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Research Article

Fredholm Weighted Composition Operators on Dirichlet Space

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This paper characterizes bounded Fredholm, bounded invertible, and unitary weighted composition operators on Dirichlet space.

1. Introduction

Let H be a Hilbert space of analytic functions on the unit disk D. For an analytic function ψ on D, we can define the multiplication operator $M_\psi : f \to \psi f,\ f \in H$. For an analytic self-mapping φ of D, the composition operator C_φ defined on H as $C_\varphi f = f \circ \varphi,\ f \in H$. These operators are two classes of important operators in the study of operator theory in function spaces [1–3]. Furthermore, for ψ and φ, we define the weighted composition operator $C_{\psi,\varphi}$ on H as

$$C_{\psi,\varphi} : f \longrightarrow \psi(f \circ \varphi), \quad f \in H. \tag{1.1}$$

Recently, the boundedness, compactness, norm, and essential norm of weighted composition operators on various spaces of analytic functions have been studied intensively, see [4–9] and so on. In this paper, we characterize bounded Fredholm weighted composition operators on Dirichlet space of the unit disk.

Recall the Dirichlet space $\mathfrak{D}$ that consists of analytic function f on D with finite Dirichlet integral:

$$D(f) = \int_D |f'|^2 dA < \infty, \tag{1.2}$$

where dA is the normalized Lebesgue area measure on D. It is well known that $\mathfrak{D}$ is the only möbius invariant Hilbert space up to an isomorphism [10]. Endow $\mathfrak{D}$ with norm

$$\|f\| = \left(|f(0)|^2 + D(f)\right)^{1/2}, \quad f \in \mathfrak{D}. \tag{1.3}$$

$\mathfrak{D}$ is a Hilbert space with inner product

$$\langle f, g \rangle = f(0)\overline{g(0)} + \int_D f'(z)\overline{g'(z)} dA(z), \quad f, g \in \mathfrak{D}. \tag{1.4}$$

Furthermore $\mathfrak{D}$ is a reproducing function space with reproducing kernel

$$K_\lambda(z) = 1 + \log \frac{1}{1 - \overline{\lambda} z}, \quad \lambda, z \in D. \tag{1.5}$$

Denote $\mathcal{M} = \{\psi : \psi \text{ is analytic on } D,\ \psi f \in \mathfrak{D} \text{ for } f \in \mathfrak{D}\}$. $\mathcal{M}$ is called the multiplier space of $\mathfrak{D}$. By the closed graph theorem, the multiplication operator M_ψ defined by $\psi \in \mathcal{M}$ is bounded on $\mathfrak{D}$. For the characterization of the element in $\mathcal{M}$, see [11].

For analytic function ψ on D and analytic self-mapping φ of D, the weighted composition operator $C_{\psi,\varphi}$ on $\mathfrak{D}$ is not necessarily bounded. Even the composition operator C_φ is not necessarily bounded on $\mathfrak{D}$, which is different from the cases in Hardy space and Bergman space. See [12] for more information about the properties of composition operators acting on the Dirichlet space.

The main result of the paper reads as the following.

Theorem 1.1. *Let ψ and φ be analytic functions on D with $\varphi(D) \subset D$. Then $C_{\psi,\varphi}$ is a bounded Fredholm operator on $\mathfrak{D}$ if and only if $\psi \in \mathcal{M}$, bounded away from zero near the unit circle, and φ is an automorphism of D.*

If $\psi(z) = 1$, then the result above gives the characterization of bounded Fredholm composition operator C_φ on $\mathfrak{D}$, which was obtained in [12].

As corollaries, in the end of this paper one gives the characterization of bounded invertible and unitary weighted composition operator on $\mathfrak{D}$, respectively. Some idea of this paper is derived from [4, 13], which characterize normal and bounded invertible weighted composition operator on the Hardy space, respectively.

2. Proof of the Main Result

In the following, ψ and φ denote analytic functions on D with $\varphi(D) \subset D$. It is easy to verify that $\psi \in \mathfrak{D}$ if $C_{\psi,\varphi}$ is defined on $\mathfrak{D}$.

Proposition 2.1. *Let $C_{\psi,\varphi}$ be a bounded Fredholm operator on $\mathfrak{D}$. Then ψ has at most finite zeroes in D and φ is an inner function.*

Proof. If $C_{\psi,\varphi}$ is a bounded Fredholm operator, then there exist a bounded operator T and a compact operator S on $\mathfrak{D}$ such that

$$T(C_{\psi,\varphi})^* = I + S, \tag{2.1}$$

where I is the identity operator.

Since

$$\begin{aligned}(C_{\psi,\varphi})^* K_w(z) &= \left\langle C^*_{\psi,\varphi} K_w, K_z \right\rangle = \langle K_w, C_{\psi,\varphi} K_z \rangle \\ &= \langle K_w, \psi K_z \circ \varphi \rangle = \overline{\psi(w) K_z(\varphi(w))} \\ &= \overline{\psi(w)} K_{\varphi(w)}(z),\end{aligned} \tag{2.2}$$

we have

$$\begin{aligned}\|T\| \left|\psi(w)\right| \frac{\left\|K_{\varphi(w)}\right\|}{\|K_w\|} &\geq \left\|T(C_{\psi,\varphi})^* k_w\right\| \\ &\geq \|k_w\| - \|Sk_w\| \\ &= 1 - \|Sk_w\|,\end{aligned} \tag{2.3}$$

where $k_w = K_w/\|K_w\|$ is the normalization of reproducing kernel function K_w.

Since S is compact and k_w weakly converges to 0 as $|w| \to 1$, $\|Sk_w\| \to 0$ as $|w| \to 1$. It follows that there exists constant r, $0 < r < 1$, such that $\|Sk_w\| < 1/2$ for all w with $r < |w| < 1$. Inequality (2.3) shows that

$$\frac{\left|\psi(w)\right|}{\|K_w\|} \geq \frac{1}{2\|T\| \left\|K_{\varphi(w)}\right\|}, \quad r < |w| < 1, \tag{2.4}$$

which implies that ψ has no zeroes in $\{w \in D,\ r < |w| < 1\}$, and, hence, ψ has at most finite zeroes in $\{w \in D,\ |w| \leq r\}$.

Since k_w weakly converges to 0 as $|w| \to 1$, $\langle \psi, k_w \rangle \to 0$ as $|w| \to 1$, that is,

$$\frac{\psi(w)}{\|K_w\|} \longrightarrow 0, \quad |w| \longrightarrow 1. \tag{2.5}$$

It follows from (2.4) that $\|K_{\varphi(w)}\| = (1 + \log(1/(1 - |\varphi(w)|^2)))^{1/2} \to \infty$ and hence $|\varphi(w)| \to 1$ as $|w| \to 1$, that is, φ is an inner function. □

For the proof of the following lemma, we cite Carleson's formula for the Dirichlet integral [14].

Let $f \in \mathfrak{D}$, $f = BSF$ be the canonical factorization of f as a function in the Hardy space, where $B = \prod_{j=1}^{\infty} (\overline{a}_j/|a_j|)((a_j - z)/(1 - \overline{a}_j z))$, is a Blaschke product, S is the singular part of f and F is the outer part of f. Then

$$D(f) = \int_{\mathrm{T}} \sum_{n=1}^{\infty} P_{\alpha_n}(\xi) |f(\xi)|^2 \frac{|d\xi|}{2\pi} + \iint_{\mathrm{T}} \frac{2}{|\zeta - \xi|^2} |f(\xi)|^2 d\mu(\zeta) \frac{|d\xi|}{2\pi} + \iint_{\mathrm{T}} \frac{\left(e^{2u(\zeta)} - e^{2u(\xi)}\right)(u(\zeta) - u(\xi))}{|\zeta - \xi|^2} \frac{|d\zeta|}{2\pi} \frac{|d\xi|}{2\pi}, \tag{2.6}$$

where T is the unit circle, $u(\xi) = \log |f(\xi)|$, $P_{\alpha}(\xi)$ is the Poisson kernel, and μ is the singular measure corresponding to S.

Lemma 2.2. *Let $C_{\psi,\varphi}$ be a bounded operator on $\mathfrak{D}$, $\psi = BF$ with B a finite Blaschke product. Then $C_{F,\varphi}$ is bounded.*

Proof. Let M_B be the multiplication operator on $\mathfrak{D}$. Then $C_{\psi,\varphi} = M_B C_{F,\varphi}$. Since B is a finite Blaschke product, by the Carleson's formula, we have

$$D(\psi(f \circ \varphi)) = D(BF(f \circ \varphi)) \geq D(F(f \circ \varphi)), \quad f \in \mathfrak{D}. \tag{2.7}$$

Since $\|f\|^2 = |f(0)|^2 + D(f)$, $f \in \mathfrak{D}$, by the inequality above it is easy to verify that $C_{F,\varphi}$ is bounded on $\mathfrak{D}$ if $C_{\psi,\varphi}$ is bounded. □

Lemma 2.3. *Let F be an analytic function on D with zero-free. If $C_{F,\varphi}$ is a bounded Fredholm operator on $\mathfrak{D}$, then φ is univalent.*

Proof. If $\varphi(a) = \varphi(b)$ for $a, b \in D$ with $a \neq b$, by a similar reasoning as [1, Lemma 3.26], there exist infinite sets $\{a_n\}$ and $\{b_n\}$ in D which is disjoint such that $\varphi(a_n) = \varphi(b_n)$. Hence,

$$(C_{F,\varphi})^* \left(\frac{K_{a_n}}{\overline{F(a_n)}} - \frac{K_{b_n}}{\overline{F(b_n)}} \right) = 0, \tag{2.8}$$

which contradicts to that kernel of $(C_{F,\varphi})^*$ is finite dimensional. □

Corollary 2.4. *If $C_{\psi,\varphi}$ is a bounded Fredholm operator on $\mathfrak{D}$, then φ is an automorphism of D and $\psi \in \mathcal{M}$.*

Proof. By Proposition 2.1, ψ has the factorization of BF with B a finite Blaschke product and F zero free in D. By Lemma 2.2, $C_{F,\varphi}$ is a bounded operator on $\mathfrak{D}$. Since $C_{\psi,\varphi} = M_B C_{F,\varphi}$ and M_B is a Fredholm operator, $C_{F,\varphi}$ is a Fredholm operator also. By Proposition 2.1 and Lemma 2.3, φ is an univalent inner function, it follows from [1, Corollary 3.28] that φ is an automorphism of D.

Since $C_{\psi,\varphi} C_{\varphi^{-1}} = M_\psi$, M_ψ is a bounded multiplication operator on $\mathfrak{D}$, which implies that $\psi \in \mathcal{M}$. □

The following lemmas is well-known. It is easy to verify by the fact $M_\psi^* K_w = \overline{\psi(w)} K_w$ also.

Lemma 2.5. *Let $\psi \in \mathcal{M}$. Then M_ψ is an invertible operator on $\mathfrak{D}$ if and only if ψ is invertible in $\mathcal{M}$.*

Lemma 2.6. *Let $\psi \in \mathcal{M}$. Then M_ψ is a Fredholm operator on $\mathfrak{D}$ if and only if ψ is bounded away from the unit circle.*

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. If $C_{\psi,\varphi}$ is a bounded Fredholm operator on $\mathfrak{D}$, by Corollary 2.4, $\psi \in \mathcal{M}$ and φ is an automorphism of D. Since C_φ is invertible, M_ψ is a Fredholm operator. So ψ is bounded away form the unit circle follows from Lemma 2.6.

On the other hand, if $\psi \in \mathcal{M}$ and bounded away from the unit circle, then M_ψ is a bounded Fredholm operator on $\mathfrak{D}$. If φ is an automorphism of D, then C_φ is invertible. Hence $C_{\psi,\varphi} = M_\psi C_\varphi$ is a bounded Fredholm operator on $\mathfrak{D}$. □

As corollaries, in the following, we characterize bounded invertible and unitary weighted composition operators on $\mathfrak{D}$.

Corollary 2.7. *Let ψ and φ be analytic functions on D with $\varphi(D) \subset D$. Then $C_{\psi,\varphi}$ is a bounded invertible operator on $\mathfrak{D}$ if and only if $\psi \in \mathcal{M}$, invertible in $\mathcal{M}$, and φ is an automorphism of D.*

Proof. Since a bounded invertible operator is a bounded Fredholm operator, the proof is similar to the proof of Theorem 1.1. □

Corollary 2.8. *Let ψ and φ be analytic functions on D with $\varphi(D) \subset D$. $C_{\psi,\varphi}$ is a bounded operator on $\mathfrak{D}$. Then $C_{\psi,\varphi}$ is a unitary operator if and only if ψ is a constant with $|\psi| = 1$ and φ is a rotation of D.*

Proof. If $C_{\psi,\varphi}$ is a unitary operator, then it must be an invertible operator. By Corollary 2.7, φ is an automorphism of D and ψ is invertible in $\mathcal{M}$.

Let n be nonnegative integer, $e_n(z) = z^n,\ z \in D$. A unitary is also an isometry, so we have

$$\|\psi\| = \|C_{\psi,\varphi}e_0\| = \|e_0\| = 1, \tag{2.9}$$

$$\|\psi\varphi^n\| = \|C_{\psi,\varphi}e_n\| = \|e_n\| = \sqrt{n}, \quad n \geq 1. \tag{2.10}$$

Let $\alpha \in D$ such that $\varphi(\alpha) = 0$. Since φ is an automorphism of D, φ^n is a finite Blaschke product with zero α of order n. By Carleson's formula for Dirichlet integral, we have

$$D(\psi\varphi^n) = n\int_{\mathbb{T}} P_\alpha(\xi)|\psi(\xi)|^2\frac{|d\xi|}{2\pi} + D(\psi). \tag{2.11}$$

Hence,

$$\begin{aligned} n = \|\psi\varphi^n\|^2 &= |\psi(0)\varphi(0)^n|^2 + D(\psi\varphi^n) \\ &= |\psi(0)\varphi(0)^n|^2 + n\int_{\mathbb{T}} P_\alpha(\xi)|\psi(\xi)|^2\frac{|d\xi|}{2\pi} + D(\psi), \quad n \geq 1. \end{aligned} \tag{2.12}$$

That is,

$$1 = \frac{\left|\psi(0)\varphi(0)^n\right|^2}{n} + \int_{\mathbb{T}} P_\alpha(\xi)\left|\psi(\xi)\right|^2 \frac{|d\xi|}{2\pi} + \frac{D(\psi)}{n}, \quad n \geq 1. \tag{2.13}$$

Let $n \to \infty$, then $1 = \int_{\mathbb{T}} P_\alpha(\xi)|\psi(\xi)|^2(|d\xi|/2\pi)$.

By (2.12), we have $D(\psi) = 0$ and $|\psi(0)\varphi(0)| = 0$. By (2.9), we obtain ψ is a constant with $|\psi| = 1$, which implies that $\varphi(0) = 0$, that is, φ is a rotation of D.

The sufficiency is easy to verify. □

Remark 2.9. The key step in the proof of the main result is to analyze zeros of the symbol ψ and univalency of φ. The following result pointed out by the referee gives a simple characterization of the symbols ψ and φ for the bounded Fredholm operator $C_{\psi,\varphi}$ on $\mathfrak{D}$.

Proposition 2.10. *Let ψ and φ be analytic functions on D with $\varphi(D) \subset D$. $C_{\psi,\varphi}$ is a bounded Fredholm operator on $\mathfrak{D}$. Then ψ has only finitely many zeros in D and φ is univalent.*

Proof. If $\psi(a) = 0$ for $a \in D$, then $C^*_{\psi,\varphi}K_a = \overline{\psi(a)}K_{\varphi(a)} = 0$, which implies that K_a is in the kernel of $C^*_{\psi,\varphi}$. Thus if ψ had infinitely many zeros, the kernel of $C^*_{\psi,\varphi}$ would be infinite dimensional and hence this operator would not be Fredholm.

If $\varphi(a) = \varphi(b)$ for $a, b \in D$ with $a \neq b$, by a similar reasoning as [1, Lemma 3.26], there exist infinite sets $\{a_n\}$ and $\{b_n\}$ in D which is disjoint such that $\varphi(a_n) = \varphi(b_n)$. Since ψ has only finitely many zeros in D, we can choose infinitely many a_n and b_n such that $\psi(a_n) \neq 0$, $\psi(b_n) \neq 0$. Hence,

$$\left(C_{\psi,\varphi}\right)^*\left(\frac{K_{a_n}}{\overline{\psi(a_n)}} - \frac{K_{b_n}}{\overline{\psi(b_n)}}\right) = 0. \tag{2.14}$$

Since $C_{\psi,\varphi}$ is a Fredholm operator, φ must be univalent. □

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