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SOME RESULTS ON *[n,m]*-PARACOMPACT AND *[n,m]*-COMPACT SPACES

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ABSTRACT. Let n and m be infinite cardinals with $n \leq m$ and n be a regular cardinal. We prove certain implications of $[n,m]$-strongly paracompact, $[n,m]$-paracompact and $[n,m]$-metacompact spaces Let X be $[n,\infty]$-compact and Y be a $[n,m]$-paracompact (resp. $[n,\infty]$-paracompact), P_n-space (resp. wP_n-space). If $m = \sum_{k<n} m^k$ we prove that $X \times Y$ is $[n,m]$-paracompact (resp $[n,\infty]$-paracompact)

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1. INTRODUCTION

Throughout this paper m and n will denote infinite cardinals with $n \leq m$ and n will be a regular cardinal. A space X is called $[n,m]$-compact (see Alexandroff [1]) if every open cover α of X with $|\alpha| \leq m$ has a subcover of cardinality $< n$. For a set A, we denote by $|A|$, the cardinality of A. A family α of subsets of X is a *locally-n* (*point-n*) family (Mansfield [2]) if for every $x \in X$, there is an open neighborhood of x in X which meets $< n$ members of α (resp x belongs to $< n$ members of α) An *open refinement* of a cover α of a space X is an open cover β such that each member of β is contained in some member of α. A space X is $[n,m]$-*paracompact* (resp. $[n,m]$-*metacompact*) if every open cover α of X with $|\alpha| \leq m$ has a locally-n (resp. point-n) open refinement. X is $[n,m]$-*strongly paracompact* if every open cover of X with $|\alpha| \leq m$, has an open refinement β such that for each $B \in \beta$,

$$|\{C \in \beta : C \cap B \neq \phi\}| < n.$$

Originally, Singal and Singal introduced the concept of (m,k)-paracompactness in [3]. Our notation is slightly different than theirs. However, we note that a space X is (m,k)-paracompact, as defined in [3], if and only if X is $[k^+,m]$-paracompact. A space X is $[n,\infty]$-compact (resp. $[n,\infty]$-paracompact, $[n,\infty]$-metacompact, $[n,\infty]$-strongly paracompact) if X is $[n,m]$-compact (resp. $[n,m]$-paracompact, $[n,m]$-metacompact, $[n,m]$-strongly paracompact for each cardinal $m \geq n$). A space X is a P_n-*space* [4] if for every family α of open subsets of X with $|\alpha| < n$, $\cap\,\alpha$ is open in X. We observe that the class of P_{ω_0}-spaces is the class of all topological spaces, where ω_0 denotes the first infinite cardinal number Also we observe that if P is any of "compact", "paracompact", or "metacompact", then the class of $[\omega_0,\infty] - P$ spaces is the same as the class of P spaces in the ordinary sense.

Morita [5] studied m-paracompact spaces. A space X is m-paracompact if and only if X is $[\omega_0, m]$-paracompact. Morita proved that if Y is an m-paracompact space and X is a compact space, then $X \times Y$ is m-paracompact. In case $m = \sum_{k<n} m^k$, we generalize Morita's result by showing that if X is an $[n, \infty]$-compact space and Y is $[n, m]$-paracompact, P_n-space, then $X \times Y$ is $[n, m]$-paracompact. We note that for $n = \omega_0$ this result implies Morita's result. A subset W of a topological space Y is called *n-open* (Hdeib [6]) if for each $y \in W$ there exists an open set V in Y such that $y \in V$ and $|V \backslash W| < n$. A subset F of Y is called *n-closed* if $Y \backslash F$ is n-open. A space Y is called a *weak P_n -space* [6] or *wP_n-space* if $\cap \alpha$ is n-open for every family α of open subsets of Y with $|\alpha| < n$. We prove that if X is a $[n, \infty]$-compact space and Y is an $[n, \infty]$-paracompact, wP_n-space, then $X \times Y$ is $[n, \infty]$-paracompact. This result is a variation of our generalization of Morita's result.

It is well known (Dungundji [7]) that if a space X is locally compact and Hausdorff, then X is paracompact if and only if X is a disjoint topological sum of σ-compact spaces. We prove that if $n > \omega_0$, then a locally $[n, \infty]$-compact, regular space X is $[n, \infty]$-paracompact if and only if X is a disjoint topological sum of $[n, \infty]$-compact spaces. A space X is, by definition, locally *$[n, \infty]$-compact* if for each point $x \in X$ and an open neighborhood G of x, there exists an $[n, \infty]$-compact neighborhood H of x such that $H \subseteq G$.

In this paper we also prove certain implications concerning $[n, m]$-paracompact, metacompact, strongly paracompact spaces.

For a space X, the *density* $d(X)$ of X is defined as the smallest cardinal number that is the cardinal number of a dense subset of X. For terminology not defined here see Engelking [8].

2. $[n, m]$-PARACOMPACT SPACES

It is clear that each $[n, m]$-strongly paracompact space is $[n, m]$-paracompact which in turn is $[n, m]$-metacompact. However, in general, the converses of these implications do not hold.

The following two theorems are interesting in this respect.

THEOREM 2.1. Let γ be an open cover of a space X such that $|\gamma| \leq m$ and $d(A) < n$ for each $A \in \gamma$. Then X is $[n, m]$-strongly paracompact if and only if X is $[n, m]$-metacompact.

PROOF. We only need to prove "if" part. Let X be $[n, m]$-metacompact. Let α be an open cover of X with $|\alpha| \leq m$. Let $\beta = \{A \cap W : A \in \gamma \text{ and } W \in \alpha\}$. Then $|\beta| \leq m$, β is an open refinement of α and $d(B) < n$ for each $B \in \beta$. Since X is $[n, m]$-metacompact, then there exists a point-n open refinement λ of β. Each $L \in \lambda$ is contained in some $B_L \in \beta$. Since L is open and $d(B_L) < n$, then $d(L) < n$. Let $L \in \lambda$ and D be a dense set in L such that $|D| < n$. Let $\triangle = \{A \in \lambda : A \cap L \neq \phi\}$. Since D is dense in L, then $A \in \triangle$ if and only if $A \cap D \neq \phi$. Thus $\triangle = \{A \in \lambda : A \cap D \neq \phi\}$. For $d \in D$ let us set $\triangle_d = \{K \in \lambda : d \in K\}$. Then $|\triangle_d| < n$ since λ is point-n. Hence

$$|\triangle| \leq \sum_{d \in D} |\triangle_d| < n.$$

Since $|D| < n$ and n is a regular cardinal, it follows that X is $[n, m]$-strongly paracompact.

COROLLARY 2.1 (Traylor [9]). Let X be a regular space with an open cover γ such that $d(G) \leq \omega_0$ for all $G \in \gamma$. Then X is strongly paracompact if and only if X is metalindelöf.

PROOF. The proof follows from Theorem 2.1 and Theorem 3, page 229 in [8].

THEOREM 2.3. Let X be a locally $[n, \infty]$-compact space. Then X is $[n, \infty]$-paracompact if and only if X is $[n, \infty]$-strongly paracompact.

PROOF. We only need to prove "only if" part. Let X be $[n, \infty]$-paracompact. Let α be an open cover of X. Since X is locally $[n, \infty]$-compact then there exists a cover σ of X such that

(i) σ refines α

(ii) $\beta = \{\text{int}\, H : H \in \sigma\}$ is a cover of X,

(iii) if $H \in \sigma$, then H is $[n,\infty]$-compact

Since X is $[n,\infty]$-paracompact, then β has a locally-n open refinement γ Now, let $G \in \gamma$ and

$$\triangle = \{L \in \gamma : G \cap L \neq \phi\}.$$

Since γ refines β, then $G \subseteq \operatorname{int} H \subseteq H$ for some $H \in \sigma$ For each $x \in H$, there is an open set W_x containing x such that W_x meets $< n$ members of γ We have

$$H = \cup\{W_x \cap H : x \in H\}.$$

Since H is $[n,\infty]$-compact, then there exists a subset T of H such that $|T| < n$ and

$$H = \cup\{W_x \cap H : x \in T\}.$$

For $x \in T$. Let us set

$$\triangle_x = \{L \in \gamma : W_x \cap L \neq \phi\}.$$

We see that

$$\triangle \subseteq \{\triangle_x : x \in T\}.$$

Hence

$$|\triangle| \leq \sum_{x \in T} |\triangle_x| < n.$$

Since $|T| < n$, $|\triangle_x| < n$ for each $x \in T$ and n is a regular cardinal.

COROLLARY 2.4. Let X be a regular, locally Lindelöf space. Then X is strongly paracompact if and only if X is paralindelöf

PROOF. The proof follows from Theorem 2.3 and Theorem 3, page 229 in [8].

It is well known in [7] that if X is a locally compact Hausdorff space, then X is paracompact if and only if X is a disjoint topological sum of σ-compact spaces. It is natural to ask when X is a locally $[n,\infty]$-compact, $[n,\infty]$-paracompact space, whether X is a disjoint topological sum of σ-$[n,\infty]$-compact spaces. The result above is the answer to the case when $n = \omega_0$ and X is Hausdorff. So we are only interested in the case when $n > \omega_0$. The following theorem provides the answer to this question

THEOREM 2.5. Let $n > \omega_0$ and X be a locally $[n,\infty]$-compact regular space. Then X is $[n,\infty]$-paracompact if and only if X is a disjoint topological sum of $[n,\infty]$-compact spaces

PROOF. It is obvious that if X is a disjoint topological sum of $[n,\infty]$-compact spaces, then X is $[n,\infty]$-paracompact. Thus let us assume that X is $[n,\infty]$-paracompact Let

$$\alpha = \{U : U \subseteq X \text{ and } U \text{ is } [n,\infty]\text{-compact}\}.$$

Then $\beta = \{\operatorname{int} U : U \in \alpha\}$ is an open cover of X since X is locally $[n,\infty]$-compact Since X is regular, then there is an open cover γ of X such that $\overline{\gamma} = \{c\ell G : G \in \gamma\}$ refines β. Since X is a locally $[n,\infty]$-compact, $[n,\infty]$-paracompact space, then by Theorem 2.3, X is $[n,\infty]$-strongly paracompact Hence there exists an open refinement σ of γ such that for each $L \in \sigma$ the set $\triangle_L = \{H \in \sigma : L \cap H \neq \phi\}$ has cardinality n. For a positive integer t, a chain of length t in σ is a sequence $L_1, ..., L_t$ in σ such that $L_i \cap L_{i+1} \neq \phi$ for $1 \leq i \leq t-1$. If $t = 1$ we simply require $L_1 \neq \phi$. For $x, y \in X$ we define $x \sim y$ if there is a chain $L_1, ..., L_t$ in σ such that $x \in L_1$ and $y \in L_t$. Clearly " $\sim$ " is an equivalence relation since σ is an open cover of X. Let R be an equivalence class and $a \in R$. If $y \in R$, then there is a chain $L_1, ..., L_t$ in σ such that $a \in L_1$ and $y \in L_t$. Clearly each point in L_t is equivalent to a with respect to " $\sim$ ", hence $L_t \subseteq R$. So R is open. Let $z \in c\ell R$. There exists $L \in \sigma$ such that $z \in L$ Since $z \in c\ell R$, then $L \cap R \neq \phi$ Thus if $w \in L \cap R$, then $z \sim w$, i.e , $z \in R$. This shows that R is also closed Let $a \in L$ and $L \in \sigma$. We know that $L \subseteq R$ For a positive integer t, let

$$\mu_t = \{H \in \gamma : \text{there is a chain } L_1, ..., L_t \text{ in } \sigma \text{ such that } L = L_1 \text{ and } L_t = H\}.$$

Clearly $\mu_1 = \{L\}$. Thus $|\mu_1| < n$. Assume that $|\mu_t| < n$ If $K \in \mu_{t+1}$, then there is a chain $L_1, L_2, ..., L_t, L_{t+1}$ in σ such that $L = L_1$ and $K = L_{t+1}$ Then $L_t \in \mu_t$ Thus

$$\mu_{t+1} \subseteq \cup\{\triangle_H : H \in \mu_t\}.$$

Hence

$$|\mu_{t+1}| \leq \sum_{H \in \mu_t} |\triangle_H| < n,$$

since $|\mu_t| < n$ and n is a regular cardinal. This inductive argument shows that $|\mu_t| < n$ for all $t \geq 1$ We show that $R = \cup\{R_i : i \geq 1\}$ where $R_t = \cup\{c\ell H : H \in \mu_t\}$. If $H \in \mu_t$, then by the definition of " $\sim$ " we get $H \subseteq R$ Since R is closed, then $c\ell H \subseteq R$. So $R \supseteq \bigcup_i R_i$ Conversely let $y \in R$ Then there is a chain $L_1, ..., L_t$ in σ such that $a \in L_1$ and $y \in L_t$. Since $a \in L_1 \cap L$, then $L, L_1, ..., L_t$ is a chain in σ Thus $L_t \in \mu_{t+1}$; and consequently $y \in \cup R_i$. This proves the result

Now, if $H \in \sigma$, then $H \subseteq c\ell E \subseteq U$ for some $G \in \gamma$ and $U \in \alpha$. Thus $c\ell G$ and consequently $c\ell H$ is $[n, \infty]$-compact. Since $|\mu_t| < n$ when t is a positive integer, then R_t is also $[n, \infty]$-compact Since $n > \omega_0$, then $R = \cup R_i$ is also $[n, \infty]$-compact. This proves the theorem since X is the disjoint topological sum of the equivalence classes of " $\sim$ ".

3. PRODUCT THEOREMS

In this section we prove theorems concerning $[n, m]$-paracompact of a product space $X \times Y$ Our first theorem is a generalization of a result by Morita [5] which states that if X is a compact space and Y is an m-paracompact space, then $X \times Y$ is an m-paracompact space.

THEOREM 3.1. Let the cardinal m satisfy $m = \Sigma\{m^k : k \text{ is a cardinal and } k < n\}$ Let X be an $[n, \infty]$-compact space and Y be an $[n, m]$-paracompact P_n-space. Then $X \times Y$ is $[n, m]$-paracompact

PROOF. Let α be an open cover of $X \times Y$ with $|\alpha| \leq m$. For each subset β of α with $|\beta| < n$, let $W_\beta = \{y \in Y : X \times \{y\} \subseteq \cup\beta\}$. Let $\beta \subseteq \alpha$ and $|\beta| < n$. Then W_β is open in X For let $y \in W_\beta$. Then $X \times \{y\}$ is contained in $G = \cup\beta$. For each $x \in X$, there exists a basic open set $B_x \times C_x$ in $X \times Y$ such that $(x, y) \in B_x \times C_x \subseteq G$. Now $\{B_x : x \in X\}$ is an open cover of X. Thus there is a subcover $\{B_x : x \in S\}$ where $|S| < n$. $C = \cap\{C_x : x \in S\}$ is open in Y, since Y is a P_n-space and $y \in G$ Moreover, $X \times C \subset \cup\{B_x \times C : x \in S\} \subseteq G$. It follows that $y \in C \subseteq W_\beta$. So W_β is open Let us set

$$\Lambda = \{W_\beta : \beta \subseteq \alpha \text{ and } |\beta| < n\}.$$

Let $y \in Y$. For each $x \in X$, there exists $A_x \in \alpha$ such that $(x, y) \in A_x$ There is a basic open set $D_x \times E_x$ in $X \times Y$ such that $(x, y) \in D_x \times E_x \subseteq A_x$. Now, $\{D_x : x \in X\}$ is an open cover of X Thus it has a subcover $\{D_x : x \in T\}$ such that $|T| < n$.

Let $\beta = \{A_x : x \in T\}$. Then $|\beta| < n$ and $X \times \{y\} \subseteq \bigcup_{x \in T} D_x \times \{y\} \subseteq \cup\beta$. Thus $y \in W_\beta$ This shows that Λ is an open cover of Y. Further notice that

$$|\Lambda| \leq \sum_{k<n} m^k = m.$$

Thus there exists a locally-n open refinement μ of Λ since Y is $[n, m]$-paracompact. For each $M \in \mu$ we pick $\beta_M \subseteq \alpha$ such that $|\beta_M| < n$ and $M \subseteq W_{\beta_M}$. For $A \in \beta_M$ we define $G(M, A) = (X \times M) \cap A$ Let $\rho = \{G(M, A) : M \in \mu, A \in \beta_M\}$ If $(x, y) \in X \times Y$, then $y \in M \subseteq W_{\beta_M}$ for some $M \in \mu$ Since $y \in W_{\beta_M}$, then $X \times \{y\} \subseteq \cup\beta_M$. Thus $(x, y) \in A$ for some $A \in \beta_M$ Hence $(x, y) \in G(M, A)$

This shows that ρ is an open cover of $X \times Y$ Clearly ρ refines α Let $(x, y) \in X \times Y$ There exists an open set N in Y such that $y \in N$ and N meets $< n$ members of μ. Let $\mu' = \{M \in \mu : N \cap M \neq \phi\}$ Thus we have $|\mu'| < n$. If $M \notin \mu'$, then $(X \times N) \cap G(M, A) = \phi$ for all $A \in \beta_M$ Thus the open neighborhood $X \times N$ of (x, y) can only meet those $G(M, A)$ with $M \in \mu'$ and $A \in \beta_M$ The cardinality of such $G(M, A)$'s is at most $\sum_{M \in \mu'} |\beta_M|$ which is less than n since $|\mu'| < n$, $|\beta_M| < n$ for each $M \in \mu'$ and n is a regular cardinal Hence ρ is a locally-n family

In Theorem 3.1 if we assume the stronger condition that Y is $[n, \infty]$-paracompact then we can show that $X \times Y$ is $[n, \infty]$-paracompact if we only assume that y is a wP_n-space Before we prove this result we first prove two theorems which are interesting in their own rights

Let A and B be topological spaces and $f : A \to B$ be a function f is called *n-closed* if for every closed subset F of A, $f(F)$ is an n-closed subset of B.

THEOREM 3.2. Let Xbe an $[n, \infty]$-compact space and Y be a wP_n-space. Then the projection mapping $P : X \times Y \to Y$ is an n-closed map

PROOF. Let F be closed in $X \times Y$ and y be in $U = Y \backslash P(F)$ Then $(x, y) \notin F$ for each $x \in X$ Hence there are open sets U_x in X and V_x in Y, for each $x \in X$, such that $(x, y) \in U_x \times V_x$ and $F \cap (U_x \times V_x) = \phi$. $\alpha = \{U_x : x \in X\}$ is an open cover of X Since X is $[n, \infty]$-compact, then there exists a subset T of X such that $|T| < n$ and $\beta = \{U_x : x \in T\}$ covers X. $W = \cap\{V_x : x \in T\}$ is n-open in Y since Y is a wP_n-space and $y \in W$ Hence there exists an open set V in Y such that $y \in V$ and $|V \backslash W| < n$. Now, we have $X \times W \cap F = \phi$ Hence $W \subseteq U$. Thus $|V \backslash U| < n$. It follows that U is n-open. Thus P is n-closed.

THEOREM 3.3. Let $f : Z \to Y$ be a continuous, n-closed mapping such that $f^{-1}(y)$ is $[n, \infty]$-compact for such $y \in Y$. If Y is $[n, \infty]$-paracompact (resp. $[n, \infty]$-compact) then Z is also $[n, \infty]$-paracompact (resp. $[n, \infty]$-compact).

PROOF. We will only prove the case when Y is $[n, \infty]$-paracompact. The $[n, \infty]$-compact case can be proved similarly.

Let α be an open cover of X For each $y \in Y$ let α_y be a subcollection of α such that $|\alpha_y| < n$ and $f^{-1}(y) \subseteq \cup \alpha_y$. Such a subcollection exists since $f^{-1}(y)$ is $[n, \infty]$-compact. For $y \in Y$, let $G_y = \cup \alpha_y$, and $W_y = Y \backslash f(X \backslash G_y)$. Then $y \in W_y$ and W_y is n-open since f is an n-closed map Thus for each $y \in Y$, there is an open set V_y in Y such that $y \in V_y$ and $|V_y \backslash W_y| < n$. $\gamma = \{V_y : y \in Y\}$ is an open cover of Y and Y is $[n, \infty]$-paracompact. Hence there exists a locally-n open refinement $\{T_i : i \in I\}$ of γ. For each $i \in I$, pick $y_i \in Y$ such that $T_i \subseteq V_{y_i}$. For $y \in Y$ let

$$\beta_y = \alpha_y \cup (\cup(\alpha_t : t \in V_y \backslash W_y)).$$

Then

$$|\beta_y| \leq |\alpha_y| + \Sigma\{|\alpha_t| : t \in V_y \backslash W_y\} < n,$$

since n is a regular cardinal. Moreover $f^{-1}(T_i) \subseteq \cup \beta_{y_i}$ since $T_i \subseteq V_{y_i}$. Let

$$\sigma = \{H \cap f^{-1}(T_i) : H \in \beta_{y_i}, i \in I\}.$$

Then clearly σ is an open refinement of α. Let $x \in X$ and $y = f(x)$. There is an open set N in Y and a subset J of I such that $|J| < n$, $y \in N$ and $N \cap T_i = \phi$ for all $i \in I \backslash J$. Let $M = f^{-1}(N)$ and $\Lambda = \{H \cap f^{-1}(T_i) : H \in \beta_{y_i}, i \in J\}$. Then $x \in M$ and $|\Lambda| \leq \sum_{i \in J} |\beta_{y_i}| < n$ since n is a regular cardinal Moreover, if $L \in \sigma \backslash \Lambda$, then $L \cap M = \sigma$. Hence σ is a locally-n family.

As a corollary of Theorem 3 2 and Theorem 3 3 we obtain the following variation of Theorem 3 1

THEOREM 3.4. Let X be an $[n, \infty]$-compact space and Y be an $[n, \infty]$-paracompact (resp $[n, \infty]$-compact) wP_n-space, then $X \times Y$ is $[n, \infty]$-paracompact (resp $[n, \infty]$-compact)

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