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A FIXED POINT THEOREM FOR NON-SELF SET-VALUED MAPPINGS

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ABSTRACT. Let X be a complete, metrically convex metric space, K a closed convex subset of X, $CB(X)$ the set of closed and bounded subsets of X. Let $F : K \to CB(X)$ satisfying definition (1) below, with the added condition that $Fx \subseteq K$ for each $x \in \partial K$. Then F has a fixed point in K. This result is an extension to multivalued mappings of a result of Ćirić [1].

KEY WORDS AND PHRASES. Fixed point, multivalued map, non-self map
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Let X be a complete metrically convex metric space. This means that, for each x, y in X, $x \neq y$, there exists a z in ∂X such that $d(x,y) = d(x,z) + d(z,y)$. Let $CB(X)$ denote the set of closed and bounded subsets of X, H denote the Hausdorff metric on $CB(X)$. Let K be a nonempty closed, convex subset of X.

Let $F : K \to CB(X)$ satisfying: for each x, y in K,

$$H(Fx,Fy) \leq h \max\left\{\frac{d(x,y)}{a}, D(x,Fx), D(y,Fy), \frac{[D(x,Fy) + D(y,Fx)]}{a+h}\right\}, \tag{1}$$

where $0 \leq h < \left(-1 + \sqrt{5}\right)/2, a \geq 1 + \left(2h^2/(1+h)\right)$, and $F(x) \subseteq K$ for each $x \in \partial K$.

Ćirić [1] proved a fixed point theorem for the single-valued version of (1). He also established a multivalued version. However, he used the δ-distance, instead of the Hausdorff distance, so that the result and proof are identical to the single-valued case. It is the purpose of this paper to prove a multivalued version. For the single-valued version of (1), one can allow h to satisfy $0 \leq h < 1$. However, the multivalued proof requires smaller values of h.

THEOREM. Let X be a complete metrically convex metric space, K a nonempty closed, convex subset of X. Let $F : K \to CB(X)$ satisfying (1), and the condition that $Fx \subseteq K$ for each $x \in \partial K$. Then F has a fixed point in K.

PROOF. We shall need the following lemma of Nadler [2].

LEMMA. Let $A, B \in CB(X), x \in A$. Then, for each positive number α, there exists a $y \in B$ such that

$$d(x,y) \leq H(A,B) + \alpha.$$

We shall assign $\alpha = h(1+h)$. We shall now construct a sequence $\{x_n\}$ in K in the following way. Let $x_0 \in K$ and define $x_1' \in Fx_0$. If $x_1' \in K$, set $x_1 = x_1'$. If not, then select a point $x_1 \in \partial K$ such that $d(x_0,x_1) + d(x_1,x_1') = d(x_0,x_1')$. Then $x_1 \in K$. By the Lemma, choose $x_2' \in Fx_1$ such that $d(x_1',x_2') \leq H(Fx_0,Fx_1) + \alpha$. If $x_2' \in K$, set $x_2 = x_2'$. Otherwise, choose x_2 so that $d(x_1,x_2) + d(x_2,x_2') = d(x_1,x_2')$. By induction we obtain sequences $\{x_n\}$, $\{x_n'\}$ such that, for $n = 1,2,\ldots$,

(i) $x_{n+1}' \in Fx_n$,

(ii) $d(x_{n+1}',x_n') \leq H(Fx_n,x_{n-1}) + \alpha^n$,

where

(iii) $x_{n+1}' = x_{n+1}$ if $x_{n+1}' \in K$, or

(iv) $d(x_n,x_{n+1}) + d(x_{n+1},x_{n+1}') = d(x_n,x_{n+1}')$ if $x_{n+1}' \notin K$ and $x_{n+1} \in \partial K$.

Now define

$$P := \{x_i \in \{x_n\} : x_i = x_i',\ i = 1,2,\cdots\};$$
$$Q := \{x_i \in \{x_n\} : x_i \neq x_i',\ i = 1,2,\cdots\}.$$

Note that, if $x_n \in Q$, for some n, then $x_{n-1} \in P$.

For $n \geq 2$ we shall consider $d(x_n,x_{n+1})$. There are three possibilities.

Case 1. $x_n, x_{n+1} \in P$. Then, from (1),

$$\begin{aligned}
&d(x_n,x_{n+1}) = d(x_n',x_{n+1}') \leq H(Fx_{n-1},Fx_n) + \alpha^n\\
&\leq h\max\left\{\frac{d(x_{n-1},x_n)}{a}, D(x_{n-1},Fx_{n-1}), D(x_n,Fx_n), \frac{D(x_{n-1},Fx_n) + D(x_n,Fx_{n-1})}{a+h}\right\} + \alpha^n\\
&\leq h\max\left\{\frac{d(x_{n-1},x_n)}{a}, d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{d(x_{n-1},x_{n+1} + d(x_n,x_n)}{a+h}\right\} + \alpha^n\\
&\leq \max\left\{hd(x_{n-1},x_n) + \alpha^n, \frac{\alpha^n}{1-h}, \frac{hd(x_{n-1},x_n) + \alpha^n(a+h)}{a}\right\}\\
&\leq hd(x_{n-1},x_n) + \max\left\{\frac{1}{1-h}, \frac{a+h}{a}\right\}\alpha^n = hd(x_{n-1},x_n) + \frac{\alpha^n}{1-h}.
\end{aligned} \tag{2}$$

Case 2. $x_n \in P, x_{n+1} \in Q$. Then, from (1),

$$\begin{aligned}
&d(x_n,x_{n+1}) \leq d(x_n,x_{n+1}') \leq H(Fx_{n-1},Fx_n) + \alpha^n\\
&\leq h\max\left\{\frac{d(x_{n-1},x_n)}{a}, D(x_{n-1},Fx_{n-1}), D(x_n,Fx_n), \frac{D(x_{n-1},Fx_n) + D(x_n,Fx_{n-1})}{a+h}\right\} + \alpha^n\\
&\leq h\max\left\{\frac{d(x_{n-1},x_n)}{a}, d(x_{n-1},x_n'), d(x_n,x_{n-1}'), \frac{d(x_{n-1},x_{n+1}'}{a+h}\right\} + \alpha^n\\
&\leq \max\left\{hd(x_{n-1},x_n) + \alpha^n, \frac{\alpha^n}{1-h}, \frac{hd(x_{n-1},x_n) + \alpha^n(a+h)}{a}\right\}\\
&\leq hd(x_{n-1},x_n) + \max\left\{\frac{1}{1-h}, \frac{a+h}{a}\right\}\alpha^n = hd(x_{n-1},x_n) + \frac{\alpha^n}{1-h}.
\end{aligned} \tag{3}$$

Case 3. $x_n \in Q, x_{n+1} \in P$. Note, that $x_n \in Q$ implies that $x_{n-1} \in P$. Using the convexity of X,

$$d(x_n,x_{n+1}) \leq \max\{d(x_{n-1},x_{n+1}), d(x_n',x_{n+1})\} \tag{4}$$

Suppose that the maximum of the right hand side of (4) is $d(x_n',x_{n+1})$. Then, from (1),

$$\begin{aligned}
&d(x_n,x_{n+1}) \leq d(x_n',x_{n+1}) \leq H(Fx_{n-1},Fx_n) + \alpha^n\\
&\leq h\max\left\{\frac{d(x_{n-1},x_n)}{a}, D(x_{n-1},Fx_{n-1}), D(x_n,Fx_n), \frac{D(x_{n-1},Fx_n) + D(x_n,Fx_{n-1})}{a+h}\right\} + \alpha^n\\
&\leq h\max\left\{\frac{d(x_{n-1},x_n)}{a}, d(x_{n-1},x_n'), d(x_n,x_{n+1}), \frac{d(x_{n-1},x_{n+1}) + d(x_n,x_n')}{a+h}\right\} + \alpha^n
\end{aligned}$$

Recall that $d(x_{n-1},x_n) \le d(x_{n-1},x'_n)$ and that $d(x_n,x'_n) \le d(x_{n-1},x'_n)$. Also, $d(x_{n-1},x_{n+1}) + d(x_n x'_n) \le d(x_{n-1},x_n) + d(x_n,x_{n+1}) + d(x_n x'_n) = d(x_{n-1}x'_n) + d(x_n,x_{n+1})$. Therefore,

$$\begin{aligned} d(x_n,x_{n+1}) &\le h \max\left\{d(x_{n-1},x'_n), d(x_n,x_{n+1}), \frac{d(x_{n-1},x'_n) + d(x_n,x_{n+1})}{a+h}\right\} + \alpha^n \\ &\le \max\left\{hd(x_{n-1},x'_n) + \alpha^n, \frac{\alpha^n}{1-h}, \frac{hd(x_{n-1},x'_n) + \alpha^n(a+h)}{a}\right\} \\ &\le hd(x_{n-1},x'_n) + \frac{\alpha^n}{1-h}. \end{aligned}$$

Since $x_{n-1} \in P$ and $x_n \in Q$, it follows from Case 2, that

$$d(x_n,x_{n+1}) \le h^2 d(x_{n-2},x_{n-1}) + \frac{h\alpha^{n-1}}{1-h} + \frac{\alpha^n}{1-h}. \tag{5}$$

If the maximum of the right hand side of (4) is $d(x_{n-1},x_{n+1})$, then, from (1),

$$\begin{aligned} d(x_n,x_{n+1}) &\le d(x_{n-1},x_{n+1}) \le d(x_{n-1},x'_n) + d(x'_n,x_{n+1}) \\ &\le d(x_{n-1},x'_n) + H(Fx_{n-1},Fx_n) + \alpha^n \\ &\le d(x_{n-1},x'_n) + h \max\left\{\frac{d(x_{n-1},x_n)}{a}, D(x_{n-1},Fx_{n-1}), D(x_n,Fx_n),\right. \\ &\qquad \left.[D(x_{n-1},Fx_n) + D(x_n,Fx_{n-1})]/(a+h)\right\} + \alpha^n \\ &\le d(x_{n-1},x'_n) + h \max\left\{\frac{d(x_{n-1},x_n)}{a}, d(x_{n-1},x'_n), d(x_n,x_{n+1}),\right. \\ &\qquad \left.[d(x_{n-1},x_{n+1}) + d(x_n,x'_n)]/(a+h)\right\} + \alpha^n \\ &\le \max\left\{(1+h)d(x_{n-1},x'_n) + \alpha^n, \frac{\alpha^n}{1-h},\right. \\ &\qquad \left.[d(x_{n-1},x_{n+1}) + d(x_n,x'_n)]/(a+h)\right\} + \alpha^n. \end{aligned} \tag{6}$$

Using (6), if the maximum of the quantity in braces is the third term, then

$$d(x_{n-1},x_{n+1}) \le \frac{hd(x_n,x'_n) + (a+h)\alpha^n}{a} \le \frac{hd(x_{n-1},x'_n) + (a+h)\alpha^n}{a}.$$

Therefore, by Case 2,

$$\begin{aligned} d(x_n,x_{n+1}) &\le \max\left\{(1+h)d(x_{n-1},x'_n) + \alpha^n, \frac{\alpha^n}{1-h}, \frac{hd(x_{n-1},x'_n) + (a+h)\alpha^n}{a}\right\} \\ &\le (1+h)d(x_{n-1},x'_n) + \frac{\alpha^n}{1-h} \\ &\le h(1+h)d(x_{n-2},x_{n-1}) + \frac{h\alpha^{n-1}}{1-h} + \frac{\alpha^n}{1-h}. \end{aligned} \tag{7}$$

Define $\delta = \alpha^{-1/2} \max\{d(x_0,x_1)d(x_1,x_2)\}$. We shall now show that

$$d(x_n,x_{n+1}) \le \alpha^{n/2}(\delta + 3n), \quad n > 1. \tag{8}$$

The proof is by induction. Note that, for $0 \le h < \left(-1+\sqrt{5}\right)/2, (1+h)/(1-h) < 3$, and $1/(1-h) < 3$.

If x_2 and x_3 are such that (3) or (4) is satisfied, then

$$d(x_2,x_3) \le hd(x_1,x_2) + \frac{\alpha^2}{1-h} \le h\alpha^{1/2}\delta + 3\alpha^2 < \alpha(\delta + 3),$$

since $h < h(1+h) = \alpha$.

Note that (5) implies (7). If x_2 and x_3 are such that (7) is satisfied, then

$$d(x_2,x_3) \leq h(1+h)d(x_1,x_2) + \frac{(1+h)\alpha}{1-h} + \frac{\alpha^2}{1-h}$$
$$\leq \alpha^{3/2}\delta + 3\alpha + 3\alpha^2 \leq \alpha(\delta+6).$$

Therefore, in aall cases, $d(x_2,x_3) \leq \alpha(\delta+6)$. Assume the induction hypothesis. If (3) or (4) are satisfied, then

$$d(x_n,x_{n+1}) \leq hd(x_{n-1},x_n) + \frac{\alpha^n}{1-h} \leq h\alpha^{(n-1)/2}\left(\delta+3(n-1)\right) + 3\alpha^n$$
$$\leq \alpha^{n/2}(\delta+3n)$$

If (7) is satisfied, then

$$d(x_n,x_{n+1}) \leq h(1+h)d(x_{n-2},x_{n-1}) + \frac{(1+h)\alpha^{n-1}}{1-h} + \frac{\alpha^n}{1-h}$$
$$\leq \alpha^{n/2}\left(\delta+3(n-2)\right) + 3\alpha^{n-1} + 3\alpha^n \leq \alpha^{n/2}(\delta+3n).$$

From (8) it follows that, for $m > n$,

$$d(x_n,x_m) \leq \sum_{i=n}^{m-1} d(x_i,x_{i+1}) \leq \delta \sum_{i=n}^{m-1} \alpha^{i/2} + 3\sum_{i=n}^{m-1} \alpha^{i/2} i,$$

and $\{x_n\}$ is Cauchy, hence convergent. Call the limit p.

Let $\{x_{n_k}\}$ denote the subsequence of $\{x_n\}$ with the property that each term of the subsequence belongs to P. Then

$$H(Fx_{n_k-1},Fp) \leq h\ \max\{d(x_{n_k-1},p)/a, D(x_{n_k-1},Fx_{n_k-1}), D(p,Fp),$$
$$[D(x_{n_k-1},Fp) + D(p,Fx_{n_k-1})]/(a+h)\}$$
$$\leq h\ \max\{d(x_{n_k-1},p)/a, d(x_{n_k-1},x_{n_k}), D(p,Fp),$$
$$[D(x_{n_k-1},Fp) + d(p,x_{n_k})]/(a+h)\}.$$

Taking the limit as $k \to \infty$ yields

$$H(p,Fp) \leq hD(p,Fp),$$

which implies, since $H(p,Fp) = D(p,Fp)$, that $p \in Fp$.

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