frac{1}{2}\|u_{k_m}(t)\|_T^2 + \left(\frac{\lambda_{1,T}}{2}-\eta_1\right)|u_{k_m}(t)|_T^2 \\
&\qquad + \varepsilon\left\|u'_{k_m}(t)\right\|_T^2 + \varepsilon k\left(\left(P_k(t)u'_{k_m}(t),u'_{k_m}(t)\right)\right)_T \\
&\qquad + k((P_k(t)u_{k_m}(t),u_{k_m}(t)))_T + \alpha_1\|u_{k_m}(t)\|_{L^p(\Omega_T)}^p \\
&\quad \leq \beta|\Omega_T| + \left(\frac{1}{4\eta_1}+\frac{1}{4\eta_2}\right)\left|g(t)\right|_T^2.
\end{aligned} \tag{4.9}$$

Denote

$$\begin{aligned}
y_{k_m}(t) := {}& |u_{k_m}(t)|_T^2 + (\varepsilon+1)\|u_{k_m}(t)\|_T^2 + (\varepsilon+1)k((P_k(t)u_{k_m}(t),u_{k_m}(t)))_T \\
&+ 2\int_{\Omega_T} F(u_{k_m}(x,t))dx.
\end{aligned} \tag{4.10}$$

Choose $\eta_2 < 1$ and η_1 small enough such that $\sigma_T < \min\{1/(\varepsilon+1), \alpha_1/\widetilde{\alpha}_2, \lambda_{1,T} - 2\eta_1\}$, we have

$$\begin{aligned}
\sigma_T y_{k_m}(t) &= \sigma_T \Bigg(|u_{k_m}(t)|_T^2 + (\varepsilon+1)\|u_{k_m}(t)\|_T^2 + (\varepsilon+1)k((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T \\
&\qquad +2\int_{\Omega_T} F(u_{k_m}(x,t))dx \Bigg) \\
&\le \sigma_T \Bigg(|u_{k_m}(t)|_T^2 + (\varepsilon+1)\|u_{k_m}(t)\|_T^2 + (\varepsilon+1)k((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T \\
&\qquad +2\int_{\Omega_T} \left(\widetilde{\beta} + \widetilde{\alpha}_2 |u_{k_m}(t)|^p\right)dx + 2\widetilde{\beta}|\Omega_T| \Bigg) \\
&\le \sigma_T \Big(|u_{k_m}(t)|_T^2 + (\varepsilon+1)\|u_{k_m}(t)\|_T^2 + (\varepsilon+1)k((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T \\
&\qquad +2\widetilde{\alpha}_2 \|u_{k_m}(t)\|_{L^p(\Omega_T)}^p dx \Big) \\
&\le 2(1-\eta_2)\left|u'_{k_m}(t)\right|_T^2 + 2\frac{1}{2}\|u_{k_m}(t)\|_T^2 + 2\left(\frac{\lambda_{1,T}}{2} - \eta_1\right)|u_{k_m}(t)|_T^2 + 2\varepsilon\left\|u'_{k_m}(t)\right\|_T^2 \\
&\quad + 2\varepsilon k\Big(\Big(P_k(t)u'_{k_m}(t), u'_{k_m}(t)\Big)\Big)_T + 2k((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T + 2\alpha_1 \|u_{k_m}(t)\|_{L^p(\Omega_T)}^p.
\end{aligned} \tag{4.11}$$

Hence, we have

$$\frac{d}{dt} y_{k_m}(t) + \sigma_T y_{k_m}(t) \le C\left(1 + \left|g(t)\right|_T^2\right). \tag{4.12}$$

By Gronwall's lemma, we get

$$y_{k_m}(t) \le e^{-\sigma_T(t-\tau)} y_{k_m}(\tau) + C\left(1 + e^{-\sigma_T t}\int_\tau^t e^{\sigma_T s}\left|g(s)\right|_T^2 ds\right). \tag{4.13}$$

Now, observe that

$$\begin{aligned}
y_{k_m}(\tau) &= |u_{\tau_m}|_T^2 + (\varepsilon+1)\|u_{\tau_m}\|_T^2 + (\varepsilon+1)k((P_k(\tau)u_{\tau_m}, u_{\tau_m}))_T \\
&\quad + 2\int_{\Omega_T} F(u_{\tau_m})dx + 2\widetilde{\beta}|\Omega_T| \\
&\le \left(\frac{1}{\lambda_{1,T}} + \varepsilon + 1\right)\|u_{\tau_m}\|_T^2 + (\varepsilon+1)k((P_k(\tau)u_{\tau_m}, u_{\tau_m}))_T
\end{aligned}$$

$$\begin{aligned}
&+2\int_{\Omega_T}\left(\widetilde{\beta}+\widetilde{\alpha}_2|u_{\tau_m}|^p\right)dx+2\widetilde{\beta}|\Omega_T| \\
&= C_T\|u_{\tau_m}\|_T^2+(\varepsilon+1)k((P_k(\tau)u_{\tau_m},u_{\tau_m}))_T+2\widetilde{\alpha}_2\|u_{\tau_m}\|^p_{L^p(\Omega_T)}+4\widetilde{\beta}|\Omega_T| \\
&\le C_T\left(1+\|u_{\tau_m}\|_T^2+\|u_{\tau_m}\|^p_{L^p(\Omega_T)}\right)+(\varepsilon+1)k((P_k(\tau)u_{\tau_m},u_{\tau_m}))_T.
\end{aligned} \tag{4.14}$$

Since

$$2\int_{\Omega_T}F(u_{k_m}(x,t))dx+2\widetilde{\beta}|\Omega_T|\ge 2\widetilde{\alpha}_1\|u_{k_m}\|^p_{L^p(\Omega_T)}, \tag{4.15}$$

so combining with (4.13) and (4.14), we have

$$\begin{aligned}
&2\widetilde{\alpha}_1\|u_{k_m}\|^p_{L^p(\Omega_T)}+\|u_{k_m}(t)\|_T^2 \\
&\quad\le C\left(e^{-\sigma_T(t-\tau)}\left(1+\|u_{\tau_m}\|_T^2+\|u_{\tau_m}\|^p_{L^p(\Omega_T)}\right)+1+e^{-\sigma_T t}\int_\tau^t e^{\sigma_T s}|g(s)|_T^2ds\right) \\
&\qquad+e^{-\sigma_T(t-\tau)}(\varepsilon+1)k((P_k(\tau)u_{\tau_m},u_{\tau_m}))_T,
\end{aligned} \tag{4.16}$$

where C is independent of t,τ,ε, and k.

Now, it is known that $u_{k_m}\rightharpoonup u_k^*$-weakly in $L^\infty(\tau,T;V_T)$ as $m\to+\infty$. Hence, by (4.16) and Lemma 4.1, we can conclude that

$$\begin{aligned}
&2\widetilde{\alpha}_1\|u_k\|^p_{L^p(\Omega_T)}+\|u_k(t)\|_T^2 \\
&\quad\le C\left(e^{-\sigma_T(t-\tau)}\left(1+\|u_\tau\|_\tau^2+\|u_\tau\|^p_{L^p(\Omega_\tau)}\right)+1+e^{-\sigma_T t}\int_\tau^t e^{\sigma_T s}|g(s)|_T^2ds\right) \\
&\qquad+e^{-\sigma_T(t-\tau)}(\varepsilon+1)k((P_k(\tau)u_\tau,u_\tau))_\tau \\
&\quad= C\left(e^{-\sigma_T(t-\tau)}\left(1+\|u_\tau\|_\tau^2+\|u_\tau\|^p_{L^p(\Omega_\tau)}\right)+1+e^{-\sigma_T t}\int_\tau^t e^{\sigma_T s}|g(s)|_T^2ds\right).
\end{aligned} \tag{4.17}$$

Finally, since $u_k\rightharpoonup u$ in $L^2(\tau,T;V_T)$ as $k\to+\infty$, we get

$$\begin{aligned}
&2\widetilde{\alpha}_1\|u(t)\|^p_{L^p(\Omega_T)}+\|u(t)\|_T^2 \\
&\quad\le C\left(e^{-\sigma_T(t-\tau)}\left(1+\|u_\tau\|_\tau^2+\|u_\tau\|^p_{L^p(\Omega_\tau)}\right)+1+e^{-\sigma_T t}\int_\tau^t e^{\sigma_T s}|g(s)|_T^2ds\right) \\
&\quad\le C\left(e^{-\sigma_T(t-\tau)}\left(\|u_\tau\|_\tau^2+\|u_\tau\|^p_{L^p(\Omega_\tau)}\right)+1+\frac{1}{1-e^{\sigma_T}}C_{g,T}\right),
\end{aligned} \tag{4.18}$$

where we have used the fact that

$$\begin{aligned}\int_{\tau}^{t} e^{-\sigma_T(t-s)}|g(s)|_T^2 ds &\le \int_{t-1}^{t} e^{-\sigma_T(t-s)}|g(s)|_T^2 ds + \int_{t-2}^{t-1} e^{-\sigma_T(t-s)}|g(s)|_T^2 ds + \cdots \\ &\le \left(1 + e^{-\sigma_T} + e^{-2\sigma_T} + \cdots\right) C_{g,T} = \frac{1}{1-e^{-\sigma_T}} C_{g,T}.\end{aligned} \tag{4.19}$$

□

Let $\mathcal{R}$ be the set of all $r(t)$ such that

$$\lim_{t \to -\infty} e^{t\sigma_t}\left(\|r(t)\|_t^2 + \|r(t)\|_{L^p(\Omega_t)}^p\right) = 0. \tag{4.20}$$

Denote by $\mathfrak{D}$ the class of all families $\widehat{\mathfrak{D}} = \{D(t); D(t) \in V_t \cap L^p(\Omega_t)), D(t) \neq \emptyset, t \in \mathbb{R}\}$ such that $D(t) \subset B(r(t))$ for some $r(t) \in \mathcal{R}$.

For each $t \in \mathbb{R}$ define

$$r_0^2(t) = 2C\left(1 + \frac{1}{1-e^{-\sigma_t}} C_{g,t}\right), \tag{4.21}$$

and consider the family of closed balls $\widehat{\mathcal{B}} = \{B(t); t \in \mathbb{R}\}$, where

$$B(t) = \{v \in V_t : \|v\|_t \le r_0(t)\}, \quad t \in \mathbb{R}. \tag{4.22}$$

Then using (4.4), it is not difficult to check that $\widehat{\mathcal{B}}$ is pullback $\mathfrak{D}$-absorbing for the process $U(\cdot,\cdot)$. Moreover, by the compactness of the injection of V_t into H_t, it is clear that $B(t)$ is a compact set of H_t for any $t \in \mathbb{R}$. Then, it follows from Theorem 2.6 that the process $U(\cdot,\cdot)$ has a pullback $\mathfrak{D}$-attractor $\widehat{\mathcal{A}}_\varepsilon = \{A_\varepsilon(t) : t \in \mathbb{R}\}$ in a family of spaces $\{H_t\}$.

5. The Upper Semicontinuity of Pullback $\mathfrak{D}$-Attractors at $\varepsilon = 0$

It is proved in [4], when $\varepsilon = 0$, the existence of a pullback $\mathfrak{D}$-attractor $\widehat{\mathcal{A}}_0 = \{A_0(t) : t \in \mathbb{R}\}$ in a family of spaces $\{H_t\}$ for problem (P_0). The aim of this section is to prove the upper semicontinuity of pullback attractors $\widehat{\mathcal{A}}_\varepsilon$ at $\varepsilon = 0$ in $\{H_t\}$, that is,

$$\lim_{\varepsilon \to 0} \sup_{t \in I} \operatorname{dist}_t(A_\varepsilon(t), A_0(t)) = 0, \tag{5.1}$$

where I is an arbitrary bounded interval in $\mathbb{R}$.

The following lemma is the key of this section.

Lemma 5.1. *For each $t \in \mathbb{R}$, each $T > 0$, and each compact subset K of V_{t-T}, we have*

$$|U_\varepsilon(t,\tau)u_\tau - U_0(t,\tau)u_\tau|_t^2 \le C\sqrt{\varepsilon}, \quad \forall \tau \in [t-T, t],\ \forall u_\tau \in K, \tag{5.2}$$

where the constant C is independent of τ and u_τ.

Proof. Denote $U_\varepsilon(t,\tau)u_\tau$ by $u(t)$, and $U_0(t,\tau)u_\tau$ by $v(t)$. Let $w(t)=u(t)-v(t)$, we have

$$w_t-\varepsilon\Delta u_t-\Delta w+f(u)-f(v)=0. \tag{5.3}$$

Multiplying this equation by w and integrating over Ω_t, we get

$$\frac{1}{2}\frac{d}{dt}|w|_t^2-\varepsilon(\Delta u_t,w)_t+\|w\|_t^2+\big(f(u)-f(v),w\big)_t=0. \tag{5.4}$$

We have

$$\begin{gathered}\big(f(u)-f(v),w\big)_t=\int_{\Omega_t}\big(f(u)-f(v)\big)(u-v)\geq-\ell|u-v|_t^2,\\ \varepsilon(\Delta u_t,w)_t=-\varepsilon(\nabla u_t,\nabla w)_t\leq\varepsilon\|u_t\|_t\cdot\|w\|_t.\end{gathered} \tag{5.5}$$

Applying (5.5) in (5.4), we have

$$\frac{d}{dt}|w|_t^2\leq 2\ell|w|_t^2+2\varepsilon\|u_t\|_t\cdot\|w\|_t. \tag{5.6}$$

Hence

$$|w(t)|_t^2\leq 2\varepsilon\int_\tau^t e^{2\ell(t-s)}\|u_t(s)\|_t\cdot\|w(s)\|_t\leq 2\varepsilon e^{2\ell T}\left(\int_\tau^t\|u_t(s)\|_t^2\right)^{1/2}\left(\int_\tau^t\|w(s)\|_t^2\right)^{1/2}. \tag{5.7}$$

Now, we estimate the term on the right-hand side of (5.7). Multiplying the first equation in (1.3) by u_t and integrating over Ω_t, we obtain

$$|u_t|_t^2+\varepsilon\|u_t\|_t^2+\frac{1}{2}\frac{d}{dt}\|u\|_t^2+\frac{d}{dt}\int_{\Omega_t}F(u)\leq\int_{\Omega_t}g(t)u_t. \tag{5.8}$$

Using Cauchy's inequality, we conclude that

$$\frac{d}{dt}\left(\|u(t)\|_t^2+2\int_{\Omega_t}F(u(t))\right)+2\varepsilon\|u_t(t)\|_t^2\leq\frac{1}{2}\big|g(t)\big|_t^2. \tag{5.9}$$

Integrating (5.9) from τ to t, $\tau\in[t-T,t]$, we find that

$$\|u\|_t^2+2\int_{\Omega_t}F(u(t))+2\varepsilon\int_\tau^t\|u_t(s)\|_t^2\leq\|u_\tau\|_t^2+2\int_{\Omega_t}F(u_\tau)+\frac{1}{2}\int_\tau^t\big|g(s)\big|_t^2. \tag{5.10}$$

Since $\int_{\Omega_t} F(u(t)) \geq -\tilde{\beta}|\Omega_t| + \tilde{\alpha}_1 \|u(t)\|^p_{L^p(\Omega_t)}$, we have

$$2\varepsilon \int_\tau^t \|u_t(s)\|_t^2 \leq \|u_\tau\|_t^2 + 2\int_{\Omega_t} F(u_\tau) + \frac{1}{2}\int_\tau^t |g(s)|_t^2 + C. \tag{5.11}$$

From (3.29), we obtain

$$\|u_\tau\|_t^2 + 2\int_{\Omega_t} F(u_\tau) \leq C\|u_\tau\|_t^2 + C\|u_\tau\|^p_{L^p(\Omega_t)} + C. \tag{5.12}$$

Combining (5.10) and (5.12), we see that

$$\begin{aligned}
\int_\tau^t \|u_t(s)\|_t^2 &\leq \frac{C}{\varepsilon}\left(1 + \|u_\tau\|_t^2 + \|u_\tau\|^p_{L^p(\Omega_t)} + \int_\tau^t |g(s)|_t^2\right) \\
&\leq \frac{C}{\varepsilon}\left(1 + \|u_\tau\|_t^2 + \|u_\tau\|^p_{L^p(\Omega_t)} + \int_{t-T}^t |g(s)|_t^2\right) \\
&\leq \frac{C(K,T,g,t)}{\varepsilon},
\end{aligned} \tag{5.13}$$

because of $u_\tau \in K$ and $g \in L^2_{\text{loc}}(\mathbb{R}^{n+1})$. Now, using (5.13) in (5.7), we get

$$|w(t)|_t^2 \leq C(K,T,g,t)\sqrt{\varepsilon}\left(\int_\tau^t \|w(s)\|_t^2\right)^{1/2}. \tag{5.14}$$

Using (4.4) and noting that $\tau \in [t-T, t]$, we have

$$\begin{aligned}
\|w(t)\|_t^2 &\leq \|u(t)\|_t^2 + \|v(t)\|_t^2 \\
&\leq 2C\left(e^{-\sigma_t(t-\tau)}\left(\|u_\tau\|_t^2 + \|u_\tau\|^p_{L^p(\Omega_t)} + 1\right) + 1 + e^{-\sigma_t t}\int_\tau^t e^{\sigma_t s}|g(s)|_t^2\right) \\
&\leq 2C\left(\|u_\tau\|_t^2 + \|u_\tau\|^p_{L^p(\Omega_t)} + 2 + e^{-\sigma_t t}\int_\tau^t e^{\sigma_t s}|g(s)|_t^2\right) \\
&\leq C(K)\left(1 + \int_\tau^t |g(s)|_t^2\right) \\
&\leq C(K)\left(1 + \int_{t-T}^t |g(s)|_t^2\right).
\end{aligned} \tag{5.15}$$

Thus,

$$\begin{aligned}\int_{\tau}^{t} \|w(t)\|_t^2 &\leq C(K)\left(t-\tau+\int_{\tau}^{t}\int_{t-T}^{t} |g(s)|_t^2\right)\\ &\leq C(K)\left(T+\iint_{t-T}^{t} |g(s)|_t^2\right)\\ &\leq C(K,T,g,t).\end{aligned} \tag{5.16}$$

Combining (5.14) and (5.16) we get

$$|w(t)|_t^2 \leq C(K,T,g,t)\sqrt{\varepsilon}. \tag{5.17}$$

The proof is complete. □

Theorem 5.2. *If $g \in L^2_{\text{loc}}(\mathbb{R}^{N+1})$ satisfies* (4.3)*, then for any bounded interval $I \in \mathbb{R}$, the family of pullback $\mathfrak{D}$-attractors $\{A_\varepsilon(\cdot) : \varepsilon \in [0,1]\}$ is upper semicontinuous in $L^2(\Omega_t)$ at 0 for any $t \in I$, that is,*

$$\lim_{\varepsilon\to 0} \sup_{t\in I} \operatorname{dist}_{L^2(\Omega_t)}(A_\varepsilon(t), A_0(t)) = 0. \tag{5.18}$$

Proof. We will verify the conditions (i)–(iii) in Theorem 2.8. First, condition (i) follows directly from Lemma 5.1.

Let $B(\cdot) = \overline{B}(r_0(\cdot))$ be the corresponding family of pullback $\mathfrak{D}$-absorbing sets of (1.3), which is uniform with respect to ε. By the definition of pullback $\mathfrak{D}$-absorbing sets, for any $t \in \mathbb{R}$, there exists $\tau_0 = \tau_0(t) \leq t$ such that

$$\bigcup_{\tau\leq\tau_0} U_\varepsilon(t,\tau)B(\tau) \subset B(t) = \overline{B}(r_0(t)). \tag{5.19}$$

By Theorem 2.6, we see that

$$A_\varepsilon(t) = \bigcap_{s\leq t}\overline{\bigcup_{\tau\leq s} U_\varepsilon(t,\tau)B(\tau)}. \tag{5.20}$$

From (5.19) and (5.20), we get

$$A_\varepsilon(t) \subset \overline{B}(r_0(t)). \tag{5.21}$$

Now, for given $t_0 \in \mathbb{R}$, we can write

$$\bigcup_{\varepsilon\in[0,1]}\bigcup_{t\leq t_0} A_\varepsilon(t) \subset \bigcup_{t\leq t_0} \overline{B}(r_0(t)). \tag{5.22}$$

Because $\lim_{t\to-\infty}\sup\, r_0(t) < +\infty$ due to (4.4), from (5.22) we have

$$\bigcup_{\varepsilon\in[0,1]}\bigcup_{t\le t_0} A_\varepsilon(t) \quad \text{is bounded in } L^2(\Omega_{t_0}) \text{ for given } t_0, \tag{5.23}$$

that is, condition (ii) is satisfied. From (5.21), we can find that, for each $t \in \mathbb{R}$,

$$\bigcup_{0<\varepsilon\le 1} A_\varepsilon(t) \subset \overline{B}(r_0(t)), \tag{5.24}$$

thus $\bigcup_{0<\varepsilon\le 1} A_\varepsilon(t)$ is bounded in $H_0^1(\Omega_t)$, hence

$$\overline{\bigcup_{0<\varepsilon\le 1} A_\varepsilon(t)} \quad \text{is compact in } L^2(\Omega_t), \tag{5.25}$$

since $H_0^1(\Omega_t) \subset L^2(\Omega_t)$ compactly. Then condition (iii) holds. □

6. Conclusion

In this paper we have proved the existence and uniqueness of variational solutions satisfying the energy equality to a class of nonautonomous nonclassical diffusion equations in noncylindrical domains. We have also proved the existence of pullback attractors $\widehat{\mathcal{A}}_\varepsilon$ of the process generated by this class of solutions and the upper semicontinuity of $\{\widehat{\mathcal{A}}_\varepsilon : \varepsilon \in [0,1]\}$ at $\varepsilon = 0$, which means that the pullback attractors $\widehat{\mathcal{A}}_\varepsilon$ of the nonclassical diffusion equations converge to the pullback attractor $\widehat{\mathcal{A}}_0$ of the reaction-diffusion equation in the sense of the Hausdorff semidistance. As far as we know, this is the first result on the existence and long-time behavior of solutions to the nonclassical diffusion equations in noncylindrical domains. The result is obtained under the assumption (1.1) of spatial domains which are expanding in time. This assumption may be replaced by the assumption that the spatial domains Ω_t in $\mathbb{R}^N$ are obtained from a bounded base domain Ω by a C^2-diffeomorphism, which is continuously differentiable in the time variable and are contained, in the past, in a common bounded domain (see [5] for the related results in the case $\varepsilon = 0$).

It is noticed that the obtained results seem to be not very satisfying because although the process $U(\cdot,\cdot)$ associated to problem (1.3) is constructed in the family of spaces $H_0^1(\Omega_t) \cap L^p(\Omega_t)$, we are only able to prove the existence and upper semicontinuity of the pullback attractor in $L^2(\Omega_t)$. It would be very interesting if one can show the existence and upper semicontinuity of the pullback attractor in the space $H_0^1(\Omega_t) \cap L^p(\Omega_t)$. For nonclassical diffusion equations in cylindrical domains, this question has been solved very recently in [15, 22] by using the so-called asymptotic *a priori* estimate method. However, in the case of non-cylindrical domains, this question seems to be more difficult and is still completely open.

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