Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 876919, 7 pages
doi:10.1155/2012/876919

Research Article

On Pure Hyperradical in Semihypergroups

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Received 24 March 2012; Accepted 23 May 2012

Academic Editor: Adolfo Ballester-Bolinches

This paper deals with a class of algebraic hyperstructures called semihypergroups, which are a generalization of semigroups. In this paper, we introduce pure hyperradical of a hyperideal in a semihypergroup with zero element. For this purpose, we define pure, semipure, and other related types of hyperideals and establish some of their basic properties in semihypergroups.

1. Introduction and Preliminaries

The applications of mathematics in other disciplines, for example, in informatics, play a key role and they represent, in the last decades, one of the purpose, of the study of the experts of hyperstructures theory all over the world. Hyperstructure theory was introduced in 1934 by the French mathematician Marty [1], at the 8th Congress of Scandinavian Mathematicians, where he defined hypergroups based on the notion of hyperoperation, began to analyze their properties, and applied them to groups. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many mathematicians. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Several books have been written on hyperstructure theory, see [2–5]. A recent book on hyperstructures [3] points out on their applications in rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Another book [4] is devoted especially to the study of hyperring theory. Several kinds of hyperrings are introduced and analyzed. The volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. Some principal notions about semihypergroups theory can be found in [6–18].

Recently, Davvaz et al. [19–23] introduced the notion of Γ-semihypergroup as a generalization of a semigroup, a generalization of a semihypergroup, and a generalization of a Γ-semigroup. They presented many interesting examples and obtained a several characterizations of Γ-semihypergroups.

In this paper, we introduce pure hyperradical of a hyperideal in a semihypergroup with zero element. For this purpose, we define pure, semipure, and other related types of hyperideals, and we establish some of their basic properties in semihypergroups.

Recall first the basic terms and definitions from the hyperstructure theory.

A map $\circ : H \times H \to \mathcal{P}^*(H)$ is called *hyperoperation* or *join operation* on the set H, where H is a nonempty set and $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$ denotes the set of all nonempty subsets of H.

A *hyperstructure* is called the pair $(H, \circ)$ where $\circ$ is a hyperoperation on the set H.

A hyperstructure $(H, \circ)$ is called a *semihypergroup* if for all $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v. \tag{1.1}$$

If $x \in H$ and A, B are nonempty subsets of H, then

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \qquad A \circ x = A \circ \{x\}, \qquad x \circ B = \{x\} \circ B. \tag{1.2}$$

A nonempty subset B of a semihypergroup H is called a *subsemihypergroup* of H if $B \circ B \subseteq B$ and H is called in this case *supersemihypergroup* of B.

Let $(H, \circ)$ be a semihypergroup. Then, H is called a *hypergroup* if it satisfies the reproduction axiom, for all $a \in H$, $a \circ H = H \circ a = H$.

A nonempty subset I of a semihypergroup H is called a *right* (*left*) *hyperideal* of H if for all $x \in H$ and $r \in I$,

$$r \circ x \subseteq I (x \circ r \subseteq I). \tag{1.3}$$

A nonempty subset I of H is called a *hyperideal* (or *two-sided hyperideal*) if it is both a left hyperideal and right hyperideal.

An element e in a semihypergroup H is called *scalar identity* if

$$x \circ e = e \circ x = \{x\}, \quad \forall x \in H. \tag{1.4}$$

An element 0 in a semihypergroup H is called *zero element* if

$$x \circ 0 = 0 \circ x = \{0\}, \quad \forall x \in H. \tag{1.5}$$

We call a semihypergroup $(H, \circ)$ a *regular semihypergroup* if for every $x \in H$, $x \in x \circ y \circ x$, for some $y \in H$. Hence, every regular semigroup is a regular semihypergroup. Notice that if $(H; \circ)$ is a hypergroup, then for every $x \in H$, $x \circ H \circ x = H$. This implies that every hypergroup is a regular semihypergroup.

2. Main Results

In this section, we introduce pure hyperradical of a hyperideal in a semihypergroup with zero element. For this purpose, we define pure, semipure, and other related types of hyperideals, and we establish some of their basic properties. In what follows, H will denote a semihypergroup with scalar identity 1, which contains a zero element.

Definition 2.1. Let H be a semihypergroup. A right hyperideal A of H is called a *right pure right hyperideal* if for each $x \in A$, there is an element $y \in A$ such that $x \in x \circ y$. If A is a two-sided hyperideal with the property that for each $x \in A$, there is an element $y \in A$ such that $x \in x \circ y$, then A is called a *right pure hyperideal*.

Left pure left hyperideals and *left pure hyperideals* are defined analogously.

Definition 2.2. Let H be a semihypergroup. A right hyperideal A of H is called a *right semipure right hyperideal* if for each $x \in A$, there is an element y belonging to some proper right hyperideal of H such that $x \in x \circ y$. If A is a two-sided hyperideal with the property that for each $x \in A$, there is an element y belonging to a proper hyperideal of H such that $x \in x \circ y$, then A is called a *right semipure hyperideal*.

Left semipure left hyperideals and *left semipure hyperideals* can be similarly defined.

Example 2.3. Let $(H, \circ)$ be a semihypergroup on $H = \{0, 1, x, y, z, t\}$ with the hyperoperation $\circ$ given by the following:

$\circ$	0	x	y	z	t	1
0	0	0	0	0	0	0
x	0	$\{1, x\}$	$\{x, y, 1\}$	$\{0, 1, x, z\}$	H	x
y	0	$\{0, y\}$	y	$\{y, t, 1\}$	$\{0, 1, y, t\}$	y
z	0	z	$\{z, t\}$	z	$\{z, t\}$	z
t	0	$\{0, t\}$	$\{0, t\}$	$\{0, t\}$	$\{0, t\}$	t
1	0	x	y	z	t	1

(2.1)

It is easy to see that $I_1 = \{0, t\}$, $I_2 = \{0, z, t\}$ are right pure right hyperideals of H.

Example 2.4. Let $(H, \circ)$ be a semihypergroup on $H = \{0, 1, a, b, c, d, e, f\}$ with the hyperoperation $\circ$ given by the following:

$\circ$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	$\{a, b\}$	c	$\{c, d\}$	e	$\{e, f\}$	a
b	0	b	b	d	d	f	f	b
c	0	c	$\{c, d\}$	c	$\{c, d\}$	c	$\{c, d\}$	c
d	0	d	d	d	d	d	d	d
e	0	e	$\{e, f\}$	c	$\{c, d\}$	e	$\{e, f\}$	e
f	0	f	f	d	d	f	f	f
1	0	a	b	c	d	e	f	1

(2.2)

Clearly, $I_1 = \{0, d\}$, $I_2 = \{0, d, f\}$, and $I_3 = \{0, b, d, f\}$ are right pure right hyperideals of H. $I_4 = \{0, c, d\}$ is a two-sided hyperideal of H which is a right pure hyperideal but not a left pure hyperideal. Also, I_4 is a right and left semipure hyperideals. $I_5 = \{0, c, d, e, f\}$ is a two-sided hyperideal of H which is a right and left pure hyperideal.

Example 2.5. Let $H = [0, 1]$. Then, H with the hyperoperation $x \circ y = [0, xy]$ is a semihypergroup. Let $t \in [0, 1]$ and $T = [0, t]$. Then, T is a semihypergroup, and, moreover, T is a two-sided hyperideal of H which is neither right pure nor left pure, but it is left and right semipure.

Proposition 2.6. *Let A be a two-sided hyperideal of H. Then A is right pure if and only for any right hyperideal B, $B \cap A = B \circ A$.*

Proof. Suppose A is a right pure hyperideal of H. Since B is a right hyperideal of H, $B \circ A \subseteq B$. Also, since A is a left hyperideal, $B \circ A \subseteq A$. Hence, $B \circ A \subseteq B \cap A$. Let $x \in B \cap A$. Since A is a right pure hyperideal, there exists $y \in A$ such that $x \in x \circ y$. As $x \in B$ and $y \in A$, $x \circ y \subseteq B \circ A$. Hence, $x \in B \circ A$. This implies that $B \cap A = B \circ A$. Conversely, assume $B \cap A = B \circ A$, for any right hyperideal B of H. We show that A is right pure. Let $x \in A$. Then, $x \circ H = x \circ H \cap A = x \circ H \circ A = x \circ A$. Since $x \in x \circ H, x \in x \circ A$. Hence, there exists $y \in A$ such that $x \in x \circ y$. This proves that A is a right pure hyperideal. □

Corollary 2.7. *If A is a right pure hyperideal, then $A = A \circ A$.*

Proposition 2.8. *(0) and H are right pure hyperideals of H. Any union and finite intersection of right pure (resp., semipure) hyperideals is right pure (resp., semipure).*

Proof. (0) and H are obviously right pure hyperideals. Let I_1 and I_2 be right pure hyperideals and let $x \in I_1 \cap I_2$. Since $x \in I_1$ and I_1 is right pure, exists $y_1 \in I_1$ such that $x \in x \circ y_1$. Similarly, exists $y_2 \in I_2$ such that $x \in x \circ y_2$. Thus, we have $x \in x \circ y_2 \subseteq (x \circ y_1) \circ y_2 = x \circ (y_1 \circ y_2)$. Since $y_1 \circ y_2 \in I_1 \cap I_2$, it follows that $I_1 \cap I_2$ is right pure. The remaining cases of this proposition can be similarly proved. □

It follows from the above proposition that if I is any hyperideal of H, then I contains a largest pure hyperideal, which is in fact the union of all pure hyperideals contained in I (such hyperideals exist, e.g., (0)) and hence a pure hyperideal. The largest pure hyperideal contained in I is denoted by $L(I)$. Similarly, each hyperideal I contains a largest semipure hyperideal, denoted by $H(I)$. $L(I)$ (resp., $H(I)$) is called the *pure* (resp., *semipure*) *part* of I.

Definition 2.9. Let I be a right pure (resp., semipure) hyperideal of H. Then, I is called *purely* (resp., *semipurely*) *maximal* if I is a maximal element in the set of proper right pure (resp., semipure) hyperideals.

In Example 2.4, $I_4 = \{0, c, d\}$, $I_5 = \{0, c, d, e, f\}$, and $I_6 = \{0, a, b, c, d, e, f\}$ are right pure hyperideals of the semihypergroup H, and it is clear that I_6 is purely maximal hyperideal.

Definition 2.10. Let I be a right pure (resp., semipure) hyperideal of H. Then, I is called *purely* (resp. *semipurely*) *prime* if it is proper and if for any right pure (resp., semipure) hyperideals I_1 and I_2, $I_1 \cap I_2 \subseteq I \Rightarrow I_1 \subseteq I$ or $I_2 \subseteq I$.

In Example 2.4, the hyperideal I_6 mentioned above is purely prime hyperideal.

The following propositions are stated for pure and semipure hyperideals simultaneously. However, the proofs are given only for one case, since the proofs are similar for the remaining cases.

Proposition 2.11. *Any purely (resp., semipurely) maximal hyperideal is purely (resp., semipurely) prime.*

Proof. Suppose I is purely maximal, and I_1, I_2 are right pure hyperideals such that $I_1 \cap I_2 \subseteq I$. Suppose $I_1 \nsubseteq I$. Then, $I_1 \cup I = H$. Now, $I_2 = I_2 \cap H = I_2 \cap (I_1 \cup I) = (I_2 \cap I_1) \cup (I_2 \cap I) \subseteq I \cup I = I$. □

Proposition 2.12. *The pure (resp. semipure) part of any maximal hyperideal is purely (resp. semipurely) prime.*

Proof. Let M be a maximal hyperideal of H. We show that $L(M)$, the pure part of M, is purely prime. Suppose $I_1 \cap I_2 \subseteq L(M)$ with I_1, I_2 pure. If $I_1 \subseteq M$, then $I_1 \subseteq L(M)$ and we are done. Suppose $I_1 \nsubseteq M$, then $I_1 \cup M = H$. Hence, $I_2 = I_2 \cap H = I_2 \cap (I_1 \cup M) = (I_2 \cap I_1 \cup (I_2 \cap M) \subseteq M \cup M = M$. Hence, $I_2 \subseteq M$. This implies that $I_2 \subseteq L(M)$, since I_2 is pure. □

Proposition 2.13. *If I is right pure (resp., semipure) hyperideal of H and $a \notin I$, then there exists a purely (resp., semipurely) prime hyperideal J such that $I \subseteq J$ and $a \notin J$.*

Proof. Consider the set X ordered by inclusion: $X = \{J : J$ is a right semipure hyperideal, $I \subseteq J, a \notin J\}$. Then, $X \neq \emptyset$, since $I \in X$. Let $(J_k)_{k \in K}$ be a totally ordered subset of X. Clearly, $\bigcup_k J_k$ is a semipure hyperideal with $a \notin \bigcup_k J_k$. Hence, X is inductively ordered. Therefore, by Zorn's Lemma, X has a maximal element J. We will show that J is semipurely prime. Suppose I_1, I_2 are right semipure hyperideals such that $I_1 \nsubseteq J$ and $I_2 \nsubseteq J$. Since I_k $(k = 1, 2)$ and J are semipure, $I_k \cup J$ is a semipure hyperideal such that $J \subseteq I_k \cup J$. We then claim that an $a \in I_k \cup J$ $(k = 1, 2)$. Because, if $a \notin I_k \cup J$, then by the maximality of J, we have $I_k \cup J \subseteq J$. But this contradicts the assumption $I_k \nsubseteq J$ $(k = 1, 2)$. Hence, $a \in (I_1 \cap I_2 \cup J)$. Since $a \notin J$, it follows that $I_1 \cap I_2 \nsubseteq J$. Hence, by contrapositivity, we conclude that J is semipurely prime. □

Proposition 2.14. *Let I be a proper right pure (resp. semipure) hyperideal of H. Then, I is contained in a purely (resp., semipurely) maximal hyperideal.*

Proof. Consider the set, ordered by inclusion, $X = \{J : J$ is a proper right semipure hyperideal and $J \supseteq I\}$. Clearly, $X \neq \emptyset$, since $I \in X$. Moreover, any $J \in X$ is a proper hyperideal because $1 \notin J$. Hence, any directed union of elements in X is still in X. So, X is inductively ordered. Hence, by Zorn's Lemma, X contains a maximal element J and any proper semipure hyperideal containing J also contains I and so it belongs to X. But such an hyperideal will be J itselft, since J is maximal in X. Therefore, J is semipurely maximal. □

Proposition 2.15. *Let I be a right pure (resp., semipure) hyperideal of H. Then, I is the intersection of purely (resp., semipurely) prime hyperideals of H containing I.*

Proof. By Propositions 2.14 and 2.11, there exists a set $\{P_\alpha : P_\alpha$ is a semipurely prime hyperideal containing $I, \alpha \in \Lambda\}$. Hence $I \subseteq \bigcap_{\alpha \in \Lambda} P_\alpha$. To prove $\bigcap_{\alpha \in \Lambda} P_\alpha \subseteq I$, assume there exists an element x such that $x \notin I$. Then by the above proposition, there exists a semipurely prime hyperideal P_{α_0} such that $I \subseteq P_{\alpha_0}$, but $x \notin P_{\alpha_0}$. This implies that $x \notin \bigcap_{\alpha \in \Lambda} P_\alpha$. This proves the proposition. □

Definition 2.16. Let A be a right hyperideal of H and let $\{K_\alpha : \alpha \in \Lambda\}$ be the set of right pure hyperideals containing A. Then, we define $P(A) = \bigcap_{\alpha\in\Lambda} K_\alpha$ and call it the *pure hyperadical* of A. Note that the set $\{K_\alpha : K_\alpha$ is a right pure hyperideal containing $A\}$ is nonempty, since H itself belongs to this set.

Proposition 2.17. *If $P(A)$ is the pure hyperradical of the hyperideal A, then each of the following statement holds true:*

(1) *$P(A)$ is either pure or semipure hyperideal containing A.*

(2) *$P(A)$ is contained in every right pure hyperideal which contains A.*

(3) *If P_α are those purely prime hyperideals which contain A, then $P(A) = \bigcap_\alpha P_\alpha$.*

Proof. (1) If the set $\{K_\alpha : \alpha \in \Lambda, K_\alpha$ is a right pure hyperideal of H containing $A\}$ consists of H alone, then $P(A) = H$. Hence, $P(A)$ is pure in this case. In case the set $\{K_\alpha : K_\alpha$ is a right pure hyperideal containing $A\}$ has only a finite number of elements, then $P(A)$ is pure by Proposition 2.8. In general, $P(A)$ is semiprime.

(2) This is obvious.

(3) Since every pure hyperideal is contained in a purely maximal hyperideal (Proposition 2.14) and every purely maximal hyperideal is purely prime (Proposition 2.11), the set $\{P_\alpha : P_\alpha$ is purely prime containing $A\}$ is nonempty. Hence, from Part (2), it follows that $P(A) \subseteq \bigcap_\alpha P_\alpha$. We prove that $\bigcap_\alpha P_\alpha \subseteq P(A)$. To prove this, assume that $x \notin P(A)$. Since $P(A) = \bigcap_\alpha K_\alpha$, where each K_α is a right pure hyperideal containing A. Hence, $x \notin K_{\alpha_0}$ for some α_0. Thus K_{α_0} is a proper pure hyperideal which contains A but misses x. Hence, by Proposition 2.13, there exists a purely prime hyperideal P_{α_0}, such that $A \subseteq K_{\alpha_0} \subseteq P_{\alpha_0}$ and $x \notin P_{\alpha_0}$. Hence, $x \notin \bigcap_\alpha P_\alpha$, where P_α's are purely prime hyperideals containing A. From this we conclude that $P(A) = \bigcap_\alpha P_\alpha$. □

Definition 2.18. Let A be a hyperideal of H. Then $H(A)$ the semipure part of A, that is, the union of all semipure hyperideals contained in A is called the *semipure hyperradical* of A.

Proposition 2.19. *For each hyperideal A, $H(A)$ is the intersection of semipurely prime hyperideals.*

Proof. It follows from Proposition 2.15. □

Note that $P(A)$ and $H(A)$ are distinct in general. For example, if $A = \{0, d\}$ of Example 2.4, then

$$\begin{gathered} P(A) = \{0, d, f\}, \\ H(A) = \{0\}. \end{gathered} \tag{2.3}$$

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