Internat. J. Math. & Math. Sci.
VOL. 20 NO. 2 (1997) 347-366

NEW APPROACH TO ASYMPTOTIC STABILITY: TIME-VARYING NONLINEAR SYSTEMS

L.T. GRUJIĆ
Ecole Nationale d'Ingenieurs de Belfort
Espace Bartholdi, Belfort Technopole, B.P. 525
90016 Belfort Cedex - FRANCE

(Received February 24, 1994 and in revised form June 30, 1995)

ABSTRACT. The results of the paper concern a broad family of time-varying nonlinear systems with differentiable motions. The solutions are established in a form of the necessary and sufficient conditions for: 1) uniform asymptotic stability of the zero state, 2) for an exact single construction of a system Lyapunov function and 3) for an accurate single determination of the (uniform) asymptotic stability domain. They permit **arbitrary selection** of a function $p(\bullet)$ from a defined functional family to determine a Lyapunov function $\mathrm{v}(\bullet)$, $[v(\bullet)]$, by solving $\mathrm{v}'(\bullet) = -p(\bullet)$ {or equivalently, $v'(\bullet) = -p(\bullet)[1 - v(\bullet)]$}, respectively. Illustrative examples are worked out.

KEY WORDS AND PHRASES: Nonlinear Dynamical Ssytems, Lyapunov and Asymptotic Stability.
1991 AMS SUBJECT CLASSIFICATION CODES: 34A34, 34C35, 34D20, 93D05, 93D20.

1. INTRODUCTION

The well known fundamental advantageous feature of the Lyapunov method consists in the use of both a positive definite function and its total derivative along system motions without knowing the motions themselves in order to investigate qualitative properties of the system behavior, among which there are various asymptotic stability properties.

Theorems established for time-varying nonlinear systems have been expressed in terms of *existence* of a Lyapunov function $\mathrm{v}(\bullet)[v(\bullet)]$ without clarifying how to determine it, that is without clarifying how to choose $p(\bullet)$ in $\mathrm{v}'(\bullet) = -p(\bullet)$ {or equivalently, in $v' = -p(\bullet)[1 - v(\bullet)]$}, and what are, with respect to a selected $p(\bullet)$, the necessary and sufficient conditions for a solution $\mathrm{v}(\bullet)\,[v(\bullet)]$, respectively, to guarantee uniform asymptotic stability of $x = 0$ and/or to determine accurately its domain of uniform asymptotic stability. Such a crucial incompleteness of the existing Lyapunov stability theory has been an inherent obstacle to broader and more effective applications of the theory than have been realized. It was overcome in [2]-[12] for different classes of time-invariant systems by proposing three distinct approaches. Their common feature is in defining a family of functions $p(\bullet)$ used to generate a function $\mathrm{v}(\bullet)$ [or, $v(\bullet)$] and in specifying the necessary and sufficient properties of $\mathrm{v}(\bullet)$ [or, $v(\bullet)$] to guarantee asymptotic stability of the zero state and/or to ensure that a set N is the domain of its asymptotic stability. This paper is aimed to establish analogous solutions for a broad family of *time-varying nonlinear systems*.

2. NOTATION

Capital italic Roman letters are used for sets, lower case block Roman characters for vectors, Greek letters and lower case italic letters denote scalars except for the empty set $\emptyset$ and subscripts. The boundary, interior and closure of a set A are designated by ∂A, In A and ClA, respectively, where A is a time-invariant set. If $A(\bullet) : R \to 2^{R^n}$ is a set-valued function then its instantaneous set value $A(t)$ at arbitrary time $t \in R$ will be called a time-varying set $A(t)$. Let $\|\bullet\| : R^n \to R_+$ be the Euclidean norm on R^n, where $R_+ = [0, \infty[= \{x : x \in R, 0 \le x < \infty\}$. B_α will be used for the open hyperball with radius α centered at the origin, $B_\alpha = \{\mathrm{x} : \mathrm{x} \in R^n, \ \| \mathrm{x} \| < \alpha\}$. An initial time $t_0 \in R_i$ and determines $R_0 = [t_0, \infty[$, where $R_i =]\sigma, +\infty[$, $\sigma \in [-\infty, \infty[$.

Let $\zeta \in R^+ =]0, \infty[$ and $p(\bullet) : R_i \times R^n \to R$. Then $P_\zeta(t)$ is the largest open connected neighborhood of $\mathrm{x} = 0$ at time $t \in R_i$ such that $p(t, \mathrm{x}) < \zeta$ for every $\mathrm{x} \in P_\zeta(t)$.

Let $\rho(\bullet) : R^n \times 2^{R^n} \to R_+$ be a distance function defined by

$$\rho(\mathrm{x}, A) = \inf[\ \| \mathrm{x} - \mathrm{y} \| \ : \mathrm{y} \in A] \ .$$

Let now $\mathrm{d}(\bullet) : 2^{R^n} \times 2^{R^n} \to R_+$ be a distance function introduced by:

$$\mathrm{d}(A, B) = \max\{[\sup \rho(\mathrm{x}, B) : \mathrm{x} \in A], [\sup \rho(\mathrm{y}, A) : \mathrm{y} \in B]\} \ .$$

A non-empty set-valued function $S(\bullet) : R \to 2^{R^n}$ is continuous at $\tau \in R$ if and only if for every $\epsilon \in R^+$ there is $\delta \in R^+$, $\delta = \delta(\tau, \epsilon)$, such that $|t - \tau| < \delta$ implies $\mathrm{d}[S(t), S(\tau)] < \epsilon$. It is continuous on R_0 (i.e. in $t \in R_0$) if and only if it is continuous at every $t \in R_0$.

Let $\mathbf{x}(\bullet\,; t_0, \mathrm{x}_0)$ be a motion (solution) of a system through $\mathrm{x}_0 \in R^n$ at $t_0 \in R_i$, and $\mathbf{x}(t\,; t_0, \mathrm{x}_0)$ be its vector value $\mathrm{x}(t)$ at time $t \in R_i$, $\mathrm{x}(t) \equiv \mathbf{x}(t\,; t_0, \mathrm{x}_0)$.

If a function $\mathrm{x}(\bullet)$ is differentiable then its total time derivative $\mathrm{dx}(\bullet)/\mathrm{d}t$ will be also denoted by $\mathrm{x}'(\bullet)$. If $v : R_i \times R^n \mapsto R$ is differentiable then its total time derivative along $\mathrm{x}(\bullet)$ is its Eulerian derivative $v'[\bullet, \mathrm{x}(\bullet)]$,

$$v'[t, \mathrm{x}(t)] = \frac{\partial v[t, \mathrm{x}(t)]}{\partial t} + \{\operatorname{grad} v[t, \mathrm{x}(t)]\}^{\mathrm{T}} \mathrm{x}'(t) \,.$$

The notation used for stability domains is explained in Definitions 1-3 (Section 4). Their importance was explained by LaSalle and Lefschetz [17].

3. SYSTEM DESCRIPTION

Nonlinear time-varying systems treated herein are described by (3.1),

$$\frac{\mathrm{dx}(t)}{\mathrm{d}t} = \mathrm{f}[t, \mathrm{x}(t)] \,, \quad \mathrm{x}(\bullet) : R \to R^n \,, \quad \mathrm{f}(\bullet) : R \times R^n \to R^n \,, \tag{3.1}$$

and by one of the following features:

Weak Smoothness Property

(i) There is an open continuous neighborhood $S(t)$ of $\mathrm{x} = 0$ for every $t \in R_i$, $S(t) \subseteq R^n$, such that $S = \cap [S(t) : t \in R_i] = S(R_i)$ is also an open neighborhood of $\mathrm{x} = 0$ and for every $(t_0, \mathrm{x}_0) \in R_i \times S(t_0)$:

  a) the system (3.1) has a unique solution $\mathbf{x}(\bullet\,; t_0, \mathrm{x}_0)$ through x_0 at t_0 on a largest interval I_0, $I_0 = I_0(t_0, \mathrm{x}_0)$, $R_i \supseteq I_0$, $I_0 \neq \emptyset$, and

  b) $\mathbf{x}(\bullet\,; t_0, \mathrm{x}_0)$ is defined, continuous and differentiable in $(t, t_0, \mathrm{x}_0) \in I_0 \times R_i \times S(t_0)$.

(ii) For every $(t_0, \mathrm{x}_0) \in R_i \times [R^n - \partial S(t_0)]$ every motion $\mathbf{x}(\bullet\,; t_0, \mathrm{x}_0)$ of the system (3.1) is continuous in $t \in I_0$.

Strong Smoothness Property

(i) The system (3.1) obeys the Weak Smoothness Property.

(ii) If the boundary $\partial S(t)$ of $S(t)$ is non-empty at any $\tau \in R_i$ then every motion of the system (3.1) passing through $\mathrm{x}_0 \in \partial S(t_0)$ at $t_0 = \tau$ obeys $\inf[\ \| \mathbf{x}(t\,; t_0, \mathrm{x}_0) \| \ : t \in I_0] > 0$ for every $(t_0, \mathrm{x}_0) \in R_i \times \partial S(t_0)$.

4. ASYMPTOTIC STABILITY DOMAINS

This section is aimed to clarify the notions of the asymptotic stability domains in the framework of time-varying systems.

The notion of the attraction domain has been mainly used in the following sense (Hahn [15], Zubov [20]):

Definition 1.

The state $x = 0$ of the system (3.1) has:

(a) **the domain of attraction at t_0** denoted by $D_a(t_0)$, $D_a(t_0) \subseteq R^n$, if and only if both 1) and 2) hold,

1) for $t_0 \in R$ and every $\zeta \in R^+$ there exists $\tau = \tau(t_0, x_0, \zeta) \in R_+$ such that $\| \mathbf{x}(t; t_0, x_0) \| < \zeta$ for all $t \in]t_0 + \tau, \infty[$ is valid provided only that $x_0 \in D_a(t_0)$,

2) the set $D_a(t_0)$ is a neighborhood of $x = 0$.

(b) **the domain $D_a(R_i)$ of uniform attraction on R_i**, $D_a(R_i) \subseteq R^n$, if and only if 1)-4) hold,

1) it has the domain $D_a(t_0)$ of attraction at every $t_0 \in R_i$,

2) $\cap [D_a(t_0) : t_0 \in R_i]$ is a neighborhood of $x = 0$,

3) $D_a(R_i) = \cap [D_a(t_0) : t_0 \in R_i]$,

4) the minimal $\tau(t_0, x_0, \zeta)$ obeying 1) of (a) and denoted by $\tau_m(t_0, x_0, \zeta)$ satisfies

$$\sup[\tau_m(t_0, x_0, \zeta) : t_0 \in R_i] < +\infty, \quad \text{for every} \quad (x_0, \zeta) \in D_a(R_i) \times R^+ .$$

The expression "***on** R_i*" is to be omitted if and only if $R_i = R$. Then and only then $D_a(R_i)$ will be denoted by D_a, $D_a = D_a(R)$.

The stability domain and the asymptotic stability domain in the Lyapunov sense were introduced in [1], and further broadened and used in [2]-[14] as follows:

Definition 2.

The state $x = 0$ of the system (3.1) has:

(a) **the domain of stability at t_0** denoted by $D_s(t_0)$, $D_s(t_0) \subseteq R^n$, if and only if 1)-3) hold,

1) for every $\epsilon \in R^+$ the motion $\mathbf{x}(\bullet; t_0, x_0)$ satisfies $\| \mathbf{x}(t; t_0, x_0) \| < \epsilon$ for all $t \in R_0$ provided only that $x_0 \in D_s(t_0, \epsilon)$,

2) the set $D_s(t_0, \epsilon)$ is a neighborhood of $x = 0$ for every $\epsilon \in R^+$,

3) the set $D_s(t_0)$ is the union of all the sets $D_s(t_0, \epsilon)$ over $\epsilon \in R^+$:

$$D_s(t_0) = \cup [D_s(t_0, \epsilon) : \epsilon \in R^+] .$$

(b) **the domain $D_s(R_i)$ of uniform stability on R_i** if and only if 1)-3) hold,

1) it has the domain $D_s(t_0)$ of stability at every $t_0 \in R_i$,

2) $\cap [D_s(t_0) : t_0 \in R_i]$ is a neighborhood of $x = 0$,

3) $D_s(R_i) = \cap [D_s(t_0) : t_0 \in R_i]$.

The expression "**on** R_i" is to be omitted if and only if $R_i = R$. Then and only then $D_s(R_i)$ will be denoted by D_s, $D_s = D_s(R)$.

Definition 3.

The state $x = 0$ of the system (3.1) has:

(a) **the domain of asymptotic stability at t_0** denoted by $D(t_0)$, $D(t_0) \subseteq R^n$, if and only if it has both $D_a(t_0)$ and $D_s(t_0)$, and $D(t_0)$ is their intersection:

$$D(t_0) = D_a(t_0) \cap D_s(t_0) .$$

(b) **the domain** $D(R_i)$ **of uniform asymptotic stability on R_i** if and only if it has both $D_a(R_i)$ and $D_s(R_i)$, and $D(R_i)$ is their intersection:

$$D(R_i) = D_a(R_i) \cap D_s(R_i) .$$

The expression "**on** R_i" is to be omitted if and only if $R_i = R$. Then and only then $D(R_i)$ will be denoted by D, $D = D(R)$.

Qualitative features of the stability domains of $x = 0$ of the system (3.1) are discovered and proved in Appendix 1. They are important for proofs of the main results of the paper (Section 6).

5. FAMILIES OF FUNCTIONS $p(\bullet)$ AND LYAPUNOV FUNCTIONS

Families $P(\bullet)$ and $P^1(\bullet)$ of functions $p(\bullet)$ were used in [10], [11] to generate Lyapunov functions $\nu(\bullet)$ obtained as solutions of $\nu'(\bullet) = -p(\bullet)$ [or, to determine Lyapunov functions $v(\bullet)$ as solutions of $v'(\bullet) = -[1 - v(\bullet)]p(\bullet)]$ in the framework of time invariant systems. In the setting of this paper they will be replaced by families $L^1(\bullet)$ and $E^1(\bullet)$ of functions $p(\bullet)$.

Definition 4.

A function $p(\bullet) : R_i \times R^n \to R$ belongs to **family** $L^1(R_i, S; f)$ if and only if:

1) $p(\bullet)$ is differentiable on $R_i \times S : p(t, x) \in C^{(1)}(R_i \times S)$,
2) the equations (5.1) with (5.1a) taken along motions of the system (3.1),

$$\nu'(t, x) = -p(t, x) , \tag{5.1a}$$

$$\nu(t, 0) = 0 , \quad \forall t \in R_i , \tag{5.1b}$$

have a solution $\nu(\bullet)$ that is continuous and differentiable in $(t, x) \in R_i \times ClB_\mu$ for some (anyhow small) $\mu \in R^+$, $\mu = \mu(f, p)$, and which obeys (5.2) for some $w_\mu(x) \in C(ClB_\mu)$,

$$\nu(t, x) \le w_\mu(x), \ \forall (t, x) \in R_i \times ClB_\mu , \tag{5.2}$$

and

3) for any $\rho \in R^+$ fulfilling $S(t) \supset ClP_\rho(t)$ for all $t \in R_i$ it holds:

$$\min\{p(t, x) : (t, x) \in R_i x[S(t) - P_\rho(t)]\} = \alpha , \quad \alpha = \alpha(\rho; p) \in R^+ .$$

Definition 5.

A function $p(\bullet) : R_i \times R^n \to R$ belongs to **family** $E^1(R_i, S; f)$ if and only if:

1) $p(\bullet)$ is differentiable on $R_i \times S : p(t, x) \in C^{(1)}(R_i \times S)$,
2) the equations (5.3) with (5.3a) taken along motions of the system (3.1),

$$v'(t, x) = -[1 - v(t, x)]p(t, x) , \tag{5.3a}$$

$$v(t, 0) = 0 , \quad \forall t \in R_i , \tag{5.3b}$$

have a solution v that is differentiable in $(t, x) \in R_i \times ClB_\mu$ for some $\mu \in R^+$, $\mu = \mu(f, p)$, and which obeys (5.4) for some $w_\mu(x) \in C(ClB_\mu)$,

$$v(t, x) \le w_\mu(x) , \quad \forall (t, x) \in R_i \times ClB_\mu , \tag{5.4}$$

and

3) for any $\rho \in R^+$ satisfying $S(t) \supset ClP_\rho(t)$ for all $t \in R_i$ it holds:

$$\min\{p(t, x) : (t, x) \in R_i x[S(t) - P_\rho(t)]\} = \alpha , \quad \alpha = \alpha(\rho; p) \in R^+ .$$

Notice that $p(\bullet) \in L^1(R_i, S, f)$ if and only if $p(\bullet) \in E^1(R_i, S, f)$, which is easy to verify. If $p(\bullet) \in L^1(R_i, S; f)$, hence $p(\bullet) \in E^1(R_i, S, f)$, then solutions $\nu(\bullet)$ and $v(\bullet)$ to (5.1) and (5.3) are interrelated by (5.5),

$$v(t, x) = 1 - \exp[-\nu(t, x)] , \tag{5.5}$$

which was pointed out by Vanelli and Vidyasagar [19]. Besides, $v(t, x) = 0$ if and only if $\nu(t, x) = 0$, and $v(t, x) \to 1$ if and only if $\nu(t, x) \to \infty$.

There is not any stability condition imposed on the system and no definiteness requirement is imposed on $p(\bullet)$, $\nu(\bullet)$ and $\upsilon(\bullet)$ in Definition 4 and Definition 5 Therefore, $L^1(R_\imath, S, \mathrm{f})$ and $E^1(R_\imath, S, \mathrm{f})$ are not dependent in general on a stability property of the system.

6. NOVEL SOLUTIONS TO THE CLASSICAL STABILITY PROBLEMS

For the sake of clearness we emphasize that the notions of a positive definite function and of a decrescent function will be used in the usual sense (c.f Hahn [15], Zubov [20]), that is that a function $\nu(\bullet) : R_\imath \times R^n \to R$

(a) is **positive definite on** $R_\imath \times A(t)$ if and only if $A = \cap [A(t) : t \in R_\imath]$ is an open connected neighborhood of $\mathrm{x} = 0$ such that there exists $w_1(\bullet) : R^n \to R$ obeying the following

1) $\nu(t, \mathrm{x})$ and $w_1(\mathrm{x})$ are uniquely determined by $(t, \mathrm{x}) \in R_\imath \times A(t)$ and continuous on $R_\imath \times A(t)$, $\nu(t, \mathrm{x})$ is also differentiable in $(t, \mathrm{x}) \in R_\imath \times A(t)$, that is $\nu(t, \mathrm{x}) \in C^{(1)}[R_\imath \times A(t)]$ and $w_1(\mathrm{x}) \in C\{\cup [A(t) : t \in R_\imath]\}$,

2) $\nu(t, 0) = 0$ for all $t \in R_\imath$, $w_1(0) = 0$,

and

3) $\nu(t, \mathrm{x}) \geq w_1(\mathrm{x})$ for all $(t, \mathrm{x}) \in R_\imath \times A(t)$.

(b) is **decrescent on** $R_\imath \times A(t)$ if and only if $A = \cap [A(t) : t \in R_\imath] = A(R_\imath)$ is open connected neighborhood of $\mathrm{x} = 0$ such that there exists $w_2(\bullet) : R^n \to R$ obeying what follows·

1) $\nu(t, \mathrm{x})$ and $w_2(\mathrm{x})$ are continuous on $R_\imath \times A$, that is $\nu(t, \mathrm{x}) \in C(R_\imath \times A)$ and $w_2(\mathrm{x}) \in C(A)$,

and

2) $\nu(t, \mathrm{x}) \leq w_2(\mathrm{x})$ for all $(t, \mathrm{x}) \in R_\imath \times A$.

The expression "R_i" is to be omitted if and only if $R_\imath = R$, and the expression "$\mathbf{x}A(t)$" is to be omitted if and only if $A(t)$ is some (anyhow small) open connected neighborhood of $\mathrm{x} = 0$ for all $t \in R_\imath$.

Solutions to the problems will depend on the smoothness properties of the system (3.1) as well as whether a function $p(\bullet)$ generating a system Lyapunov function is selected from $L^1(R_\imath, S, \mathrm{f}_r)$ or from $E^1(R_\imath, S, \mathrm{f})$

THEOREM 1. For the state $\mathrm{x} = 0$ of the system (3.1) with the Strong Smoothness Property to have the domain $D(R_\imath)$ of uniform asymptotic stability on $R_\imath$, for a set $N(t_0)$, $N(t_0) \subseteq R^n$, to be the domain of its asymptotic stability at $t_0 \in R_\imath : N(t_0) \equiv D(t_0)$, and for a set N, $N \subseteq R^n$, to be the domain of its uniform asymptotic stability on $R_\imath$, $N = D(R_\imath)$, it is both necessary and sufficient that

1) the set $N(t_0)$ is an open neighborhood of $\mathrm{x} = 0$ and $N(t_0) \subseteq S(t_0)$ for every $t_0 \in R_\imath$,

2) the set N is a connected neighborhood of $\mathrm{x} = 0$ and $N \subseteq S$,

3) $\mathrm{f}(t, \mathrm{x}) = 0$ for $(t, \mathrm{x}) \in R_\imath \times N(t)$ if and only if $\mathrm{x} = 0$,

and

4) for any differentiable decrescent positive definite function $p(\bullet)$ on $R_\imath \times S(t)$ obeying:

(a) $p(\bullet) \in L^1(R_\imath, S; \mathrm{f})$ the equations (5.1) have the unique solution function $\nu(\bullet)$ with the following properties.

(i) $\nu(\bullet)$ is a decrescent positive definite function on $R_\imath \times N(t)$,

(ii) if the boundary $\partial N(t)$ of $N(t)$ is nonempty then $\mathrm{x} \to \partial N(t), \mathrm{x} \in N(t)$, implies $\nu(t, \mathrm{x}) \to \infty$ for every $t \in R_\imath$, and

(iii) $N = \cap [N(t) : t \in R_\tau] = N(R_\tau)$,

or obeying:

(b) $p(\bullet) \in E^1(R_\tau, S; f)$ the equations (5.3) have the unique solution function $v(\bullet)$ with the following properties:

(i) $v(\bullet)$ is a decrescent positive definite function on $R_\tau \times N(t)$,

(ii) if the boundary $\partial N(t)$ of $N(t)$ is nonempty then $x \to \partial N(t), x \in N(t)$, implies $v(t, x) \to 1$ for every $t \in R_\tau$,

and

(iii) $N = \cap [N(t) : t \in R_\tau] = N(R_\tau)$.

PROOF. Necessity. Let $x = 0$ of the system (3.1) possessing the Strong Smoothness Property have the uniform asymptotic stability domain $D(R_\tau)$ on R_τ. Hence, it has also the asymptotic stability domain $D(t_0)$ at every $t_0 \in R_\tau$ (Definition 1). Definitions 1 and 3 show that it has also the uniform attraction domain $D_a(R_\tau)$ and the attraction domain $D_a(t_0)$ at every $t_0 \in R_\tau$. Obviously, $D_a(t_0) \supseteq D(t_0)$ for all $t_0 \in R_\tau$ and $D_a(R_\tau) \supseteq D(R_\tau)$. Besides, $D_a(t_0)$ is a neighborhood of $x = 0$ at every $t_0 \in R_\tau$ and $D_a(R_\tau)$ is also a neighborhood of $x = 0$ (Definition 1) The set $S(t_0)$ is a neighborhood of $x = 0$ at every $t_0 \in R_\tau$ and S is also a neighborhood of $x = 0$ (the Weak Smoothness Property). Hence, $D_a(t_0) \cap S(t_0) \neq \emptyset$ for all $t_0 \in R_\tau$ and $D_a(R_\tau) \cap S \neq \emptyset$. Let us prove $S(t_0) \supseteq D_a(t_0)$ for every $t_0 \in R_\tau$. If $\partial S(t_0) \neq \emptyset$ then we will consider separately $u \in \partial S(t_0)$ and $w \in [R^n - ClS(t_0)]$. If $u \in \partial S(t_0)$ then $u \notin \partial D_a(t_0)$ due to (ii) of the Strong Smoothness Property. Therefore, $D_a(t_0) \cap \partial S(t_0) = \emptyset, \forall t_0 \in R_\tau$. If $w \in [R^n - ClS(t_0)]$ then for $\mathbf{x}(t; t_0, w) \to 0$ as $t \to \infty$ it is necessary that there is $t^* \in R_0$ such that $\mathbf{x}(t^*; t_0, w) \in \partial S(t^*)$ because $D_a(t)$ and $S(t)$ are neighborhoods of $x = 0$ for all $t_0 \in R_i$, both $S(t)$ and $\mathbf{x}(t; t_0, w)$ are continuous in $t \in R_\tau$ and $S = \cap [S(t) : t \in R_\tau]$ is an open neighborhood of $x = 0$ (the Weak Smoothness Property). However, $\mathbf{x}(t^*; t_0, w) \in \partial S(t^*)$ implies $\inf [\, \| \mathbf{x}(t; t_0, w) \| \, : t \in R_0] > 0$ because of (ii) of the Strong Smoothness Property. This yields $w \notin \partial D_a(t_0)$ and $[R^n - ClS(t_0)] \cap D(t_0) = \emptyset, \forall t_0 \in R_\tau$. Altogether, $D_a(t_0) \cap S(t_0) \neq \emptyset$, $D_a(t_0) \cap \partial S(t_0) = \emptyset$, $D_a(t_0) \cap [R^n - ClS(t_0)] = \emptyset$, $\forall t_0 \in R_\tau$, which prove $S(t_0) \supseteq D_a(t_0)$, $\forall t_0 \in R_\tau$. Therefore, $S(t_0) \supseteq D(t_0)$ due to $D_a(t_0) = D(t_0)$ (Lemma A.2). Let $N(t_0) \equiv D(t_0)$ so that $S(t_0) \supseteq N(t_0)$. Hence, $N(t_0)$ is an open neighborhood of $x = 0$ (Lemma A.1), which proves necessity of the condition 1). Now, $N = D$ and Lemma A.1 prove necessity of the condition 2). From $D_s(t) \supseteq D_a(t) \equiv D(t) \equiv N(t)$ and Definitions 1-3 it results that $x = 0$ is the unique equilibrium state of $x = 0$ of the system (3 1) in $N(t)$, $\forall t_0 \in R_\tau$, which implies $f(t, x) = 0$ for $(t, x) \in R_\tau \times N(t)$ if and only if $x = 0$ (Proposition 7 in Grujić et al. [12]) This proves necessity of the condition 3). From $N(t_0) \equiv D(t_0)$ it follows that the interval I_0 of existence of $\mathbf{x}(\bullet; t_0, x_0)$ satisfies $I_0 \subseteq R_0$, $\forall (t_0, x_0) \in R_\tau \times N(t_0)$ due to Definitions 1-3. Let $p(\bullet) \in L^1(R_\tau, S, f)$ be an arbitrarily selected positive definite decrescent function on $R_\tau \times S(t)$ Hence, there is $\mu > 0$ such that there exists a solution function $v(\bullet)$ to the equations (5.1), which is continuous in $(t, x) \in R_\tau \times ClB_\mu$ and satisfies (5.2). Therefore,

$$|v(t, x)| < \infty, \quad \forall (t, x) \in R_\tau \times ClB_\mu, \tag{6 1a}$$

$$|\partial v(t,x)/\partial t| < \infty, \quad \forall (t, x) \in R_\tau \times ClB_\mu, \tag{6.1b}$$

and

$$\| \operatorname{grad} v(t,x) \| < \infty , \quad \forall (t,x) \in R_i \times ClB_\mu . \tag{6.1c}$$

Let $\beta \in]1, \infty[$ and $\zeta \in R^+$ be such that

$$ClB_\beta \cap ClB_\mu \cap S(t) \supset P_\zeta(t) , \quad \forall t \in R_i . \tag{6.2}$$

Existence of such ζ is guaranteed by positive definiteness of $p(\bullet)$ on $R_i \times S(t)$ and the fact that $\cap [S(t) : t \in R_i]$ is a neighborhood of $x = 0$. Let $t_0 \in R_i$ be arbitrary and $\tau \in R_+$, $\tau = \tau(t_0, x_0; f, p, \zeta)$, be such that for any $x_0 \in N(t_0)$ the condition (6.3) holds,

$$\mathbf{x}(t; t_0, x_0) \in ClP_\zeta(t) , \quad \forall t \in [t_0 + \tau, \infty[. \tag{6.3}$$

Such τ exists due to Definitions 1 and 3, $x_0 \in N(t_0)$ and $D_a(t_0) \equiv D(t_0) \equiv N(t_0)$. Notice that $x_0 \in N(t_0)$ yields also

$$\mathbf{x}(\infty; t_0, x_0) = 0 . \tag{6.4}$$

Let (5.1a) be integrated from $t \in R_0$ to ∞,

$$v[\infty, \mathbf{x}(\infty; t_0, x_0)] - v[t, \mathbf{x}(t; t_0, x_0)] = -\int_t^\infty p[\sigma, \mathbf{x}(\sigma; t_0, x_0)] d\sigma, \quad \forall (t, x_0) \in R_0 \times N(t_o) , \tag{6.5}$$

Now, (5.1b) and (6.4) yield (6.5) in the next form,

$$v[t, \mathbf{x}(t; t_0, x_0)] = \int_t^\tau p[\sigma, \mathbf{x}(\sigma; t_0, x_0)] d\sigma + \int_\tau^\infty p[\sigma, \mathbf{x}(\sigma; t_0, x_0)] d\sigma, \quad \forall (t, x_0) \in R_0 \times N(t_o) . \tag{6.6}$$

Invariance of $D_a(t)$ with respect to system motions on R_i (Lemma A.1), $S(t) \supseteq D(t) \equiv D_a(t) \equiv N(t)$, differentiability of the motions $\mathbf{x}(t; t_0, x_0)$ in $(t; t_0, x_0) \in I_0 \times R_i \times S(t_0)$ [(i-b) of the Weak Smoothness Property], continuity, positive definiteness and decrescency of $p(\bullet)$ on $R_i \times S(t)$, the definition of τ and compactness of $[t, \tau]$ for any $t \in R_0$ imply

$$\left| \int_t^\tau p[\sigma, \mathbf{x}(\sigma; t_0, \mathbf{x}_0)] d\sigma \right| < \infty, \quad \forall (t, t_0, x_0) \in R_0 \times R_i \times N(t_0) , \tag{6.7a}$$

$$\left| \frac{\partial}{\partial t} \int_t^\tau p[\sigma, \mathbf{x}(\sigma; t_0, x_0)] \, d\sigma \right| = |p[t, \mathbf{x}(t; t_0, x_0)]| < \infty, \quad \forall (t, t_0, x_0) \in R_0 \times R_i \times N(t_0) , \tag{6.7b}$$

and

$$\left\| \int_t^\tau \operatorname{grad} p[\sigma, \mathbf{x}(\sigma; t_0, x_0)] d\sigma \right\| < \infty, \quad \forall (t, t_0, x_0) \in R_0 \times R_i \times N(t_0) . \tag{6.7c}$$

Now, (6.1)-(6.3), (6.6) and (6.7) yield

$$|v[t, \mathbf{x}(t; t_0, x_o)]| < \infty , \quad \forall (t, t_0, x_0) \in R_0 \times R_i \times N(t_0) , \tag{6.8a}$$

$$\left| \frac{\partial}{\partial t} v[t, \mathbf{x}(t; t_0, x_0)] \right| < \infty , \quad \forall (t, t_0, x_0) \in R_0 \times R_i \times N(t_0) , \tag{6.8b}$$

and

$$\| \operatorname{grad} v[t, \mathbf{x}(t; t_0, x_0)] \| < \infty , \quad \forall (t, t_0, x_0) \in R_0 \times R_i \times N(t_0) . \tag{6.8c}$$

Let $t = t_0$ and $x = x_0$ be set in (6.8) Then,

$$|v(t,x)| < \infty , \quad \forall(t,x) \in R_t \times N(t) , \tag{6 9a}$$

$$\left|\frac{\partial}{\partial t} v(t,x)\right| < \infty , \quad \forall(t,x) \in R_t \times N(t) , \tag{6 9b}$$

and

$$\|\text{grad}\, v(t,x)\| < \infty , \quad \forall(t,x) \in R_t \times N(t) \tag{6 9c}$$

Differentiability of $p(\bullet)$ on $R_t \times S(t)$, $S(t) \supseteq N(t)$, (6 6) and (6.9) prove

$$v(t,x) \in C^{(1)}[R_t \times N(t)] . \tag{6 10}$$

Invariance of both $D_a(t)$ and $N = D(R_t)$ on R_t, $D_a(t) \equiv D(t) \equiv N(t)$, continuity of $\mathbf{x}(t;t_0,x_0)$ in $(t;t_0,x_0) \in I_0 \times R_t \times D(t_0)$, positive definiteness and decrescency of $p(\bullet)$ on $R_t \times N(t)$, $p(\bullet) \in L(R_t, S; f)$, (5.2), the definition of τ and compactness of $[t,\tau]$ guarantee existence of $\zeta_i(\bullet) : R^n \mapsto R$, $\zeta_1(x) \in C\{\cup[N(t) : t \in R_t]\}$ and $\zeta_2(x) \in C(N)$ such that

$$\zeta_i(0) = 0 , \quad i = 1,2 , \tag{6.11a}$$

$$0 < \zeta_1(x_0) \leq \int_t^{\infty} \psi_1[\mathbf{x}(\sigma;t_0,x_0)]d\sigma , \quad \forall(t,t_0,x_0 \neq 0) \in R_0 \times R_t \times N(t_0) , \tag{6 11b}$$

$$\infty > \zeta_2(x_0) \geq \int_t^{\tau} \psi_2[\mathbf{x}(\sigma;t_0,x_0)]d\sigma , \quad \forall(t,t_0,x_0) \in R_0 \times R_t \times N , \tag{6.11c}$$

where $\psi_i(\bullet) : R^n \mapsto R$, $i = 1,2$, obey

$$\psi_i(x) \in C\{\cup[N(t) : t \in R_t]\} \quad \text{and} \quad \psi_2(x) \in C(N) , \tag{6.12a}$$

$$\psi_i(0) = 0 , \quad i = 1,2 , \tag{6.12b}$$

$$\psi_1(x) > 0 , \quad \forall(x \neq 0) \in \{\cup[N(t) : t \in R_t]\} \quad \text{and} \quad \psi_2(x) > 0 , \quad \forall(x \neq 0) \in N , \tag{6.12c}$$

$$\psi_1(x) \leq p(t,x) , \quad \forall(t,x) \in R_t \times N(t) , \tag{6.12d}$$

$$p(t,x) \leq \psi_2(x) , \quad \forall(t,x) \in R_t \times N . \tag{6 12e}$$

Such functions $\psi_i(\bullet)$ exist due to decrescency and positive definiteness of $p(\bullet)$ on $R_t \times S(t)$ and $S(t) \supseteq N(t)$ Let $w_i(\bullet) : R^n \mapsto R$, $w_i(x) \in C(R^n)$, $w_i(0) = 0$, $i = 1,2$, be such that

$$0 < w_1(x) \leq \zeta_1(x) , \quad \forall(x \neq 0) \in \cup[N(t) : t \in R_t] , \tag{6 13a}$$

$$w_2(x) \geq \zeta_2(x) + w_\mu(x_\tau) , \quad x_\tau = \mathbf{x}(\tau;t_0,x) , \quad \forall(t_0,x) \in R_t \times N , \tag{6.13b}$$

where $w_\mu(\bullet)$ is defined by (5.2) Now (5.2), (6 6) and (6.11)-(6 13) yield the following for $(t_0,x_0) = (t,x)$

$$w_1(\mathrm{x}) \leq v(t,\mathrm{x}), \quad \forall (t,\mathrm{x}) \in R_i \times N(t), \tag{6.14a}$$

$$v(t,\mathrm{x}) \leq w_2(\mathrm{x}), \quad \forall (t,\mathrm{x}) \in R_i \times N. \tag{6.14b}$$

From $p(\bullet) \in L^1(R_i, S; \mathrm{f})$, (5.1b), (6.10) and (6.14) it follows that a function $v(\bullet)$ defined by (5.1) is decrescent positive definite on $R_i \times N(t)$. Let it be assumed that there exist two such solutions $v_1(\bullet)$ and $v_2(\bullet)$ of (6.1). Hence,

$$v_1(t_0,\mathbf{x}_0) - v_2(t_0,\mathbf{x}_0) = \int_{t_0}^{\infty} \{p[\sigma,\mathbf{x}_1(\sigma;t_0,\mathbf{x}_0)] - p[\sigma,\mathbf{x}_2(\sigma;t_0,\mathrm{x}_0)]\}d\sigma, \forall (t_0,\mathrm{x}_0) \in R_i \times N(t_0). \tag{6.15}$$

Uniqueness of the motions $\mathbf{x}(\bullet;t_0,\mathrm{x}_0)$, $\forall (t_0,\mathrm{x}) \in R_i \times S(t_0)$ (the Weak Smoothness Property), $S(t_0) \supseteq N(t_0)$ and uniqueness of $p(t,\mathrm{x})$ for every $(t,\mathrm{x}) \in R_i \times S(t)$ due to positive definiteness of $p(\bullet)$ on $S(t)$ imply

$$\int_{t_0}^{\infty} \{p[\sigma,\mathbf{x}_1(\sigma;t_0,\mathrm{x}_0)] - p[\sigma,\mathbf{x}_2(\sigma;t_0,\mathrm{x}_0)]\}d\sigma \equiv$$

$$\int_{t_0}^{\infty} \{p[\sigma,\mathbf{x}(\sigma;t_0,\mathrm{x}_0)] - p[\sigma,\mathbf{x}(\sigma;t_0,\mathrm{x}_0)]\}d\sigma = 0\ , \quad \forall (t_0,\mathrm{x}_0) \in R_i \times N(t_0)\ .$$

This and (6.15) prove

$$v_1(t_0,\mathrm{x}_0) \equiv v_2(t_0,\mathrm{x}_0)\ .$$

Hence, the function $v(\bullet)$ is the unique solution to (5). This completes the proof of necessity of the condition 4-a-1). Let $t_0 \in R_i$ be arbitrary and x_k, $\mathrm{k} = 1,2,\ldots$, be a sequence converging to u, $\mathrm{x}_k \mapsto \mathrm{u}$ as $\mathrm{k} \mapsto \infty$, $\mathrm{x}_k \in N(t_0)$ for all $\mathrm{k} = 1,2,\ldots$, and $\mathrm{u} \in \partial N(t_0)$ in case $\partial N(t_0) \neq \phi$. Let $\rho \in R^+$ be arbitrarily chosen so that $N(t) \supset ClP_\rho(t)$ for all $t \in R_i$. Such ρ exists because $p(\bullet)$ is positive definite and defines $ClP_\rho(t)$, and because $\cap [N(t) : t \in R_i]$ is a neighborhood of $\mathrm{x} = 0$. Let τ_k, $\tau_k = \tau(\mathrm{x}_k, \rho) \in R_+$, be the first instant satisfying (6.16),

$$\mathbf{x}(t;t_0,\mathrm{x}_k) \in ClP_\rho(t)\ , \quad \forall t \in [t_0 + \tau_k, \infty[\ . \tag{6.16}$$

Existence of such τ_k is ensured by $\mathrm{x}_k \in N(t_0)$, $N(t) \equiv D(t)$ and by the fact that $\cap [P_\rho(t) : t \in R_i]$ is a neighborhood of $\mathrm{x} = 0$ due to decrescency of $p(\bullet)$ on $R_i \times N(t)$ (Grujić et al. [12]). Continuity of the motions $\mathbf{x}(t;t_0,\mathrm{x})$ in $(t,t_0,\mathrm{x}) \in I_0 \times R_i \times S(t_0)$ (the Weak Smoothness Property), $S(t_0) \supseteq D(t_0) \equiv N(t_0)$, positive invariance of $D(t)$ [(a) of Lemma A.1] and $\mathrm{x}_k \in N(t_0)$ imply

$$\tau_k \mapsto \infty \ \text{ as } \ \mathrm{k} \mapsto \infty\ . \tag{6.17}$$

Let $\mathrm{m} \in \{1,2,\ldots\}$ be such that $\mathrm{x}_k \in [N(t_0) - ClP_\rho(t_0)]$ for all $\mathrm{k} = \mathrm{m}, \mathrm{m}+1,\ldots$, which exists because $N(t_0)$ is open (Lemma 1), $N(t_0) \supset ClP_\rho(t_0)$ and $\mathrm{x}_k \mapsto \partial N(t_0)$ as $\mathrm{k} \mapsto \infty$. Let α be defined by

$$\alpha = \min\{p(t,\mathrm{x}) : (t,\mathrm{x}) \in R_i \times [S(t) - P_\rho(t)]\}\ . \tag{6.18}$$

Since $p(\bullet) \in L^1(R_i, S; \mathrm{f})$ then $\alpha \in R^+$. Hence, (6.16), (6.18) and the definitions of α and τ_k yield $v(t_0,\mathrm{x}_k) \geq \alpha\tau_k$, $\forall t_0 \in R_i$, which together with (6.17) proves necessity of the condition 4-a-ii). The fact: $N(t) \equiv D(t)$ implies the condition 4-a-iii) due to Definitions 1-3. The conditions under 4-b) follow from 4-a) due to (5.1), (5.3) and (5.5). This completes the proof of the necessity part.

Sufficiency Let all the conditions of Theorem 1 hold and $t_0 \in R_\imath$ be arbitrary Then $x = 0$ of the system (3.1) is uniformly asymptotically stable (Grujić et al [12], Hahn [15], Lakshmikantham and Leela [16], Miller and Michel [18], Zubov [20]). Hence, $x = 0$ has both $D(t_0)$ at t_0 and $D(R_\imath)$ (Definitions 1-3) so that $D_a(t_0) = D(t_0)$ and $D_a(R_\imath) = D(R_\imath)$ (Lemma A.2). Besides, $S(t_0) \supseteq N(t_0)$ Since the function $\nu(\bullet)$ is solution to (5.1), [or, $\nu(\bullet)$ is solution to (5.3)], and it is positive definite and decrescent on $R_\imath \times N(t)$, $p(\bullet) \in L^1(R_\imath, S; \mathrm{f})$, [or, $p(\bullet) \in E^1(R_\imath, S; \mathrm{f})$], and $p(\bullet)$ is decrescent positive definite on $R_\imath \times N(t)$, then $N(t_0) \equiv D(t_0)$ as shown in the proof of the necessity part. Hence, $D(R_\imath) = N$ (Definition 3), which completes the proof.

Conditions of Theorem 1 slightly change for the system (3.1) possessing the Weak Smoothness Property rather than the Strong Smoothness Property

THEOREM 2. In order for the state $x = 0$ of the system (3 1) with the Weak Smoothness Property to have the domain $D(R_\imath)$ of uniform asymptotic stability on $R_\imath$, for a set $N(t_0)$, $N(t_0) \subseteq S(t_0)$ for all $t_0 \in R_\imath$, to be the domain of its asymptotic stability at $t_0 \in R_\imath : N(t_0) \equiv D(t_0)$, and for a set $N, N \subseteq S_\imath$, to be the domain of its uniform asymptotic stability on $R_\imath$, $N = D(R_\imath)$, it is both necessary and sufficient that

1) the set $N(t_0)$ is an open neighborhood of $x = 0$ for all $t_0 \in R_\imath$,

2) the set N is a connected neighborhood of $x = 0$,

3) $\mathrm{f}(t, x) = 0$ for $(t, x) \in R_\imath x N(t)$ if and only if $x = 0$,

and

4) for any differentiable decrescent positive definite function $p(\bullet)$ on $R_\imath \times R^n$ obeying:

   (a) $p(\bullet) \in L^1(R_\imath, R^n; \mathrm{f})$ the equations (5.1) have the unique solution function $v(\bullet)$ with the following properties

      (i) $v(\bullet)$ is decrescent positive definite on $R_\imath \times N(t)$,

      (ii) if the boundary $\partial N(t)$ of $N(t)$ is nonempty then $x \mapsto \partial N(t)$, $x \in N(t)$, implies $v(t, x) \mapsto \infty$ for every $t \in R_\imath$,

      and

      (iii) $N = \cap [N(t) : t \in R_\imath]$,

   or obeying

   (b) $p(\bullet) \in E^1(R_\imath, R^n; \mathrm{f})$ the equations (5.3) have the unique solution function $v(\bullet)$ with the following properties.

      (i) $v(\bullet)$ is decrescent positive definite on $R_\imath \times N(t)$,

      (ii) if the boundary $\partial N(t)$ of $N(t)$ is nonempty then $x \mapsto \partial N(t)$, $x \in N(t)$, implies $v(t, x) \mapsto 1$ for every $t \in R_\imath$,

      and

      (iii) $N = \cap [N(t) : t \in R_\imath]$.

PROOF. *Necessity.* Let the system (3.1) possess the Strong Smoothness Property. Let $x = 0$ have the uniform asymptotic stability domain $D(R_\imath)$ on $R_\imath$ so that it has also the asymptotic stability domain $D(t_0)$ at every $t_0 \in R_\imath$. Let $S(t_0) \supseteq D(t_0)$ and let $N(t_0) = D(t_0)$ for all $t_0 \in R_\imath$ so that $D(R_\imath) \subseteq S$ and $N = D(R_\imath)$ Let a positive definite decrescent function $p(\bullet)$ on $R_\imath \times R^n$ be arbitrarily selected so that $p(\bullet) \in L^1(R_i, R^n; f)$ $[p(\bullet) \in E^1(R_i, R^n; f)]$. From now on we should repeat the proof of necessity of the conditions of Theorem 1 in order to complete this proof.

Sufficiency. Let the system (3.1) possess the Weak Smoothness Property and let the conditions 1)-4) hold. Hence, $x = 0$ of the system is uniformly asymptotically stable (Grujić et al. [12], Hahn [15], Lakshmikantham and Leela [16], Miller and Michel [18], Zubov [20]) so that it has both the domain $D(R_i)$ of uniform asymptotic stability and the domain $D(t_0)$ of asymptotic stability at $t_0 \in R_i$ (Definition 3). Let $x_0 \in [R^n - N(t_0)]$ and t_0 be arbitrary. Continuity of $\mathbf{x}(t; t_0, x_0)$in $t \in R_0$ (the Weak Smoothness Property), positive definiteness of $p(\bullet)$ on $R_i \times R^n$ and the condition 4-a-ii), [4-b-ii)] imply $\mathbf{x}(t; t_0, x_0) \in [R^n - N(t)]$ for all $t \in I_0$. Therefore, $D(t_0) \subseteq ClN(t_0)$ and $D(R_i) \subseteq N$. Since $\nu(\bullet)$ is generated via (5.1) [$v(\bullet)$ is generated via (5.3)], then (as shown in the proof of the necessity part of Theorem 1) $\nu(t, x) \mapsto \infty$ as $x \mapsto \partial D(t)$, $x \in D(t)$ [$v(t, x) \mapsto 1$ as $x \mapsto \partial D(t)$, $x \in D(t)$], for every $t \in R_i$, which, together with the condition 4-a-1), [4-b-i)], proves $\partial D(t) \cap N(t) = \phi$ for every $t \in R_i$. This result, $D(t) \subseteq ClN(t)$, and the fact that both $N(t)$ and $D(t)$ are open neighborhoods of $x = 0$ [condition 1 and Lemma A.1] imply $N(t) = D(t)$ and $N = D(R_i)$, which complete the proof.

Theorems 1 and 2 are based on the usage of $p(\bullet) \in L^1(\bullet)$, $[p(\bullet) \in E^1(\bullet)]$. This means that $p(\bullet)$ should obey the condition 3) of Definition 4, [3) of Definition 5], which was used to generate $\nu(t, x) \mapsto \infty$ as $x \mapsto \partial N(t)$, $x \in N(t)$ [$v(t, x) \mapsto 1$ as $x \mapsto \partial N(t)$, $x \in N(t)$], for every $t \in R_i$, in order to determine exactly $D(t)$ and $D(R_i)$. If we are interested only in uniform asymptotic stability of $x = 0$, then such a requirement need not be imposed on $p(\bullet)$ as explained in what follows:

THEOREM 3. In order for the state $x = 0$ of the system (3.1) possessing the Weak Smoothness Property to be uniformly asymptotically stable on R_i it is both necessary and sufficient that

1) $f(t, 0) = 0$ for all $t \in R_i$,

and

2) for any differentiable decrescent positive definite function $p(\bullet)$ on R_i obeying the conditions 1) and 2) of Definition 4 the equations (5.1) have the unique solution function $\nu(\bullet)$ that is differentiable, decrescent and positive definite on R_i.

PROOF. *Necessity*. Let the system (3.1) possess the Weak Smoothness Property. Let $x = 0$ be uniformly asymptotically stable on R_i so that it has the domain $D(R_i)$ of uniform asymptotic stability (Definitions 1-3). Necessity of the condition 1) is proved in the same way as in the proof of Theorem 1. Since $D(R_i)$ and S are neighborhoods of $x = 0$ then $D(R_i) \cap S \neq \phi$. Let A be an open connected neighborhood of $x = 0$, which obeys $A \subseteq D(R_i) \cap S$, and let $p(\bullet)$ be arbitrary decrescent positive definite function on $R_i \times A$ obeying the conditions 1) and 2) of Definition 4. Hence, there exist positive definite functions $\psi_i(\bullet) : R^n \mapsto R$, $i = 1, 2$, which satisfy (6.19),

$$\psi_1(x) \leq p(t, x) \leq \psi_2(x), \quad \forall (t, x) \in R_i \times A \ . \tag{6.19}$$

From the conditions 1) and 2) of Definition 4 it results that there is a solution $\nu(\bullet)$ to (5.1), which is well defined and continuous on ClB_μ and obeys (5.2). The set $L = A \cap B_\mu$ is also an open and connected neighborhood of $x = 0$ and $L \subseteq D(R_i)$. Let $\epsilon \in R^+$ be arbitrarily selected so that $B_\epsilon \subseteq L$. Hence, $B_\epsilon \subseteq D(R_i)$. Let $\rho \in R^+$ obeying $B_\rho \subseteq D_s(\epsilon) = \cap [D_s(t_0, \epsilon) : t_0 \in R_i]$ (Definitions 2 and 3), be arbitrarily selected. By following the proofs of (6.9) and (6.10) we prove that the function $\nu(\bullet)$ has the next property since $B_\rho \subseteq D_s(\epsilon) \subseteq B_\epsilon \subseteq L \subseteq A$,

$$|\nu(t, x)| < \infty, \quad \forall (t, x) \in R_i \times B_\rho \ . \tag{6.20}$$

By following the proof of (6.14) we show that there are $w_i(\mathrm{x}) \in C(B_\rho)$, $w_i(0) = 0$ and $w_i(\mathrm{x}) > 0$, $\forall(\mathrm{x} \neq 0) \in B_\rho$, $i = 1, 2$, such that

$$w_i(\mathrm{x}) \le v(t, \mathrm{x}) \le w_2(\mathrm{x}), \quad \forall(t, \mathrm{x}) \in R_i \times B_\rho . \tag{6.21}$$

(6.20), (6.21), $w_i(\mathrm{x}) \in C(B_\rho)$, and $w_i(0) = 0$ prove that the solution $v(\bullet)$ is decrescent positive definite on $R_i \times B_\rho$. Its uniqueness is proved in the same way as in the proof of the necessity part of Theorem 1. Hence, all the conditions are necessary for uniform asymptotic stability of $\mathrm{x} = 0$ on R_i.

Sufficiency. Sufficiency of the conditions of Theorem 3 for uniform asymptotic stability of $\mathrm{x} = 0$ on R_i of the system (3.1) is well known (Grujić et al. [12], Hahn [15], Lakshmikantham and Leela [16], Miller and Michel [18], Zubov [20]).

7. EXAMPLES

Example 1.

Let $\mathrm{x} = (x_1\ x_2)^T \in R^2$, $\mathrm{f}(\bullet) = [f_1(\bullet)\ f_2(\bullet)]^T$ and

$$\frac{dx_1}{dt} = [2(1+e^{-t})]^{-1}\{e^{-t}x_1 + [4 - (1+e^{-t})x_1^2]^2[-(5+e^{-t}\sin t)x_1 + x_2]\}, \tag{7.1a}$$

$$\frac{dx_2}{dt} = [2(1+e^{-t})]^{-1}\{e^{-t}x_2 - [10 - (1+e^{t})x_2^2][4x_1 + (1+t^2)(1+2t^2)^{-1}x_2]\}. \tag{7.1b}$$

The system possesses the Weak Smoothness Property on $R_i \times R^2$, where $R_i =]-1, \infty[$ and $S = R^2$. The set S_e of the equilibrium states is singleton, $S_e = \{0\}$. The function $v(\bullet)$ (7.2),

$$v(t, \mathrm{x}) = \frac{(1+e^{-t})x_1^2}{4 - (1+e^{-t})x_1^2} - \ln\left[\frac{10 - (1+e^{-t})x_2^2}{10}\right], \tag{7.2}$$

and the function $p(\bullet)$ (7.3),

$$p(t, \mathrm{x}) = 4(5 + e^{-t}\sin t)x_1^2 + (1+t^2)(1+2t^2)^{-1}x_2^2, \tag{7.3}$$

satisfy the equations (5.1). For the function $p(\bullet)$ (7.3) we find that $\psi_1(\bullet)$ and $\psi_2(\bullet)$, $\psi_1(\mathrm{x}) = (x_1^2 + 2^{-1}x_2^2)$ and $\psi_2(\mathrm{x}) = (32x_1^2 + x_2^2)$, satisfy

$$\psi_1(\mathrm{x}) \le p(t, \mathrm{x}), \quad \forall(t, \mathrm{x}) \in R \times R^2, \tag{7.4a}$$

$$p(t, \mathrm{x}) \ge \psi_2(\mathrm{x}), \quad \forall(t, \mathrm{x}) \in R \times R^2 . \tag{7.4b}$$

Similarly, for the function $v(\bullet)$ (7.2) we find that $w_1(\bullet)$ and $w_2(\bullet)$, where $w_1(\mathrm{x}) = \{[x_1^2(4 - x_1^2)^{-1}] - \ln[(10 - x_2^2)10^{-1}]\}$ and $w_2(x) = \{4x_1^2[4 - (1+e)x_1^2]^{-1} - \ln[10 - (1+e)x_2^2]\ 10^{-1}\}$, obey

$$w_1(\mathrm{x}) \le v(t, \mathrm{x}), \quad \forall(t, \mathrm{x}) \in R_i \times N(t), \tag{7.5a}$$

$$v(t, \mathrm{x}) \ge w_2(\mathrm{x}), \quad \forall(t, \mathrm{x}) \in R_i \times N, \tag{7.5b}$$

where

$$N(t) = \{\mathrm{x} : \mathrm{x} \in R^2, x_1^2 < 4(1+e^{-t})^{-1}, x_2^2 < 10(1+e^{-t})^{-1}\}, \tag{7.6}$$

and

$$N = \{\mathrm{x} : \mathrm{x} \in R^2, x_1^2 < 4(1+e)^{-1}, x_2^2 < 10(1+e)^{-1}\}. \tag{7.7}$$

The functions $v(\bullet)$ (7.2) and $p(\bullet)$ (7.3) are differentiable. We may now conclude that $p(\bullet) \in L^1(R_i, S, f)$. For such $p(\bullet)$ the function $v(\bullet)$ (7.3) and the set $N(t)$ (7.7), as well as the system (7.1) itself, fulfill all the conditions of Theorem 2. Hence, $x = 0$ of the system (7.1) is uniformly asymptotically stable with the domain $D(t)$ of asymptotic stability: $D(t) = N(t)$ (7.6) on R_i and with the domain D of uniform asymptotic stability on $R_i : D = N$ (7.7). Simulation results are shown in Fig. 1a,b. They illustrate an influence of the initial time t_0 on $D(t_0)$ and on system solutions with the same initial state at different initial instants. They illustrate also an influence of the initial state x_0 on system solutions at every initial time $t_0 \in \{0s, 0.3s\}$. The initial states $x_0 = (1.4 \quad 2.236)^T \in D(t_0)$ and $x_0 = (-1.4 \quad 2.236)^T \in D(t_0)$ for every $t_0 \in \{0s, 0.3s\}$. The initial state $x_0 = (1.42 \quad 2.236)^T$ is in $D(t_0)$ for $t_0 = 0.3s$, but not for $t_0 = 0s$.

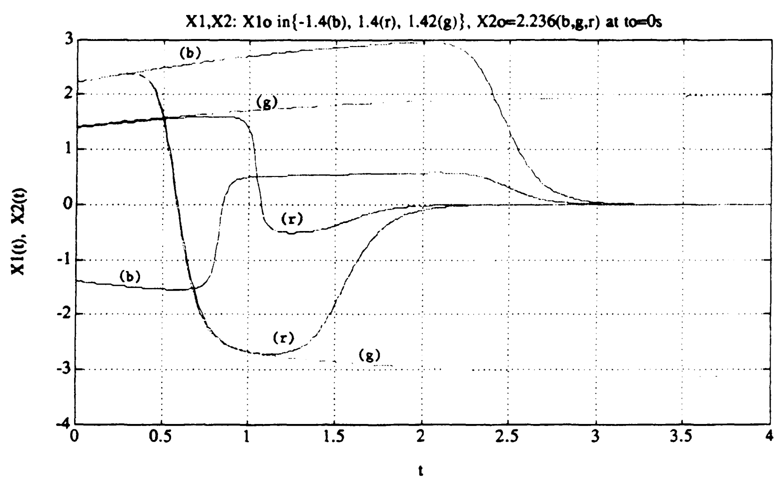
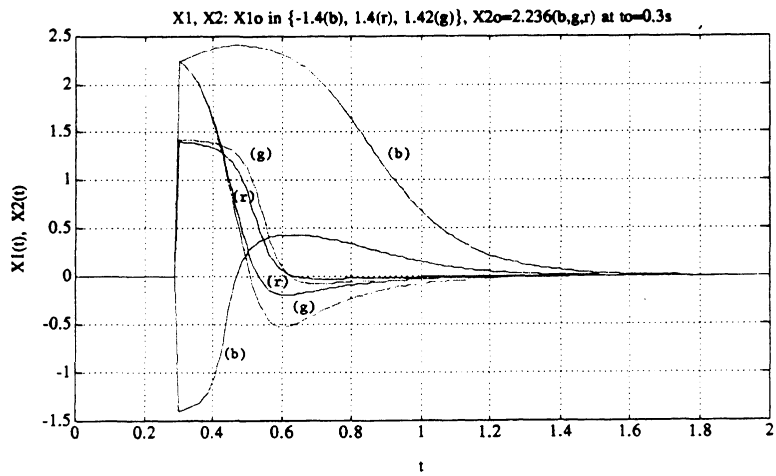


Fig. 1: a) $x_{01} = (-1.4 \quad 2.236)^T \in D(0)$ and $x_{02} = (1.4 \quad 2.236)^T \in D(0)$ but $x_{03} = (1.42 \quad 2.236)^T \notin D(0)$.



b) $x_{01} = (-1.4 \quad 2.236)^T \in D(0.3)$, $x_{02} = (1.4 \quad 2.236)^T \in D(0.3)$ and $x_{03} = (1.42 \quad 2.236)^T \in D(0.3)$.

Example 2.

Let $x = (x_1\ x_2\ x_3)^T \in R^3$, $f(\bullet) = [f_1(\bullet)\ f_2(\bullet) f_3(\bullet)]^T$ and

$$\frac{dx}{dt} = -\frac{\cos t + (2+\cos t)[99 - (2+\sin t)x^T Hx]^2}{2(2+\sin t)} x = f(t,x), \tag{7.8a}$$

$$H = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 4 & 0 \\ 2 & 0 & 6 \end{bmatrix}. \tag{7.8b}$$

The matrix H is symmetric and positive definite. The system (7.8) has the Weak Smoothness Property with $S = R^3$ and has the single equilibrium state at $x = 0$ so that $f(t,x) = 0$ for all $t \in R$ iff $x = 0$. For the function $p(\bullet)$,

$$p(t,x) = 99(2+\cos t)x^T Hx, \tag{7.9}$$

the function $v(\bullet)$,

$$v(t,x) = \frac{2(2+\sin t)x^T Hx}{99 - (2+\sin t)x^T Hx}, \tag{7.10}$$

satisfies (5.1). The following comparison functions are found for them,

$$\begin{aligned} \psi_1(x) &= 99x^T Hx, \\ \psi_2(x) &= 297x^T Hx, \\ w_1(x) &= x^T Hx(99 - x^T Hx)^{-1}, \\ w_2(x) &= 3x^T Hx(99 - 3x^T Hx)^{-1}, \end{aligned}$$

so that

$$\psi_1(x) \le p(t,x) \le \psi_2(x), \quad \forall (t,x) \in R \times R^3,$$

$$w_1(x) \le v(t,x), \quad \forall (t,x) \in R \times N(t),$$

and

$$w_2(x) \ge v(t,x), \quad \forall (t,x) \in R \times N,$$

where

$$N(t) = \left\{ x : x \in R^3, x^T Hx < \frac{99}{2+\sin t} \right\} \equiv \text{In}\, N(t); \quad 0 \in N(t), \quad \forall t \in R,$$

and

$$N = \{x : x \in R^3, x^T Hx < 33\}.$$

The system (7.8), the functions $p(\bullet)$ (7.9) and $v(\bullet)$ (7.10), and the set $N(t)$ satisfy all the conditions of Theorem 2 for $R_i = R$. The equilibrium state $x = 0$ of the system is uniformly asymptotically stable with

$$D(t) = \left\{ \mathrm{x} : \mathrm{x} \in R^3, \mathrm{x}^T \mathrm{Hx} < \frac{99}{2 + \sin t} \right\}, \quad \forall t \in R \,,$$

and

$$D = \{ \mathrm{x} : \mathrm{x} \in R^3, \mathrm{x}^T \mathrm{Hx} < 33 \} \,.$$

The simulations completely verified the above results. They are shown for two initial states, $\mathrm{x}_{01} = (6.6 \;\; 1 \;\; -2)^T$, Fig. 2, and $\mathrm{x}_{02} = (7.96 \;\; 1 \;\; -2)^T$ that was preserved unchanged at three different initial instants $t_0 \in \{0s, \;\; 3.893s, \;\; 3.895s\}$, Fig. 3a,b,c. In this example, $\mathrm{x}_{01} = (6.6 \;\; 1 \;\; -2)^T \in D(0)$, Fig. 2. However, $\mathrm{x}_{02} \notin D(0)$, but $\mathrm{x}_{02} \in D(3.893) \cap D(3.895)$, Fig. 3a,b,c.

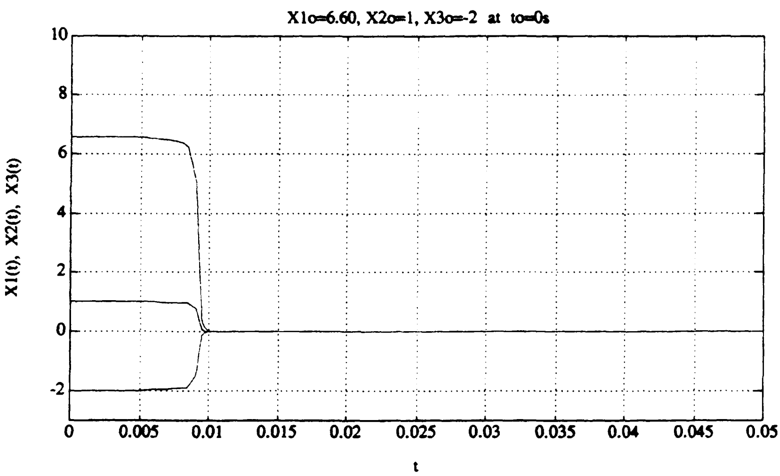


Fig. 2: $\mathrm{x}_{01} = (6.6 \;\; 1 \;\; -2)^T \in D(0)$.

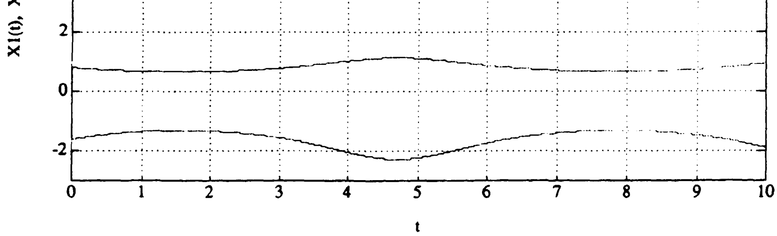


Fig. 3: a) $\mathrm{x}_{02} = (7.96 \;\; 1 \;\; -2)^T \notin D(0)$.

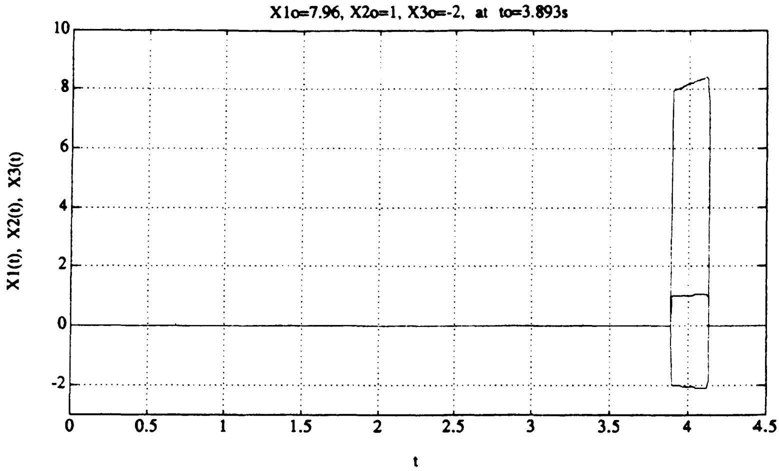


Fig. 3: b) $x_{02} = (7.96 \quad 1 \quad -2)^T \in D(3.893)$.

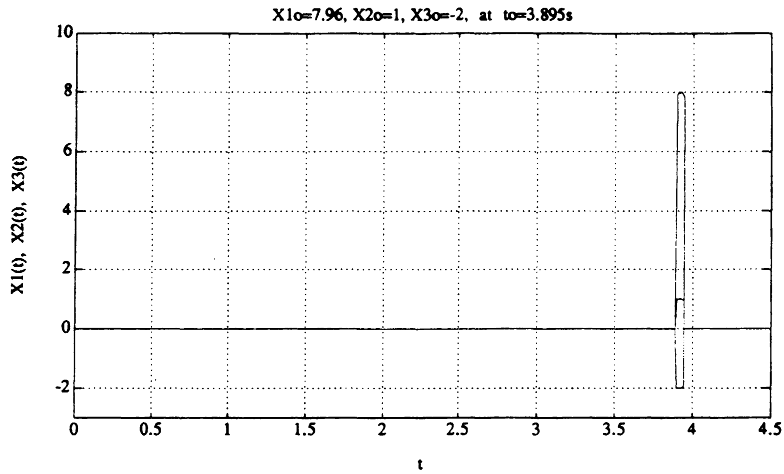


Fig. 3: c) $x_{02} = (7.96 \quad 1 \quad -2)^T \in D(3.895)$. The rate of convergence is much higher than when $t_0 = 3.893s$.

Example 3.

Let $x = (x_1 \;\; x_2)^T \in R^2$, $f(\bullet) = [f_1(\bullet) \;\; f_2(\bullet)]^T$ and

$$\frac{dx_1}{dt} = \frac{1}{(1+2t^2)[1+t^2-4(1+2t^2)x_2^2]}\{-t-[1+t^2-(1+2t^2)(x_1^2+2x_2^2)]^2\}x_1\,, \quad (7.11a)$$

$$\frac{dx_2}{dt} = \frac{1}{(1+2t^2)[1+t^2-2(1+2t^2)x_1^2]}\{-t+[1+t^2-(1+2t^2)(x_1^2+2x_2^2)]^2\}x_2\,, \quad (7.11b)$$

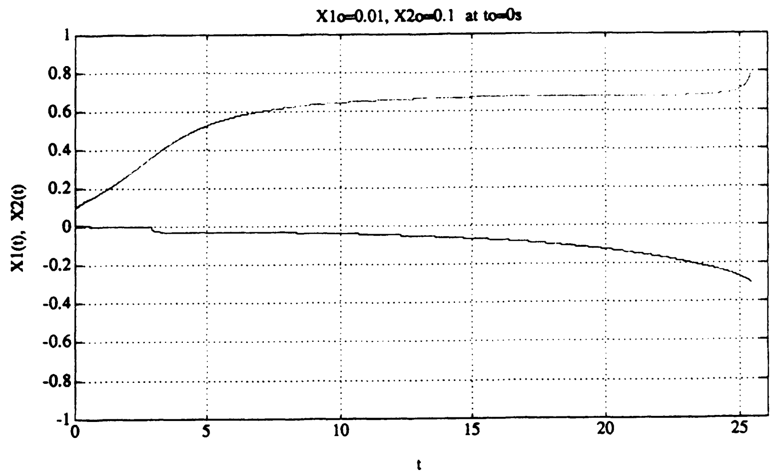
The system (7.11) has the Weak Smoothness Property with

$$S(t) = \{x : x \in R^2, x_1^2 < (1+t^2)[2(1+2t^2)]^{-1}, x_2^2 < (1+t^2)[4(1+2t^2)]^{-1}\}\ .$$

The system has the unique equilibrium state $x = 0$. The function $p(\bullet)$,

$$p(t,\mathrm{x}) \equiv p(\mathrm{x}) = 2(x_1^2 + 2x_2^2)\,, \quad \text{and the function } \nu(\bullet)\,,$$
$$\nu(t,\mathrm{x}) = \frac{(1+2t^2)(x_1^2 - 2x_2^2)}{1+t^2-(1+2t^2)(x_1^2+2x_2^2)}\,,$$

show that $p(\bullet) \in L^1(R, S, \mathfrak{f})$ so that we may apply Theorem 3. Since the solution function $\nu(\bullet)$ to (5.1) is not positive definite then Theorem 3 is not satisfied. Hence, $\mathrm{x} = 0$ is not uniformly asymptotically stable. Simulation results shown in Fig. 4a,b will illustrate that $\mathrm{x} = 0$ is not uniformly asymptotically stable. This example illustrates importance and usefulness of the necessity of the conditions of Theorem 3, as well as of Theorems 1 and 2. Since it is not satisfied there is not any sense to try constructing a more suitable function $\nu(\bullet)$ in order to prove uniform asymptotic stability of $\mathrm{x} = 0$ because it is impossible. This example illustrates the usefulness of the single step construction of the function $\nu(\bullet)$ by using any of the above theorems.



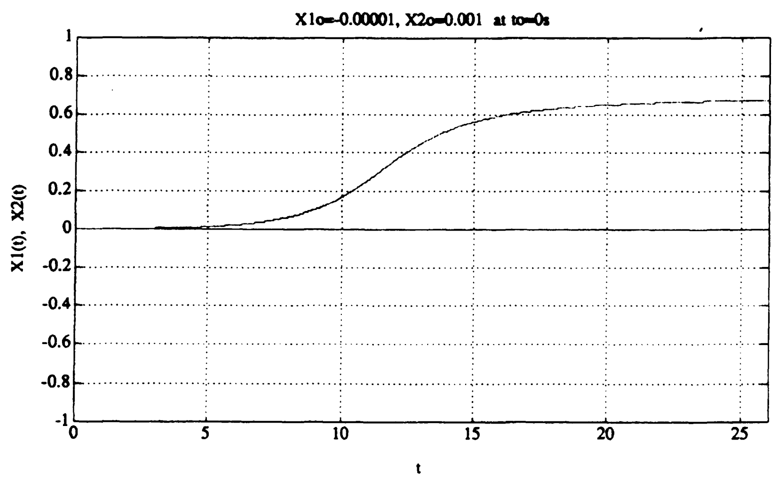


Fig. 4: Motions for two different small initial conditions under a) and b) express instability of $\mathrm{x} = 0$ at $t_0 = 0s$.

8. CONCLUSION

The new methodology for a single simultaneous exact construction of a system Lyapunov function and accurate determination of the domain of asymptotic stability of $x = 0$ established in [4]-[11] for time invariant systems is essentially broadened to time-varying systems The conditions are presented as both necessary and sufficient, and in terms of **arbitrary choice of** a differentiable decrescent positive definite function $p(\bullet) \in L^1(\bullet)$, {or, $p(\bullet) \in E^1(\bullet)$}, and in terms of properties of a solution function $v(\bullet)$ to $v'(\bullet) = -p(\bullet)$ (5.1), {or in terms of properties of a solution function $\upsilon(\bullet)$ to $\upsilon'(\bullet) = -[1-\upsilon(\bullet)]p(\bullet)$ (5.3)}, respectively, obtained **for arbitrarily selected function** $p(\bullet)$. The families $L^1(\bullet)$ and $E^1(\bullet)$ are determined by Definitions 4 and 5 If so obtained $v(\bullet)$, $\{\upsilon(\bullet)\}$, is also differentiable decrescent positive definite then (Theorem 3) $x = 0$ *IS uniformly asymptotically stable* If $v(\bullet)$, $\{\upsilon(\bullet)\}$, does not have any of these properties then $x = 0$ *IS NOT uniformly asymptotically stable* The solution to the problem of uniform asymptotic stability is obtained under a single application of Theorem 3 The same conclusion is valid for the determination of the domain of (uniform) asymptotic stability of $x = 0$ via Theorems 1 and 2 Numerous simulations were carried out They completely verified the theoretical results. Few of them are presented in the paper

APPENDIX

LEMMA A.1. Let the system (3.1) possess the Weak Smoothness Property and let $x = 0$ be uniformly attractive on R_t with the instantaneous domain $D_a(t)$ of attraction obeying $D_a(t) \subseteq S(t)$ for all $t \in R_t$ and with the domain $D_a(R_t)$ of uniform attraction on R_t.

a) If $R_t \subset R$ then

1) $(t_0, x_0) \in R_t \times D_a(t_0)$ implies $\mathbf{x}(t; t_0, x_0) \in D_a(t)$ for all $t \in R_t$, that is that $D_a(t)$ is invariant on R_t,

2) $D_a(t)$ is an open neighborhood of $x = 0$ at any $t \in R_t : D_a(t) \equiv \mathrm{In}D_a(t)$,

and

3) $D_a(R_t)$ is a connected neighborhood of $x = 0$ such that $(t_0, x_0) \in R_t \times D_a(R_t)$ implies $\mathbf{x}(t; t_0, x_0) \in D_a(R_t)$ for every $t \in R_i$, that is that $D_a(R_t)$ is invariant on R_t in case $D_a(t) = D_a(R_t)$ for all $t \in R_t$ Otherwise, $D_a(R_t)$ is connected neighborhood of $x = 0$

b) If $R_t = R$ then

1) $D_a(t)$ is invariant, that is that $(t_0, x_0) \in R \times D_a(t_0)$ implies $x(t; t_0, x_0) \in D_a(t)$ for all $t \in R$,

2) $D_a(t)$ is an open neighborhood of $x = 0$ at any $t \in R : D_a(t) \equiv \mathrm{In}D_a(t)$,

and

3) D_a is an invariant connected neighborhood of $x = 0 : (t_0, x_0) \in R \times D_a(R)$ implies $\mathbf{x}(t; t_0, x_0) \in D_a(R)$ for all $t \in R$ in case $D_a(t) = D_a(R)$ for all $t \in R$ Otherwise, $D_a = D_a(R)$ is connected neighborhood of $x = 0$.

PROOF. Let the system (3.1) possess the Weak Smoothness Property and let $x = 0$ be uniformly attractive on R_t with the instantaneous domain $D_a(t)$ of attraction obeying $D_a(t) \subseteq S(t)$ for all $t \in R_t$ and with the domain $D_a(R_t)$ of uniform attraction on R_t. Hence, $D_a(R_t) = \cap[D_a(t_0) : t_0 \in R_t]$ (Definition 1).

a) Let t_0 and $\tau_0 \in R_t$, $t_0 \neq \tau_0$. Let $x^* = \mathbf{x}(\tau_0; t_0, x_0)$ for any $x_0 \in D_a(t_0)$ Then, $\mathbf{x}(t; t_0, x_0) \to 0$ as $t \to \infty$ Since $\mathbf{x}(t; \tau_0, x^*) \equiv \mathbf{x}[t; \tau_0, \mathbf{x}(\tau_0; t_0, x_0)] = \mathbf{x}(t; t_0, x_0)$ due to (i) of the Weak Smoothness Property and $D_a(t_0) \subseteq S(t_0)$ then $\mathbf{x}(t; \tau_0, x^*) \to 0$ as $t \to \infty$. Hence, $x^* = \mathbf{x}(\tau_0; t_0, x_0) \in D_a(\tau_o)$ that proves the statement under a-1). Let $\zeta \in R^+$ be such that $B_{2\zeta} \subset D_a(R_t)$. Let there exist $t_0' \in R_t$ and $x_0' \in \partial D_a(t_0') \cap D_a(t_0')$. Let $\epsilon \in]0, \zeta/2[$. Then, (i) of the Weak Smoothness Property, $D_a(t_0') \subseteq S(t_0')$ and (a) of Definition 1 imply existence of $\rho \in R^+$, $\rho = \rho(t_0', x_0, \epsilon)$, such that $||x_0 - x'_0|| < \rho$ ensures $||\mathbf{x}(t'_0 + 2\sigma'; t'_0, x_0) - \mathbf{x}(t'_0 + 2\sigma'; t'_0, x'_0)|| < \epsilon$, where $\sigma' = \tau(t'_0, x'_0, \zeta)$ (Definition 1). Since $\epsilon < \zeta/2$ and $||\mathbf{x}(t'_0 + 2\sigma'; t'_0, x'_0)|| < \zeta$ then $\mathbf{x}(t'_0 + 2\sigma'; t'_0, x_0) \in B_{2\zeta} \subset D_a(R_i)$. Hence, $x_0 \in D_a(t'_0)$. However, x_0 may be chosen in a ρ-neighborhood of x'_0 out of $D_a(t'_0)$ that is contradicted by the obtained $x_0 \in D_a(t'_0)$. Since the former is true then the latter is wrong so that there are not $t'_0 \in R_i$ and $x'_0 \in \partial D_a(t'_0) \cap D_a(t'_0)$. Hence, if $x'_0 \in \partial D_a(t'_0)$ then $x'_0 \notin D_a(t'_0)$. The set $D_a(t_0)$ is open for all $t_0 \in R_i$ and it is neighborhood of $x = 0$ due to Definition 1. Hence, the statement under a-2) is correct. Furthermore, $D_a(R_i)$ is a neighborhood of $x = 0$ by definition (Definition 1). Its connectedness is proved as follows. Let us assume that it is not connected. Then, there are disjoint sets D_{ai}, $i = 1, 2, ..., N$, such that $D_a(R_i) = \cup [D_{ai} : i = 1, 2, ..., N]$. One of D_{ai} is not a neighborhood of $x = 0$. Let it be D_{a1}. Then $x_0 \in D_{a1}$ implies $\mathbf{x}(t; t_0, x_0) \rightarrow 0$ as $t \rightarrow \infty$, $\forall t_0 \in R_i$. Hence, there is $t_1 \in R_0$ such that $\mathbf{x}(t_1; t_0, x_0) \notin D_a(R_i)$ because of continuity of $\mathbf{x}(t; t_0, x_0)$ in $t \in R_0$, $\forall\, t_0 \in R_i$, and because D_{a1} is disjoint subset of $D_a(R_i)$ that is not a neighborhood of $x = 0$, which is impossible due to $\mathbf{x}[t; t_1, \mathbf{x}(t_1; t_0, x_0)] = \mathbf{x}(t; t_0, x_0) \rightarrow 0$ as $t \rightarrow \infty$. Hence, the assumption on disconnectedness of $D_a(R_i)$ is incorrect. In order to prove that $(t_o, x_o) \in R_i \times D_a(R_i)$ implies $\mathbf{x}(t; t_0, x_0) \in D_a(R_i)$ for all $t \in R_i$ let $(t_o, x_o) \in R_i \times D_a(R_i)$ be arbitrarily selected. The condition b-4) of Definition 1 guarantees $\sup[\tau_m(t_o, x_o, \zeta) : t_0 \in R_i] = \alpha < \infty$, $\forall \zeta \in \mathrm{R}^+$. Let $t_2 \in R_i$ be arbitrary. Evidently, $\beta = t_o - t_2$ obeys $|\beta| = |t_o - t_2| < \infty$. Let $x_2 = \mathbf{x}(t_2; t_0, x_0)$ so that $x_2 \in D_a(t_2)$ and $\sup[\tau_m(t_2, x_2, \zeta) : t_0 \in R_i] = \sup[\tau_m(t_0, x_0, \zeta) + \beta : t_0 \in R_i] = \alpha + \beta < \infty$ that proves $x_2 = \mathbf{x}(t_2; t_0, x_0) \in D_a(R_i)$ in view of the condition b) of Definition 1 and $\mathbf{x}(t; t_2, x_2) = \mathbf{x}(t; t_0, x_0)$. This completes the proof of all the statements under a).

b) The assertions under b) directly follow from those under a) in case $R_i = R$.

LEMMA A.2.

(a) If the state $x = 0$ of the system (3.1) possessing the Weak Smoothness Property is asymptotically stable and its domain $D_a(t_0)$ of attraction at $t_0 \in R_i$ obeys $D_a(t_0) \subseteq S(t_0)$ for all $t_0 \in R_i$ then its domains $D_a(t_0)$, $D_s(t_0)$ and $D(t_0)$ are interrelated by $D_a(t_0) \subseteq D_s(t_0)$ and $D(t_0) = D_a(t_0)$, for all $t_0 \in R_i$.

(b) If the state $x = 0$ of the system (1) possessing the Weak Smoothness Property is uniformly asymptotically stable and its domain $D_a(R_i)$ of uniform asymptotic stability on R_i satisfies $D_a(R_i) \subseteq S(R_i)$ then its domains $D_a(R_i)$, $D_s(R_i)$ and $D(R_i)$ are interrelated by $D_a(R_i) \subseteq D_s(R_i)$ and $D(R_i) = D_a(R_i)$.

PROOF. Let the system (3.1) possess the Weak Smoothness Property. Let $x = 0$ be asymptotically stable and $D_a(t_0) \subseteq S(t_0)$ for all $t_0 \in R_i$. Let $(t_0, x_0) \in R_i \times D_a(t_0)$ be arbitrary. Continuity of $\mathbf{x}(t; t_0, x_0)$ in $(t_0, x_0) \in R_i \times S(t_0)$, $D_a(t_0) \subseteq S(t_0)$ and $x_0 \in D_a(t_0)$ imply $\max[||\mathbf{x}(t; t_0, x_0)|| : t \in R_0] < \infty$. Let $\epsilon = 2 \max[||\mathbf{x}(t; t_0, x_0)|| : t \in R_0]$. Hence, $x_0 \in D_s(t_0, \epsilon)$ due to (a-1) of Definition 2, which implies $x_0 \in D_s(t_0)$ in view of (a-3) of Definition 2. Altogether, $x_0 \in D_a(t_0)$ yields $x_0 \in D_s(t_0)$ that proves $D_a(t_0) \subseteq D_s(t_0)$ for all $t_0 \in R_i$. This result and (a) of Definition 3 complete the proof of the statement under (a).

Let $x = 0$ be uniformly asymptotically stable on R_i and $D_a(R_i) \subseteq S(R_i)$. Let $x_0 \in D_a(R_i)$ be arbitrary. Hence, $\max[||\mathbf{x}(t; t_0, x_0)|| : (t, t_0) \in R_0 \times R_i] < \infty$ due to continuity of $\mathbf{x}(t; t_0, x_0)$ in $(t, t_0) \in R_0 \times R_i$ and $x_0 \in D_a(R_i)$. Let $\epsilon = 2 \max[||\mathbf{x}(t; t_0, x_0)|| : (t, t_0) \in R_0 \times R_i] \in R^+$ so that obviously $x_0 \in D_s(\epsilon, R_i) = \cap [D_s(t_0, \epsilon) : t_0 \in R_i]$ and therefore $x_0 \in D_s(R_i)$ (Definition 3). The result that $x_0 \in D_a(R_i)$ implies $x_0 \in D_s(R_i)$ proves $D_a(R_i) \subseteq D_s(R_i)$ and $D(R_i) = D_a(R_i)$ (due to Definition 3). This completes the proof.

REFERENCES

[1] GRUJIĆ, L., Novel development of Lyapunov stability of motion, *Int. Journal of Control*, 22, 4 (1975), 525-549.

[2] GRUJIĆ, L. T., On absolute stability and the Aizerman conjecture, *Automatica*, 17, 2 (1981), 335-349.

[3] GRUJIĆ, L. T., Concepts of stability domains (in Serbo-Croatian), *Automatika, Zagreb*, 26, 1-2 (1985), 5-10.

[4] GRUJIĆ, L. T., Solutions to Lyapunov stability problems: Nonlinear systems with globally differentiable motions, in *The Lyapunov Functions Method and Applications*, Eds. P. Borne and V. Matrosov, J. C. Baltzer, AG, Scientific Publishing Co., *IMACS* (1990), 19-27.

[5] GRUJIĆ, L. T., Solutions to Lyapunov stability problems: Nonlinear systems with differentiable motions, in *Proc. 13th IMACS World Congress on Computation and Applied Mathematics*, 3 (1991), 1228-1231.

[6] GRUJIĆ, L. T., The necessary and sufficient conditions for the exact construction of a Lyapunov function and the asymptotic stability domain, *Proc. 30th IEEE Conference on Decision and Control*, 3 (1991), 2885-2888.

[7] GRUJIĆ, L. T., On solutions of Lyapunov stability problems, *Facta Universitatis, Ser.: Mechanics, Automatic Control and Robotics*, University of Niš, Niš, Serbia, 1, 2 (1992), 121-138.

[8] GRUJIĆ, L. T., Exact determination of a Lyapunov function and the asymptotic stability domain, *Int. J. Systems Science*, 23, 11 (1992), 1874-1888.

[9] GRUJIĆ, L. T., Exact both construction of Lyapunov function and asymptotic stability domain determination, *Proc. 1993 IEEE International Conference on Systems, Man and Cybernetics*, 1 (1993), 331-336.

[10] GRUJIĆ, L. T., Solutions to Lyapunov stability problems: nonlinear systems with continuous motions, *Int. J. of Mathematics and Mathematical Sciences*, 17, 3 (1994), 587-596.

[11] GRUJIĆ, L. T., Solutions to Lyapunov stability problems of sets: nonlinear systems with differentiable motions, *Int. J. of Mathematics and Mathematical Sciences*, 17, 1 (1994), 103-112.

[12] GRUJIĆ, L. T., MARTYNYUK, A. A. and RIBBENS-PAVELLA, M., *Large-Scale Systems Stability under Structural and Singular Perturbations* (in Russian), Naukova dumka, Kiev, 1984 (English translation: Springer Verlag, New York, 1987).

[13] GRUJIĆ, L. T. and MICHEL, A. N., Exponential stability and trajectory bounds of neural networks under structural variations, *IEEE Transactions on Circuits and Systems*, 38, 10 (1991), 1182-1192.

[14] GRUJIĆ, L. T. and RIBBENS-PAVELLA, M., Asymptotic stability of large-scale systems with application to power systems: Part 1; domain estimation, *Eleo. Power and Energy Systems*, 1, 3 (1979), 151-157.

[15] HAHN, W., *Stability of Motion*, Springer-Verlag, Berlin, 1967.

[16] LAKSHMIKANTHAM, V. and LEELA, S., *Differential and Integral Inequalities*, Academic Press, New York, 1969.

[17] LaSALLE, J. P. and LEFSCHETZ, S., *Stability by Liapunov's Direct Method*, Academic Press, New York, 1961.

[18] MILLER, R. K. and MICHEL, A. N., *Ordinary Differential Equations*, Academic Press, New York, 1982.

[19] VANELLI, A. and VIDYASAGAR, M., Maximal Lyapunov functions and domains of attraction for autonomous nonlinear systems, *Automatica*, 21, 1 (1985), 69-80.

[20] ZUBOV, V. I., *Methods of A. M. Liapunov and Their Applications*, P. Noordhoff Ltd., Groningen, 1964.