Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 927607, 16 pages
doi:10.1155/2012/927607

Research Article

On the Solvability of Discrete Nonlinear Two-Point Boundary Value Problems

Blaise Kone[1] and Stanislas Ouaro[2]

[1] *Laboratoire d'Analyse Mathématique des Equations (LAME), Institut Burkinabé des Arts et Métiers, Université de Ouagadougou, 03 BP 7021, Ouagadougou 03, Burkina Faso*
[2] *Laboratoire d'Analyse Mathématique des Equations (LAME), UFR Sciences Exactes et Appliquées, Université de Ouagadougou, 03 BP 7021, Ouagadougou 03, Burkina Faso*

Correspondence should be addressed to Stanislas Ouaro, ouaro@yahoo.fr

Received 22 March 2012; Accepted 27 April 2012

Academic Editor: Paolo Ricci

We prove the existence and uniqueness of solutions for a family of discrete boundary value problems by using discrete's Wirtinger inequality. The boundary condition is a combination of Dirichlet and Neumann boundary conditions.

1. Introduction

In this paper, we study the following nonlinear discrete boundary value problem:

$$\begin{gathered} -\Delta(a(k-1,\Delta u(k-1))) = f(k), \quad k \in \mathbb{Z}[1,T], \\ u(0) = \Delta u(T) = 0, \end{gathered} \tag{1.1}$$

where $T \geq 2$ is a positive integer and $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator. Throughout this paper, we denote by $\mathbb{Z}[a,b]$ the discrete interval $\{a, a+1, \ldots, b\}$, where a and b are integers and $a < b$.

We consider in (1.1) two different boundary conditions: a Dirichlet boundary condition ($u(0) = 0$) and a Neumann boundary condition ($\Delta u(T) = 0$). In the literature, the boundary condition considered in this paper is called a mixed boundary condition.

We also consider the function space

$$W = \{v : \mathbb{Z}[0,T+1] \longrightarrow \mathbb{R};\ \text{such that}\ v(0) = \Delta v(T) = 0\}, \tag{1.2}$$

where W is a T-dimensional Hilbert space [1, 2] with the inner product

$$(u,v)=\sum_{k=1}^{T}u(k)v(k),\quad \forall u,v\in W. \tag{1.3}$$

The associated norm is defined by

$$\|u\|=\left(\sum_{k=1}^{T}|u(k)|^{2}\right)^{1/2}. \tag{1.4}$$

For the data f and a, we assume the following.

(H_1) $f:\mathbb{Z}[1,T]\to\mathbb{R}$.

(H_2) $a(k,\cdot):\mathbb{R}\to\mathbb{R}$ for all $k\in\mathbb{Z}[0,T]$ and their exists a mapping $A:\mathbb{Z}[0,T]\times\mathbb{R}\to\mathbb{R}$ which satisfies $a(k,\xi)=(\partial/\partial\xi)A(k,\xi)$, for all $k\in\mathbb{Z}[0,T]$ and $A(k,0)=0$, for all $k\in\mathbb{Z}[0,T]$.

(H_3) $(a(k,\xi)-a(k,\eta))\cdot(\xi-\eta)>0$ for all $k\in\mathbb{Z}[0,T]$ and $\xi,\eta\in\mathbb{R}$ such that $\xi\neq\eta$.

(H_4) $|\xi|^{p(k)}\le a(k,\xi)\xi\le p(k)A(k,\xi)$ for all $k\in\mathbb{Z}[0,T]$ and $\xi\in\mathbb{R}$.

(H_5) $p:\mathbb{Z}[0,T]\to(1,+\infty)$.

The theory of difference equations occupies now a central position in applicable analysis. We just refer to the recent results of Agarwal et al. [1], Yu and Guo [3], Koné and Ouaro [4], Guiro et al. [5], Cai and Yu [6], Zhang and Liu [7], Mihăilescu et al. [8], Candito and D′Agui [9], Cabada et al. [10], Jiang and Zhou [11], and the references therein. In [7], the authors studied the following problem:

$$\begin{gathered}\Delta^{2}y(k-1)+\lambda f(y(k))=0,\quad k\in\mathbb{Z}[1,T],\\ y(0)=y(T+1)=0,\end{gathered} \tag{1.5}$$

where $\lambda>0$ is a parameter, $\Delta^{2}y(k)=\Delta(\Delta y(k))$, $f:[0,+\infty)\to\mathbb{R}$ a continuous function satisfying the condition

$$f(0)=-a<0,\quad \text{where } a \text{ is a positive constant}. \tag{1.6}$$

The problem (1.5) is referred as the “semipositone” problem in the literature, which was introduced by Castro and Shivaji [2]. Semipositone problems arise in bulking of mechanical systems, design of suspension bridges, chemical reactions, astrophysics, combustion, and management of natural resources.

The studies regarding problems like (1.1) or (1.5) can be placed at the interface of certain mathematical fields such as nonlinear partial differential equations and numerical analysis. On the other hand, they are strongly motivated by their applicability in mathematical physics as mentioned above.

In [11], Jiang and Zhou studied the following problem:

$$\Delta^2 u(k-1) = f(k, u(k)), \quad k \in \mathbb{Z}[1,T],$$
$$u(0) = \Delta u(T) = 0, \tag{1.7}$$

where T is a fixed positive integer, $f : \mathbb{Z}[1,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Jiang and Zhou proved an existence of nontrivial solutions for (1.7) by using strongly monotone operator principle and critical point theory.

In this paper, we consider the same boundary conditions as in [11] but the main operator is more general than the one in [11] and involves variable exponent.

Problem (1.1) is a discrete variant of the variable exponent anisotropic problem

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u}{\partial x_i}\right) = f(x) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \Gamma_1 \tag{1.8}$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_2,$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary, $\Gamma_1 \cup \Gamma_2 = \partial\Omega$, $f \in L^\infty(\Omega)$, p_i continuous on $\overline{\Omega}$ such that $1 < p_i(x) < N$ and $\sum_{i=1}^{N}(1/p_i^-) > 1$ for all $x \in \overline{\Omega}$ and all $i \in \mathbb{Z}[1,N]$, where $p_i^- := \operatorname{ess\,inf}_{x\in\Omega} p_i(x)$.

The first equation in (1.8) was recently analyzed by Koné et al. [12] and Ouaro [13] and generalized to a Radon measure data by Koné et al. [14] for an homogeneous Dirichlet boundary condition ($u = 0$ on $\partial\Omega$). The study of (1.8) will be done in a forthcoming work. Problems like (1.8) have been intensively studied in the last decades since they can model various phenomena arising from the study of elastic mechanics (see [15, 16]), electrorheological fluids (see [15, 17–19]), and image restoration (see [20]). In [20], Chen et al. studied a functional with variable exponent $1 \leq p(x) \leq 2$ which provides a model for image denoising, enhancement, and restoration. Their paper created another interest for the study of problems with variable exponent.

Note that Mihăilescu et al. (see [21, 22]) were the first authors who studied anisotropic elliptic problems with variable exponent. In general, the interested reader can find more information about difference equations in [1–11, 23–25], more information about variable exponent in [12–22, 26].

Our goal in this paper is to use a minimization method in order to establish some existence results of solutions of (1.1). The idea of the proof is to transfer the problem of the existence of solutions for (1.1) into the problem of existence of a minimizer for some associated energy functional. This method was successfully used by Bonanno et al. [27] for the study of an eigenvalue nonhomogeneous Neumann problem, where, under an appropriate oscillating behavior of the nonlinear term, they proved the existence of a determined open interval of positive parameters for which the problem considered admits infinitely many weak solutions that strongly converges to zero, in an appropriate Orlicz-Sobolev space. Let us point out that, to our best knowledge, discrete problems like (1.1) involving anisotropic exponents have been discussed for the first time by Mihăilescu et al. (see [8]), in a second time by Koné and Ouaro [4], and in a third time by Guiro et al. [5]. In

[8], the authors proved by using critical point theory, existence of a continuous spectrum of eigenvalues for the problem

$$-\Delta\left(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\right) = \lambda|u(k)|^{q(k)-2}u(k), \quad k \in \mathbb{Z}[1,T],$$
$$u(0) = u(T+1) = 0, \tag{1.9}$$

where $T \geq 2$ is a positive integer and the functions $p : \mathbb{Z}[0,T] \rightarrow [2,\infty)$ and $q : \mathbb{Z}[1,T] \rightarrow [2,\infty)$ are bounded while λ is a positive constant.

In [4], Koné and Ouaro proved, by using minimization method, existence and uniqueness of weak solutions for the following problem:

$$-\Delta(a(k-1,\Delta u(k-1))) = f(k), \quad k \in \mathbb{Z}[1,T],$$
$$u(0) = u(T+1) = 0, \tag{1.10}$$

where $T \geq 2$ is a positive integer.

The function $a(k-1,\Delta u(k-1))$ which appears in the left-hand side of problem (1.1) is more general than the one which appears in (1.9). Indeed, as examples of functions which satisfy the assumptions (H_2)–(H_5), we can give the following.

(i) $A(k,\xi) = (1/p(k))|\xi|^{p(k)}$, where $a(k,\xi) = |\xi|^{p(k)-2}\xi$, for all $k \in \mathbb{Z}[0,T]$ and $\xi \in \mathbb{R}$.

(ii) $A(k,\xi) = 1/p(k)[(1+|\xi|^2)^{p(k)/2} - 1]$, where $a(k,\xi) = (1+|\xi|^2)^{(p(k)-2)/2}\xi$, for all $k \in \mathbb{Z}[0,T]$ and $\xi \in \mathbb{R}$.

In [5], Guiro et al. studied the following two-point boundary value problems

$$-\Delta(a(k-1,\Delta u(k-1))) + |u(k)|^{p(k)}u(k) = f(k), \quad k \in \mathbb{Z}[1,T],$$
$$\Delta u(0) = \Delta u(T) = 0. \tag{1.11}$$

The function $a(k-1,\Delta u(k-1))$ has the same properties as in [4], but the boundary conditions are different. For this reason, Guiro et al. defined a new norm in the Hilbert space considered in order to get, by using minimization methods, existence of a unique weak solution (which is also a classical solution since the Hilbert space associated is of finite dimension). Indeed, they used the following norm:

$$\|u\| = \left(\sum_{k=1}^{T+1}|\Delta u(k-1)|^2 + \sum_{k=1}^{T}|u(k)|^2\right)^{1/2}, \tag{1.12}$$

which is associated to the Hilbert space

$$W = \{v : \mathbb{Z}[0,T+1] \longrightarrow \mathbb{R}; \text{ such that } \Delta v(0) = \Delta v(T) = 0\}. \tag{1.13}$$

In order to get the coercivity of the energy functional, the authors of [5] assumed the following on the exponent:

$$p : \mathbb{Z}[0,T] \longrightarrow (2,+\infty). \tag{1.14}$$

The assumption above allowed them to exploit the convexity property of the map $x \to x^{p^-/2}$.

Problem (1.11) is a discrete variant of the following problem:

$$\begin{aligned} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u}{\partial x_i}\right) &= f(x,u) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \Omega, \end{aligned} \tag{1.15}$$

which was studied by Boureanu and Radulescu in [26] with an additional condition that $u \geq 0$. Note that, in [26], the Neumann condition is more general than the one in problem (1.11). In this paper, we use the discrete Wirtinger inequality (see [23]) which allows us to assume that the exponent $p : \mathbb{Z}[0,T] \to (1,+\infty)$. The discrete Wirtinger inequality is a discrete variant of the well-known Poincaré-Wirtinger inequality (see [28]). Another difference of the present paper compared to [5] is on the boundary condition.

The remaining part of this paper is organized as follows. Section 2 is devoted to mathematical preliminaries. The main existence and uniqueness result is stated and proved in Section 3. In Section 4, we discuss some extensions, and, finally, in Section 5, we apply our theoretical results to an example.

2. Preliminaries

From now, we will use the following notations:

$$p^- = \min_{k\in\mathbb{Z}[0,T]} p(k), \qquad p^+ = \max_{k\in\mathbb{Z}[0,T]} p(k). \tag{2.1}$$

Moreover, it is useful to introduce other norms on W, namely,

$$|u|_m = \left(\sum_{k=1}^{T} |u(k)|^m\right)^{1/m} \quad \forall u \in W,\ m \geq 2. \tag{2.2}$$

We have the following inequalities (see [6, 8]) which are used in the proof of Lemma 2.1:

$$T^{(2-m)/(2m)}\,|u|_2 \leq |u|_m \leq T^{1/m}|u|_2 \quad \forall u \in W,\ m \geq 2. \tag{2.3}$$

In the sequel, we will use the following auxiliary result.

Lemma 2.1 (see [5]). *There exist two positive constants C_1, C_2 such that*

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq C_1 \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^2 \right)^{p^-/2} - C_2, \tag{2.4}$$

for all $u \in W$ with $\|u\| > 1$.

We have the following result.

Lemma 2.2 (discrete Wirtinger's inequality, see Theorem 12.6.2, page 860 in [23]). *For any function $u(k)$, $k \in \mathbb{Z}[0, T]$ satisfying $u(0) = 0$, the following inequality holds:*

$$4 \sin^2 \left(\frac{\pi}{2(2T+1)} \right) \sum_{k=1}^{T} |u(k)|^2 \leq \sum_{k=0}^{T-1} |\Delta u(k)|^2. \tag{2.5}$$

3. Existence and Uniqueness of Weak Solution

In this section, we study the existence and uniqueness of weak solution of (1.1).

Definition 3.1. A weak solution of (1.1) is a function $u \in W$ such that

$$\sum_{k=1}^{T+1} a(k-1, \Delta u(k-1)) \Delta v(k-1) = \sum_{k=1}^{T} f(k) v(k) \quad \text{for any } v \in W. \tag{3.1}$$

Note that, since W is a finite dimensional space, the weak solutions coincide with the classical solutions of the problem (1.1).

We have the following result.

Theorem 3.2. *Assume that* (H_1)–(H_5) *hold. Then, there exists a unique weak solution of* (1.1).

The energy functional corresponding to problem (1.1) *is defined by* $J : W \to \mathbb{R}$ *such that*

$$J(u) = \sum_{k=1}^{T+1} A(k-1, \Delta u(k-1)) - \sum_{k=1}^{T} f(k) u(k). \tag{3.2}$$

We first present some basic properties of J.

Proposition 3.3. *The functional J is well defined on W and is of class $C^1(W, \mathbb{R})$ with the derivative given by*

$$\langle J'(u), v \rangle = \sum_{k=1}^{T+1} a(k-1, \Delta u(k-1)) \Delta v(k-1) - \sum_{k=1}^{T} f(k) v(k), \tag{3.3}$$

for all $u, v \in W$.

The proof of Proposition 3.3 can be found in [5].

We now define the functional I by

$$I(u)=\sum_{k=1}^{T+1}A(k-1,\Delta u(k-1)). \tag{3.4}$$

We need the following lemma for the proof of Theorem 3.2.

Lemma 3.4. *The functional I is weakly lower semicontinuous.*

Proof. A is convex with respect to the second variable according to (H_2). Thus, it is enough to show that I is lower semicontinuous. For this, we fix $u\in W$ and $\epsilon>0$. Since I is convex, we deduce that, for any $v\in W$,

$$\begin{aligned}
I(v)&\geq I(u)+\langle I'(u),v-u\rangle\\
&\geq I(u)+\sum_{k=1}^{T+1}a(k-1,\Delta u(k-1))(\Delta v(k-1)-\Delta u(k-1))\\
&\geq I(u)-\sum_{k=1}^{T+1}|a(k-1,\Delta u(k-1))||\Delta v(k-1)-\Delta u(k-1)|\\
&\geq I(u)-\sum_{k=1}^{T+1}|a(k-1,\Delta u(k-1))||v(k)-u(k)+u(k-1)-v(k-1)|\\
&\geq I(u)-\sum_{k=1}^{T+1}|a(k-1,\Delta u(k-1))|(|v(k)-u(k)|+|v(k-1)-u(k-1)|).
\end{aligned} \tag{3.5}$$

We define H and B by

$$\begin{aligned}
H&=\sum_{k=1}^{T+1}|a(k-1,\Delta u(k-1))||v(k)-u(k)|,\\
B&=\sum_{k=1}^{T+1}|a(k-1,\Delta u(k-1))||v(k-1)-u(k-1)|.
\end{aligned} \tag{3.6}$$

By using Schwartz inequality, we get

$$\begin{aligned}
H&\leq\left(\sum_{k=1}^{T+1}|a(k-1,\Delta u(k-1))|^2\right)^{1/2}\left(\sum_{k=1}^{T+1}|v(k)-u(k)|^2\right)^{1/2}\\
&\leq\left(\sum_{k=1}^{T+1}|a(k-1,\Delta u(k-1))|^2\right)^{1/2}\|v-u\|.
\end{aligned} \tag{3.7}$$

The same calculus gives

$$B \leq \left(\sum_{k=1}^{T+1} |a(k-1, \Delta u(k-1))|^2\right)^{1/2} \|v-u\|. \tag{3.8}$$

Finally, we have

$$I(v) \geq I(u) - \left(1 + 2\sum_{k=1}^{T+1} |a(k-1, \Delta u(k-1))|^2\right)^{1/2} \|v-u\| \geq I(u) - \epsilon \tag{3.9}$$

for all $v \in W$ with $\|v-u\| < \delta = \epsilon / K(T,u)$, where $K(T,u) = (1 + 2\sum_{k=1}^{T+1} |a(k-1, \Delta u(k-1))|^2)^{1/2}$.

We conclude that I is weakly lower semicontinuous. □

We also have the following result.

Proposition 3.5. *The functional J is bounded from below, coercive, and weakly lower semicontinuous.*

Proof. By Lemma 3.4, J is weakly lower semicontinuous. We will only prove the coerciveness of the energy functional since the boundedness from below of J is a consequence of coerciveness. The other proofs can be found in [5]. By (H_4), we deduce that

$$\begin{aligned} J(u) &= \sum_{k=1}^{T+1} A(k-1, \Delta u(k-1)) - \sum_{k=1}^{T} f(k)u(k) \\ &\geq \sum_{k=1}^{T+1} \frac{1}{p(k)} |\Delta u(k-1)|^{p(k-1)} - \sum_{k=1}^{T} |f(k)u(k)| \\ &\geq \frac{1}{p^+} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} - \left(\sum_{k=1}^{T} |f(k)|^2\right)^{1/2} \left(\sum_{k=1}^{T} |u(k)|^2\right)^{1/2}. \end{aligned} \tag{3.10}$$

To prove the coercivity of J, we may assume that $\|u\| > 1$, and we get from the above inequality and Lemma 2.1, the following:

$$J(u) \geq \frac{C_1}{p^+} \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^2\right)^{p^-/2} - C_2 - \left(\sum_{k=1}^{T} |f(k)|^2\right)^{1/2} \left(\sum_{k=1}^{T} |u(k)|^2\right)^{1/2}. \tag{3.11}$$

Using Wirtinger's discrete inequality, we obtain

$$\begin{aligned} J(u) &\geq \frac{C_1}{p^+}\left(4\sin^2\left(\frac{\pi}{2(2T+1)}\right)\sum_{k=1}^{T}|u(k)|^2\right)^{p^-/2} - C_2 - \left(\sum_{k=1}^{T}|f(k)|^2\right)^{1/2}\left(\sum_{k=1}^{T}|u(k)|^2\right)^{1/2} \\ &\geq \frac{C_1 2^{p^-}}{p^+}\sin^{p^-}\left(\frac{\pi}{2(2T+1)}\right)\left(\sum_{k=1}^{T}|u(k)|^2\right)^{p^-/2} - C_2 - \left(\sum_{k=1}^{T}|f(k)|^2\right)^{1/2}\left(\sum_{k=1}^{T}|u(k)|^2\right)^{1/2} \\ &\geq \frac{C_1 2^{p^-}}{p^+}\sin^{p^-}\left(\frac{\pi}{2(2T+1)}\right)\|u\|^{p^-} - K_1\|u\| - C_2, \end{aligned} \tag{3.12}$$

where K_1 is positive constant. Hence, since $p^- > 1$, J is coercive. □

We can now give the proof of Theorem 3.2.

Proof of Theorem 3.2. By Proposition 3.5, J has a minimizer which is a weak solution of (1.1).

In order to end the proof of Theorem 3.2, we will prove the uniqueness of the weak solution.

Let u_1 and u_2 be two weak solutions of problem (1.1), then we have

$$\begin{aligned} \sum_{k=1}^{T+1} a(k-1,\Delta u_1(k-1))\Delta(u_1-u_2)(k-1) &= \sum_{k=1}^{T} f(k)(u_1-u_2)(k), \\ \sum_{k=1}^{T+1} a(k-1,\Delta u_2(k-1))\Delta(u_1-u_2)(k-1) &= \sum_{k=1}^{T} f(k)(u_1-u_2)(k). \end{aligned} \tag{3.13}$$

Adding the two equalities of (3.13), we obtain

$$\sum_{k=1}^{T+1}[a(k-1,\Delta u_1(k-1)) - a(k-1,\Delta u_2(k-1))]\Delta(u_1-u_2)(k-1) = 0. \tag{3.14}$$

Using (H_3), we deduce from (3.14) that

$$\Delta u_1(k-1) = \Delta u_2(k-1) \quad \forall k = 1,\dots,T+1. \tag{3.15}$$

Therefore, by using discrete's Wirtinger inequality, we get

$$4\sin^2\left(\frac{\pi}{2(2T+1)}\right)\sum_{k=1}^{T}|(u_1-u_2)(k)|^2 \leq \sum_{k=1}^{T+1}|\Delta(u_1-u_2)(k-1)|^2 = 0, \tag{3.16}$$

which implies that $\left(\sum_{k=1}^{T}|(u_1-u_2)(k)|^2\right)^{1/2} = 0$. It follows that $u_1 = u_2$. □

4. Some Extensions

4.1. Extension 1

In this section, we show that the existence result obtained for (1.1) can be extended to more general discrete boundary value problem of the form

$$\begin{gathered} -\Delta(a(k-1,\Delta u(k-1)))+|u(k)|^{q(k)-2}u(k)=f(k), \quad k\in\mathbb{Z}[1,T] \\ u(0)=\Delta u(T)=0, \end{gathered} \tag{4.1}$$

where $T\geq 2$ is a positive integer, and we assume that

(H_6) $q:\mathbb{Z}[1,T]\longrightarrow(1,+\infty)$.

By a weak solution of problem (4.1), we understand a function $u\in W$ such that, for any $v\in W$,

$$\sum_{k=1}^{T+1}a(k-1,\Delta u(k-1))\Delta v(k-1)+\sum_{k=1}^{T}|u(k)|^{q(k)-2}u(k)v(k)=\sum_{k=1}^{T}f(k)v(k). \tag{4.2}$$

We have the following result.

Theorem 4.1. *Under assumptions (H_1)–(H_6), there exists a unique weak solution of problem* (4.1).

Proof . For $u\in W$,

$$J(u)=\sum_{k=1}^{T+1}A(k-1,\Delta u(k-1))+\sum_{k=1}^{T}\frac{1}{p(k)}|u(k)|^{q(k)}-\sum_{k=1}^{T}f(k)u(k) \tag{4.3}$$

is such that $J\in C^1(W;\mathbb{R})$ and is weakly lower semicontinuous, and we have

$$\begin{aligned} \langle J'(u),v\rangle=&\sum_{k=1}^{T+1}a(k-1,\Delta u(k-1))\Delta v(k-1) \\ &+\sum_{k=1}^{T}|u(k)|^{q(k)-2}u(k)v(k)-\sum_{k=1}^{T}f(k)v(k), \end{aligned} \tag{4.4}$$

for all $u,v\in W$.

This implies that the weak solutions of problem (4.1) coincide with the critical points of J. We then have to prove that J is bounded below and coercive in order to complete the proof.

As

$$\sum_{k=1}^{T}\frac{1}{q(k)}|u(k)|^{q(k)}\geq 0, \tag{4.5}$$

then

$$J(u) \geq \sum_{k=1}^{T+1} A(k-1, \Delta u(k-1)) - \sum_{k=1}^{T} f(k)v(k). \tag{4.6}$$

Using Proposition 3.5, we deduce that J is bounded below and coercive.

Let u_1 and u_2 be two weak solutions of problem (4.1), then we have

$$\begin{aligned}
&\sum_{k=1}^{T+1} a(k-1, \Delta u_1(k-1))\Delta(u_1-u_2)(k-1) + \sum_{k=1}^{T} |u_1(k)|^{q(k)-2} u_1(k)(u_1(k)-u_2(k)) \\
&\quad = \sum_{k=1}^{T} f(k)(u_1-u_2)(k), \\
&\sum_{k=1}^{T+1} a(k-1, \Delta u_2(k-1))\Delta(u_1-u_2)(k-1) + \sum_{k=1}^{T} |u_2(k)|^{q(k)-2} u_2(k)(u_1(k)-u_2(k)) \\
&\quad = \sum_{k=1}^{T} f(k)(u_1-u_2)(k).
\end{aligned} \tag{4.7}$$

Adding these two equalities, we obtain

$$\begin{aligned}
&\sum_{k=1}^{T+1} [a(k-1, \Delta u_1(k-1)) - a(k-1, \Delta u_2(k-1))]\Delta(u_1-u_2)(k-1) \\
&\quad + \sum_{k=1}^{T} \left(|u_1(k)|^{q(k)-2} u_1(k) - |u_2(k)|^{q(k)-2} u_2(k)\right)(u_1(k)-u_2(k)) = 0.
\end{aligned} \tag{4.8}$$

We deduce that

$$\sum_{k=1}^{T} \left(|u_1(k)|^{q(k)-2} u_1(k) - |u_2(k)|^{q(k)-2} u_2(k)\right)(u_1(k)-u_2(k)) = 0, \tag{4.9}$$

which implies that

$$u_1(k) - u_2(k) = 0 \quad \forall k = 1, \ldots, T, \tag{4.10}$$

and we get $u_1 = u_2$. □

4.2. Extension 2

In this section, we show that the existence result obtained for (1.1) can be extended to more general discrete boundary value problem of the form

$$-\Delta(a(k-1,\Delta u(k-1)))+\lambda|u(k)|^{\beta^+-2}u(k)=f(k,u(k)),\quad k\in\mathbb{Z}[1,T],$$
$$u(0)=\Delta u(T)=0, \tag{4.11}$$

where $T\geq 2$ is a positive integer, $\lambda\in\mathbb{R}^+$, and $f:\mathbb{Z}[1,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function with respect to the second variable for all $(k,z)\in\mathbb{Z}[1,T]\times\mathbb{R}$.

For every $k\in\mathbb{Z}[1,T]$ and every $t\in\mathbb{R}$, we put $F(k,t)=\int_0^t f(k,\tau)d\tau$.

By a weak solution of problem (4.11), we understand a function $u\in W$ such that

$$\sum_{k=1}^{T+1}a(k-1,\Delta u(k-1))\Delta v(k-1)+\lambda\sum_{k=1}^{T}|u(k)|^{\beta^+-2}u(k)v(k)$$
$$=\sum_{k=1}^{T}f(k,u(k))v(k),\quad \text{for any } v\in W. \tag{4.12}$$

We assume the following.

(H_7) $f(k,\cdot):\mathbb{R}\to\mathbb{R}$ is continuous for all $k\in\mathbb{Z}[1,T]$.

(H_8) There exists a positive constant C_3 such that $|f(k,t)|\leq C_3(1+|t|^{\beta(k)-1})$, for all $k\in\mathbb{Z}[1,T]$ and $t\in\mathbb{R}$.

(H_9) $1<\beta^-<p^-$.

Remark 4.2. The hypothesis (H_8) implies that there exists one constant $C'>0$ such that $|F(k,t)|\leq C'(1+|t|^{\beta(k)})$.

We have the following result.

Theorem 4.3. *Under assumptions* (H_2)–(H_5) *and* (H_7)–(H_9), *there exists* $\lambda^*>0$ *such that, for* $\lambda\in[\lambda^*,+\infty)$, *the problem* (4.11) *has at least one weak solution.*

Proof. Let $g(u)=\sum_{k=1}^{T}F(k,u(k))$, then $g':W\to W$ is completely continuous, and, thus, g is weakly lower semicontinuous.

Therefore, for $u\in W$,

$$J(u)=\sum_{k=1}^{T+1}A(k-1,\Delta u(k-1))+\frac{\lambda}{\beta^+}\sum_{k=1}^{T}|u(k)|^{\beta^+}-\sum_{k=1}^{T}F(k,u(k)) \tag{4.13}$$

is such that $J \in C^1(W;\mathbb{R})$ and is weakly lower semicontinuous, and we have

$$\begin{aligned}\langle J'(u), v\rangle = &\sum_{k=1}^{T+1} a(k-1, \Delta u(k-1))\Delta v(k-1) + \lambda \sum_{k=1}^{T} |u(k)|^{\beta^+ -2} u(k) v(k) \\ &- \sum_{k=1}^{T} f(k, u(k)) v(k),\end{aligned} \tag{4.14}$$

for all $u, v \in W$.

This implies that the weak solutions of problem (4.11) coincide with the critical points of J. We then have to prove that J is bounded below and coercive in order to complete the proof.

Then, for $u \in W$ such that $\|u\| > 1$,

$$\begin{aligned} J(u) &\geq \frac{C_1 2^{p^-}}{p^+}\sin^{p^-}\left(\frac{\pi}{2(2T+1)}\right)\|u\|^{p_-} + \frac{\lambda}{\beta^+}\sum_{k=1}^{T}|u(k)|^{\beta^+} - C_2 - \sum_{k=1}^{T}F(k,u(k)) \\ &\geq \frac{C_1 2^{p^-}}{p^+}\sin^{p^-}\left(\frac{\pi}{2(2T+1)}\right)\|u\|^{p_-} + \frac{\lambda}{\beta^+}\sum_{k=1}^{T}|u(k)|^{\beta^+} - C_2 - C'\sum_{k=1}^{T}\left(1 + |u(k)|^{\beta(k)}\right) \\ &\geq \frac{C_1 2^{p^-}}{p^+}\sin^{p^-}\left(\frac{\pi}{2(2T+1)}\right)\|u\|^{p_-} + \frac{\lambda}{\beta^+}\sum_{k=1}^{T}|u(k)|^{\beta^+} - C_2 - C'T - C'\left(\sum_{k=1}^{T}|u(k)|^{\beta(k)}\right) \\ &\geq \frac{C_1 2^{p^-}}{p^+}\sin^{p^-}\left(\frac{\pi}{2(2T+1)}\right)\|u\|^{p_-} + \left(\frac{\lambda}{\beta^+} - C'\right)\sum_{k=1}^{T}|u(k)|^{\beta^+} - C_2 - C'T - C'\|u\|^{\beta^-} \\ &\geq \frac{C_1 2^{p^-}}{p^+}\sin^{p^-}\left(\frac{\pi}{2(2T+1)}\right)\|u\|^{p_-} - C_2 - C'T - C'\|u\|^{\beta^-}, \end{aligned} \tag{4.15}$$

where we put $\lambda^* = C'\beta^+$ with C' a positive constant.

Furthermore, by the fact that $1 < \beta^- < p^-$, it turns out that

$$J(u) \geq \frac{C}{p^+}\|u\|^{p_-} - C_2 - C'T - C'\|u\|^{\beta^-} \longrightarrow +\infty \quad \text{as } \|u\| \longrightarrow +\infty. \tag{4.16}$$

Therefore, J is coercive. □

4.3. Extension 3

We consider the problem

$$\begin{aligned} -\Delta(a(k-1, \Delta u(k-1))) = f(k, u(k)), &\quad k \in \mathbb{N}[1,T] \\ u(0) = \Delta u(T) = 0, & \end{aligned} \tag{4.17}$$

where $T \geq 2$.

We suppose the following.

(H_{10}) There exist two positive constants C_5 and C_6 such that $f^+(k,t) \leq C_5 + C_6|t|^{\beta-1}$, for all $(k,t) \in \mathbb{Z}[1,T] \times \mathbb{R}$, where $1 < \beta < p^-$.

Definition 4.4. A weak solution of problem (4.17) is a function $u \in W$ such that

$$\sum_{k=1}^{T+1} a(k-1, \Delta u(k-1))\Delta v(k-1) = \sum_{k=1}^{T} f(k, u(k))v(k), \quad \forall v \in W. \tag{4.18}$$

We have the following result.

Theorem 4.5. *Under the hypothesis (H_2)–(H_5) and (H_{10}), problem* (4.3) *admits at least one weak solution.*

Proof. We consider

$$J(u) = \sum_{k=1}^{T+1} A(k-1, \Delta u(k-1)) - \sum_{k=1}^{T} F(k, u(k)), \quad \forall u \in H. \tag{4.19}$$

J is such that $J \in C^1(W, \mathbb{R})$ and

$$\langle J'(u), v\rangle = \sum_{k=1}^{T+1} a(k-1, \Delta u(k-1))\Delta v(k-1) - \sum_{k=1}^{T} f(k, u(k))v(k), \tag{4.20}$$

for all $u, v \in W$.

As $f = f^+ - f^-$, then $F^-(k,t) = \int_0^T f^+(k,\tau)d\tau$.

By (H_{10}), there exists $C > 0$ such that

$$|F^+(k,t)| \leq C\left(1 + |t|^{\beta}\right). \tag{4.21}$$

For all $u \in W$ such that $\|u\| > 1$, we have

$$\begin{aligned} J(u) &= \sum_{k=1}^{T+1} A(k-1, \Delta u(k-1)) - \sum_{k=1}^{T} F(k, u(k)) \\ &= \sum_{k=1}^{T+1} A(k-1, \Delta u(k-1)) - \sum_{k=1}^{T} F^+(k, u(k)) + \sum_{k=1}^{T} F^-(k, u(k)) \\ &\geq \sum_{k=1}^{T+1} A(k-1, \Delta u(k-1)) - \sum_{k=1}^{T} F^+(k, u(k)). \end{aligned} \tag{4.22}$$

Therefore, similar to the proof of Theorem 4.3, Theorem 4.5 follows immediately. □

5. Example

We consider the following problem:

$$\begin{gathered}
-\Delta\left(|\Delta u(0)|^2 \Delta u(0)\right)+\lambda|u(1)|^2 u(1)=\frac{1}{5}(u(1))^2, \\
-\Delta\left(|\Delta u(1)|^3 \Delta u(1)\right)+\lambda|u(2)|^2 u(2)=\frac{1}{5}(u(2))^3, \\
u(0)=0, \quad u(2)=u(3).
\end{gathered} \tag{5.1}$$

Then, $T=2$, $p(0)=4$, $p(1)=5$, $\beta(1)=3$, $\beta(2)=4$, $p^-=4$, $p^+=5$, $\beta^-=3$, $\beta^+=4$,$f(1,t)=(1/5)t^2$, and $f(2,t)=(1/5)t^3$. Thus,

$$F(1,t)=\frac{1}{15}t^3, \qquad F(2,t)=\frac{1}{20}t^4. \tag{5.2}$$

After computation, we can take $C'=1/15$ and we deduce that $\lambda^*=4/15$.

Therefore, by Theorem 4.3, for any $\lambda \geq 4/15$, Problem (5.1) admits at least one weak solution.

Acknowledgment

The authors want to express their deepest thanks to the editor and anonymous referees for comments and suggestions on the paper.

References

[1] R. P. Agarwal, K. Perera, and D. O'Regan, "Multiple positive solutions of singular and nonsingular discrete problems via variational methods," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 58, no. 1-2, pp. 69–73, 2004.
[2] A. Castro and R. Shivaji, "Nonnegative solutions for a class of radically symmetric nonpositone problems," *Proceedings of the American Mathematical Society*, vol. 106, pp. 735–740, 1989.
[3] J. Yu and Z. Guo, "On boundary value problems for a discrete generalized Emden-Fowler equation," *Journal of Differential Equations*, vol. 231, no. 1, pp. 18–31, 2006.
[4] B. Koné and S. Ouaro, "Weak solutions for anisotropic discrete boundary value problems," *Journal of Difference Equations and Applications*, vol. 17, no. 10, pp. 1537–1547, 2011.
[5] A. Guiro, I. Nyanquini, and S. Ouaro, "On the solvability of discrete nonlinear Neumann problems involving the $p(x)$-Laplacian," *Advances in Difference Equations*, vol. 32, 14 pages, 2011.
[6] X. Cai and J. Yu, "Existence theorems for second-order discrete boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 320, no. 2, pp. 649–661, 2006.
[7] G. Zhang and S. Liu, "On a class of semipositone discrete boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 1, pp. 175–182, 2007.
[8] M. Mihăilescu, V. Rădulescu, and S. Tersian, "Eigenvalue problems for anisotropic discrete boundary value problems," *Journal of Difference Equations and Applications*, vol. 15, no. 6, pp. 557–567, 2009.
[9] P. Candito and G. D'Agui, "Three solutions for discrete nonlinear Neumann problem involving p-Laplacien," *Advances in Difference Equations*, vol. 2010, Article ID 862016, 11 pages, 2010.
[10] A. Cabada, A. Iannizzotto, and S. Tersian, "Multiple solutions for discrete boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 356, no. 2, pp. 418–428, 2009.
[11] L. Jiang and Z. Zhou, "Existence of nontrivial solutions for discrete nonlinear two point boundary value problems," *Applied Mathematics and Computation*, vol. 180, no. 1, pp. 318–329, 2006.

[12] B. Koné, S. Ouaro, and S. Traoré, "Weak solutions for anisotropic nonlinear elliptic equations with variable exponents," *Electronic Journal of Differential Equations*, vol. 144, pp. 1–11, 2009.

[13] S. Ouaro, "Well-posedness results for anisotropic nonlinear elliptic equations with variable exponent and L^1-data," *Cubo. A Mathematical Journal*, vol. 12, no. 1, pp. 133–148, 2010.

[14] B. Koné, S. Ouaro, and S. Soma, "Weak solutions for anisotropic nonlinear elliptic problem with variable exponent and measure data," *International Journal of Evolution Equations*, vol. 5, no. 3, pp. 1–23, 2011.

[15] M. Ruzicka, *Electrorheological Fluids: Modeling and Mathematical Theory*, vol. 1748 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2000.

[16] V. Zhikov, "Averaging of functionals in the calculus of variations and elasticity," *Mathematics of the USSR-Izvestiya*, vol. 29, pp. 33–66, 1987.

[17] L. Diening, *Theoretical and numerical results for electrorheological fluids [Ph.D. thesis]*, University of Freiburg, 2002.

[18] T. C. Halsey, "Electrorheological fluids," *Science*, vol. 258, pp. 761–766, 1992.

[19] K. R. Rajagopal and M. Ruzicka, "Mathematical modeling of electrorheological materials," *Continuum Mechanics and Thermodynamics*, vol. 13, pp. 59–78, 2001.

[20] Y. Chen, S. Levine, and M. Rao, "Variable exponent, linear growth functionals in image restoration," *SIAM Journal on Applied Mathematics*, vol. 66, no. 4, pp. 1383–1406, 2006.

[21] M. Mihăilescu, P. Pucci, and V. Rădulescu, "Nonhomogeneous boundary value problems in anisotropic Sobolev spaces," *Comptes Rendus Mathématique*, vol. 345, no. 10, pp. 561–566, 2007.

[22] M. Mihăilescu, P. Pucci, and V. Rădulescu, "Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent," *Journal of Mathematical Analysis and Applications*, vol. 340, no. 1, pp. 687–698, 2008.

[23] R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods, and Applications*, vol. 228 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2nd edition, 2000.

[24] R. P. Agarwal and P. J. Y. Wong, *Advanced Topics in Difference Equations*, vol. 404 of *Mathematics and Its Applications*, Kluwer Academic Publishers Group, Dordrecht, The Netherlands, 1997.

[25] L. Jiang and Z. Zhou, "Three solutions to Dirichlet boundary value problems for p-Laplacian difference equations," *Advances in Difference Equations*, vol. 2008, Article ID 345916, 10 pages, 2008.

[26] M.-M. Boureanu and V. Radulescu, "Anisotropic Neumann problems in Sobolev spaces with variable exponent," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 75, no. 12, pp. 4471–4482, 2012.

[27] G. Bonanno, G. Molica Bisci, and V. Rădulescu, "Arbitrarily small weak solutions for a nonlinear eigenvalue problem in Orlicz–Sobolev spaces," *Monatshefte für Mathematik*, vol. 165, no. 3-4, pp. 305–318, 2012.

[28] H. Brézis, *Analyse Fonctionnelle: Théorie et Applications*, Masson, Paris, France, 1983.