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RESEARCH NOTES

ON THE DISTANCE BETWEEN TWO CHEBYSHEV SETS IN BANACH SPACES

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ABSTRACT. The paper answers a question concerning the distance between two Chebyshev sets in some Banach spaces

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1. INTRODUCTION

Let E be a real Banach space with norm $\|.\|$ and A be a subset of E. For every $z \in E$, the distance $d(z, E)$ is defined as

$$\text{dist}(z, A) = \inf\{\rho(z, y) : y \in A\}.$$

If B is another set of E, then the distance $d(A, B)$ between A and B is defined as

$$\text{dist}(A, B) = \inf\{\|a - b\| : a \in A, b \in B\}.$$

A subset A of E is said to be a Chebyshev set if for every $z \in E$ there exists a unique $a \in A$ such that $\|z - a\| = \text{dist}(z, A)$ So, we can define the metric projection P_A of E onto A which assigns for each $z \in E$, a point $a \in A$ such that $\|z - a\| = \text{dist}(z, A)$

It is well known [3] that if E is reflexive and strictly convex, then every closed convex subset of E is a Chebyshev set. Therefore, for every closed convex subset A of a reflexive and strictly convex Banach space E we can define the metric projection $P_A : E \to A$.

In this paper we investigate the following question:

Let A and B be convex Chebyshev sets in a Banach space E. Assume that $a \in A$ is the fixed point of the mapping $P_A P_B$. Does this imply that $\|a - P_B a\| = \text{dist}(A, B)$?

2. THE DISTANCE BETWEEN TWO CHEBYSHEV SETS

In this section we attempt to answer the question raised in Section 1. We first recall that the existence of the fixed point of the mapping $P_A P_B$ is proved in [4].

In order to answer this question we shall need a few auxiliary facts. First, let us recall (from [1]) that the duality map from a real Banach space E into the family 2^{E^*} of subsets of the dual space E^* is defined by

$$F(z) = \left\{z^* \in E^* : z^*(z) = \|z\|^2 = \|z^{*2}\|\right\}.$$

By Hahn-Banach theorem, the set $F(z)$ is nonempty for every $z \in E$.

The following result will be useful in future considerations, for a proof see [2]

LEMMA 2.1. Let A be a convex Chebyshev set in E and consider $z \in E$ and $a \in A$. Then the following conditions are equivalent:

1. $a = P_A z$.

2. For any $x \in A$, there exists $z^* \in F(z - a)$ such that $z^*(x - a) \leq 0$

By means of the duality map F we can define the so-called semi-inner products

$$(z, x)_- = \inf\{x^*(z) : x^* \in F(x)\},$$
$$(z, x)_+ = \sup\{x^*(z) : x^* \in F(x)\}.$$

The properties of these semi-inner products are given in [1]. For our further purposes we will need the following lemma, see [1]

LEMMA 2.2. Let E be a real Banach space such that E^* is strictly convex Then the duality map F is single-valued and the relation

$$(z, x)_- = (z, x)_+ \quad \text{for all} \quad x, z \in E$$

holds Moreover, we have

1 $(x + z, a) \leq (x, a) + (z, a)$,

2. $|(x, z)| \leq \|x\| \, \|z\|$,

3. $(x + z, z) = (x, z) + \|z\|^2$,

4. $(-x, -z) = (x, z)$.

Now we give our main result that partly answers the above question.

THEOREM 2.1. Let E be a real Banach space with strictly convex dual E^*. Assume that A and B are two Chebyshev sets in E and $a \in A$ is a fixed point of the map $P_A P_B$. Then

$$|a - P_B a| = \text{dist}(A, B).$$

PROOF. For convenience, let $b = P_B a$. Now suppose the converse. This means that there exist points $x \in A$ and $y \in B$ such that

$$\|x - y\| < \|a - b\|. \tag{2.1}$$

Now, since $a = P_A b$ and $b = P_B a$ and applying Lemma 1 and Lemma 2 we obtain

$$(x - a, b - a) \leq 0,$$
$$(y - b, a - b) \leq 0.$$

Hence by Lemma 2 we obtain

$$\begin{aligned} 0 &\geq (x - a, b - a) + (y - b, a - b) \\ &\geq (x - a, b - a) + (b - y, b - a) \\ &\geq (x - a + b - y, b - a) \\ &\geq (x - y + b - a, b - a) \\ &\geq (x - y, b - a) + \|b - a\|^2. \end{aligned}$$

Consequently

$$\|b - a\|^2 \leq -(x - y, b - a) \leq \|x - y\| \, \|b - a\|,$$

which implies that

$$\|b - a\| \leq \|x - y\|.$$

The last inequality contradicts inequality (2.1) and completes the proof.

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