Internat. J. Math. & Math. Sci.
VOL. 20 NO. 3 (1997) 573-576

INEQUALITIES VIA LAGRANGE MULTIPLIERS

W. T. SULAIMAN

Zarka Private University
P O. Box 150863
Zarka 13115, JORDAN

(Received August 8, 1993 and in revised form June 6, 1996)

ABSTRACT. An easy method is obtained to prove many inequalities using Lagrange mutipliers.

KEY WORDS AND PHRASES: Inequalities.
1991 AMS SUBJECT CLASSIFICATION CODES: 90C50, 980C39.

1. INTRODUCTION

Let us assume that $d_1, ..., d_n$ are unit perpendicular vectors in an n-dimensional space X. In particular d_1, d_2, and d_3 are the unit perpendicular vectors i, j, and k in the 3-dimensional space. Any vector v in X is usually uniquely written in the form

$$v = \sum_{i=1}^{n} \lambda_i d_i$$

for scalars λ_i. We define

$$\nabla f(x_1, ..., x_n) = \sum_{i=1}^{n} f_{x_i}(x_1, ..., x_n) d_i, \quad f_x = \frac{\partial}{\partial x}.$$

Kapur and Kumar (1986), have used the principle of dynamic programming to prove major inequalities due to Shannon, Renyi, and Holder, see [1]. In this note we give a new method using Lagrange multipliers.

2. SHANNON'S INEQUALITY

THEOREM 2.1. Given $\sum_{i=1}^{n} p_i = a$, $\sum_{i=1}^{n} q_i = b$, then

$$a \ln(a/b) \le \sum_{i=1}^{n} p_i \ln(p_i/q_i), \quad p_i, q_i \ge 0.$$

The equality holds iff $p_i = q_i$ for each i.

PROOF. Let the q_i's and a be fixed; set

$$f(p_1, ..., p_n) = \sum_{i=1}^{n} p_i \ln(p_i/q_i); \quad p_i, q_i \ge 0,$$

we aim to minimize f subject to the constraint

$$g(p_1, ..., p_n) = \sum_{i=1}^{n} p_i - a = 0.$$

There is a minimum achieved where $\nabla f = \lambda \nabla g$ because g is linear and f is convex, since its second order partials are all non-negative

$$\begin{aligned}\nabla f = \lambda \nabla g &\Rightarrow \sum_{i=1}^{n} \{1 + \ln(p_i/q_i)\} d_i = \lambda \sum_{i=1}^{n} d_i \\ &\Rightarrow 1 + \ln(p_i/q_i) = \lambda \\ &\Rightarrow \frac{p_1}{q_1} = \frac{p_2}{q_2} = \cdots = \frac{p_n}{q_n} = \frac{\Sigma a_i}{\Sigma b_i} = \frac{a}{b}.\end{aligned}$$

Therefore

$$\min \sum_{i=1}^{n} p_i \ln(p_i/q_i) = \ln(a/b) \sum_{i=1}^{n} p_i = a \ln(a/b),$$

or

$$a \ln(a/b) \le \sum_{i=1}^{n} p_i \ln(p_i/q_i).$$

If $a = b = 1$, we get Shannon's inequality

$$\sum_{i=1}^{n} p_i \ln(p_i/q_i) \ge 0 \quad \text{and} \quad \sum_{i=1}^{n} p_i \ln(p_i/q_i) = 0 \quad \text{iff} \quad p_i = q_i \quad \text{for each} \quad i.$$

3. RENYI'S INEQUALITY

THEOREM 3.1. Given $\sum_{i=1}^{n} a_i = a$, $\sum_{i=1}^{n} b_i = b$, then

$$\frac{1}{\alpha - 1}(a^\alpha b^{1-\alpha} - a) \le \sum_{i=1}^{n} \frac{1}{\alpha - 1}(p_i^\alpha q_i^{1-\alpha} - p_i), \quad p_i, q_i \ge 0, 0 < \alpha \ne 1.$$

The equality holds iff $p_i = q_i$ for each i.

PROOF. Let the q_i's and a be fixed and write

$$f(p_1, \ldots, p_n) = \sum_{i=1}^{n} \frac{1}{\alpha - 1} p_i^\alpha q_i^{1-\alpha}, \quad g(p_1, \ldots, p_n) = \sum_{i=1}^{n} p_i - a = 0$$

$$\begin{aligned}\nabla f = \lambda \nabla g &\Rightarrow \sum_{i=1}^{n} \frac{\alpha}{\alpha - 1} p_i^{\alpha-1} q_i^{1-\alpha} d_i = \lambda \sum_{i=1}^{n} d_i \\ &\Rightarrow (p_i/q_i)^{\alpha-1} = \lambda \left(\frac{\alpha - 1}{\alpha}\right) \\ &\Rightarrow \frac{p_1}{q_1} = \cdots = \frac{p_n}{q_n} = \frac{a}{b} \\ &\Rightarrow \min f(p_1, \ldots, p_n) = \frac{1}{\alpha - 1} a^\alpha b^{1-\alpha},\end{aligned}$$

by the convexity of f and linearity of g. Hence

$$\frac{1}{\alpha - 1} a^\alpha b^{1-\alpha} \le \sum_{i=1}^{n} \frac{1}{\alpha - 1} p_i^\alpha q_i^{1-\alpha}.$$

If $a = b = 1$, we get Renyi's inequality

$$\frac{1}{\alpha - 1}\left(\sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha} - 1\right) \ge 0.$$

4. HOLDER'S INEQUALITY

THEOREM 4.1. Given $\sum_{i=1}^{n} a_i^p = A$, $\sum_{i=1}^{n} b_i^q = B$, $\sum_{i=1}^{n} a_i b_i = C$, $a_i, b_i \ge 0$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$C \leq A^{1/p} B^{1/q}. \tag{4 1}$$

PROOF. This follows from Renyi's inequality, taking $\alpha = 1/p$, $a_i = p_i^p$, $b_i = q_i^q$, or, we prove the result directly as follows:

let the a_i's and C be fixed and write

$$f(b_1, ..., b_n) = A^{q/p} \sum_{i=1}^{n} b_i^q, g(b_1, ..., b_n) = \sum_{i=1}^{n} a_i b_i - C = 0$$

$$\nabla f = \lambda \nabla g \Rightarrow q A^{q/p} \sum_{i=1}^{n} b_i^{q-1} d_i = \lambda \sum_{i=1}^{n} a_i d_i$$

$$\Rightarrow A^{q/p} b_i^{q-1} = (\lambda/q) a_i \tag{4.2}$$

$$(4.2) \Rightarrow A^{q/p} = (\lambda/q) C, \tag{4.3}$$

and

$$A^q B = (\lambda/q) A, \quad \text{as} \quad p(q-1) = q \tag{4.4}$$

$$(4.3) \ \& \ (4.4) \Rightarrow \lambda/q = C^{q-1}.$$

Therefore, by the convexity of f and linearity of g,

$$\min\left(A^{q/p} B\right) = C^q,$$

or

$$C \leq A^{1/p} B^{1/q}.$$

5. GENERALIZATIONS OF HOLDER'S INEQUALITY

THEOREM 5.1. Given $\sum_{i=1}^{n} a_i^p = A$, $\sum_{i=1}^{n} b_i^q = B$, $\sum_{i=1}^{n} c_i^r = C$, and $\sum_{i=1}^{n} a_i b_i c_i = D$, $a_i, b_i, c_i \geq 0$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, then

$$D \leq a^{1/p} B^{1/q} C^{1/r}.$$

PROOF. This follows by an easy application of Holder's inequality:

$$\begin{aligned}
\sum_{i=1}^{n} a_i b_i c_i &\leq \left[\sum_{i=1}^{n} (a_i b_i)^{\frac{r}{r-1}}\right]^{1-\frac{1}{r}} C^{\frac{1}{r}} \\
&= \left[\sum_{i=1}^{n} (a_i b_i)^{\frac{pq}{p+q}}\right]^{\frac{1}{p}+\frac{1}{q}} C^{\frac{1}{r}} \\
&\leq \left[\sum_{i=1}^{n} \left(a_i^{\frac{pq}{p+q}}\right)^{\frac{p+q}{q}}\right]^{\frac{p}{p+q}\left(\frac{p+q}{pq}\right)} \left[\sum_{i=1}^{n} \left(b_i^{\frac{pq}{p+q}}\right)^{\frac{p+q}{p}}\right]^{\frac{p}{p+q}\left(\frac{p+q}{pq}\right)} C^{\frac{1}{r}} \\
&= A^{\frac{1}{p}} B^{\frac{1}{q}} C^{\frac{1}{r}}.
\end{aligned}$$

6. MINKOWSKI'S INEQUALITY

THEOREM 6.1. Given $\sum_{i=1}^{n} a_i^p = A$, $\sum_{i=1}^{n} b_i^p = B$, and $\sum_{i=1}^{n} (a_i + a_i)^p = C$, $a_i, b_i \geq 0$, $p \geq 1$, then

$$C^{\frac{1}{p}} \leq A^{\frac{1}{p}} + B^{\frac{1}{p}}.$$

PROOF. Let the b_i's and A be fixed and write

$$f(a_1, ..., a_n) = \sum_{i=1}^{n} (a_i + b_i)^p, \quad g(a_1, ..., a_n) = \sum_{i=1}^{n} a_i^p - A = 0$$

$$\begin{aligned}\nabla f = \mu \nabla g \Rightarrow \sum_{i=1}^{n} p(a_i + b_i)^{p-1} d_i &= \mu \sum_{i=1}^{n} p a_i^{p-1} d_i \\ \Rightarrow (a_i + b_i)^{p-1} &= \mu a_i^{p-1} \\ \Rightarrow \frac{b_1}{a_1} = \cdots = \frac{b_n}{a_n} &= C.\end{aligned}$$

Therefore,

$$\begin{aligned}\max C^{\frac{1}{p}} &= \left[\sum_{i=1}^{n} (a_i + c a_i)^p\right]^{\frac{1}{p}} \\ &= (1 + c) A^{\frac{1}{p}} \\ &= A^{\frac{1}{p}} + c A^{\frac{1}{p}} \\ &= A^{\frac{1}{p}} + B^{\frac{1}{p}},\end{aligned}$$

or

$$C^{\frac{1}{p}} \leq A^{\frac{1}{p}} + B^{\frac{1}{p}}.$$

7. ARITHMETIC-GEOMETRIC-MEAN INEQUALITY

THEOREM 7.1.

$$\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{i=1}^{n} x_i.$$

PROOF. Write

$$f(x_1, ..., x_n) = x_1 x_2 ... x_n = y, \quad g(x_1, ..., x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i - C = 0.$$

Let C be fixed, we have

$$\begin{aligned}\nabla f = \mu \nabla g \Rightarrow \sum_{i=1}^{n} \frac{y}{x_i} d_i &= \frac{\mu}{n} \sum_{i=1}^{n} d_i \\ \Rightarrow x_i &= \frac{n}{\mu} y \\ \Rightarrow C &= \frac{n}{\mu} y.\end{aligned}$$

Therefore

$$\max y^{\frac{1}{n}} = \frac{n}{\mu} y = C,$$

or

$$\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{i=1}^{n} x_i.$$

ACKNOWLEDGMENT. The author is so grateful to the referee for his kind remarks, suggestions, and improvements of this paper.

REFERENCES

[1] KAPUR, J.N., KUMAR, V. and KUMAR, U., A measure of mutual divergence among a number of probability distributions, *Internat. J. Math. & Math. Sci.*, **10** (1987), 597-608.