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A FORMULA TO CALCULATE THE SPECTRAL RADIUS OF A COMPACT LINEAR OPERATOR

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ABSTRACT. There is a formula (Gelfand's formula) to find the spectral radius of a linear operator defined on a Banach space. That formula does not apply even in normed spaces which are not complete. In this paper we show a formula to find the spectral radius of any linear and compact operator T defined on a complete topological vector space, locally convex. We also show an easy way to find a non-trivial T-invariant closed subspace in terms of Minkowski functional.

KEY WORDS AND PHRASES: Compact linear operator, spectral radius, locally convex topological linear space, invariant subspace, nets, ultimately bounded nets.
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1. INTRODUCTION

In all that follows E stands for a linear space of infinite dimension over the field $\mathbf{C}$ of the complex numbers. $E[t]$ will denote a complete topological vector space, locally convex, with topology t and $T : E \rightarrow E$ will be a linear map. Finally, $v(t)$ will be the filter of all balanced, convex and closed t-neighborhoods of zero (in E).

DEFINITION 1. The linear operator $T : E[t] \rightarrow E[t]$ is said to be a bounded (compact) operator, if there is a neighborhood $U \in v(t)$ such that $T(U)$ is a bounded (relatively compact) set

REMARK 2. It is easy to show that any compact operator is a bounded operator and any bounded operator is continuous.

DEFINITION 3. For a topological vector space $X[\theta]$ and a linear operator $S : X[\theta] \rightarrow X[\theta]$ we define the resolvent of S as

$$\rho_\theta(S) = \{\xi \in \mathbf{C} | \xi I - S : X[\theta] \rightarrow X[\theta] \text{ is bijective and has a continuous inverse}\}.$$

The spectrum of S is defined by $\sigma_\theta(S) = \mathbf{C} \backslash \rho_t(S)$ (the set theoretic complement) and the spectral radius is $sr_\theta(S) = \sup\{|\lambda| \mid \lambda \in \sigma_\theta(S)\}$.

DEFINITION 4. A net $\{x_\alpha\}_J \subset E[t]$ is said to be t-ultimately bounded (t-ub) if given any $V \in v(t)$ there is a positive real number r and an index $\alpha_0 \in J$, both depending on V, such that $x_\alpha \in rV \;\; \forall \;\; \alpha \geq \alpha_0$. Let us denote by Γ the set of all t-ub nets in $E[t]$.

REMARK 5. Any bounded net is a t-ub net. For more details about t-ub nets we refer the reader to DeVito [2]

From now on $T : E[t] \to E[t]$ will denote a compact operator and $U \in v(t)$ will be the zero-neighborhood such that $T(U)$ is a t-relatively compact set. P_U will stand for the Minkowski functional generated by U (see Cotlar [1]), which is a seminorm on E. Let us denote by $E[P_U]$ the linear space E with the topology given by the seminorm P_U.

DEFINITION 6 (Γ_t-convergence). Let $\xi \in \mathbf{C}$, with $\xi \neq 0$ We will say that $\frac{1}{\xi^n} T^n \xrightarrow{\Gamma_t} 0$ ($T^n = T \circ T \circ \ldots \circ T$ n times) if given both $V \in v(t)$ and $\{x_\alpha\}_J \in \Gamma$ there exist $\alpha_0 \in J$ and $n_0 \in \mathbf{N}$ such that $\frac{1}{\xi^n} T^n(x_\alpha) \in V \, \forall \, \alpha \geq \alpha_0$ and $\forall \, n \geq n_0$.

DEFINITION 7 $\gamma_t(T) = \inf\left\{ |\xi| \mid \frac{1}{\xi^n} T^n \xrightarrow{\Gamma_t} 0 \right\}$.

REMARK 8. It is shown by Vera [7] that for a bounded operator T,

(a) $\gamma_t(T) < \infty$, and for any $\xi \in \mathbf{C}$ such that $\gamma_t(T) < |\xi|$, $\frac{1}{\xi^n} T^n \xrightarrow{\Gamma_t} 0$

(b) $sr_t(T) \leq \gamma_t(T)$, where $sr_t(T)$ is the spectral radius of T.

(c) If $E[t]$ is a Banach space then $\gamma_t(T) = sr_t(T)$

The main theorem (Theorem 28) in this paper states that for a compact linear operator on any complete topological vector space, locally convex; we have $sr_t(T) = \gamma_t(T)$, even when the topology t is not given by a norm. In fact, Remark 8 (c) will be used to prove our result.

2. MAIN RESULTS

Now, we will state a well known theorem about compact operators

THEOREM 9 (see Nikol'skij [5]). The spectrum of a compact operator T on an infinite-dimensional linear topological space E consists of zero and no more than a countable set of eigenvalues different from zero. The unique accumulation point of this set, if it is infinite, is zero.

REMARK 10. The topology on E given by the seminorm P_U is coarser than the topology t ($P_U \leq t$).

PROPOSITION 11. $T : E[P_U] \to E[P_U]$ is a compact operator.

PROOF. Since $T(U)$ is t-relatively compact and $P_U \leq t$, $T(U)$ is also P_U-relatively compact.

DEFINITION 12. $\gamma_{P_U} = \inf\left\{ |\xi| \mid \frac{1}{\xi^n} T^n \xrightarrow{\Gamma_{P_U}} 0 \right\}$. Here, the meaning of $\frac{1}{\xi^n} T^n \xrightarrow{\Gamma_{P_U}} 0$ is given by Definition 6 where the topology P_U is used instead of t.

It is easy to show that Γ_{P_U}-convergence means that given any net $\{x_\alpha\}_J \subset E$ such that for all $\alpha \in J$, $P_U(x_\alpha) \leq r$ for some $r \in \mathbf{R}^+$ (these kinds of nets are said to be P_U-bounded nets), then $P_U\left(\frac{1}{\xi^n} T^n x_\alpha\right) \to 0$ as a net in $\mathbf{R}$ whose set of indices is $\mathbf{N} \times J$.

PROPOSITION 13 $\gamma_{P_U}(T) = \gamma_t(T)$.

PROOF. Let $\xi \in \mathbf{C}$ be such that $\gamma_{P_U}(T) < |\xi|$, let $V \in v(t)$ and $\{x_\alpha\}_J \in \Gamma$ be given Since $\frac{1}{\xi} T(U)$ is a bounded set, there is a positive real number r_1 such that $\frac{1}{r_1 \xi} T(U) \subset V$ DeVito [2] shows that $\{x_\alpha\}_J \in \Gamma \Rightarrow \{r_1 x_\alpha\}_J \in \Gamma$. This implies that there exist both $\alpha_0 \in J$ and $r_2 > 0$ such that $r_1 x_\alpha \in r_2 U \, \forall \, \alpha \geq \alpha_0$. This means that $P_U(r_1 x_\alpha) \leq r_2$, hence the net $\{x_\alpha\}_{\alpha \geq \alpha_0}$ is a P_U-bounded net. Therefore, $\exists \, \alpha_1 \in J$, $\alpha_1 \geq \alpha_0$ and $n_1 \in \mathbf{N}$ such that $P_U\left(\frac{1}{\xi^n} T^n(x_\alpha)\right) < 1 \, \forall \, \alpha \geq \alpha_1$, $n \geq n_1$, that is, $\frac{1}{\xi^n} T^n(x_\alpha) \in U$ for those indices. Hence

$$\frac{1}{\xi^{n+1}} T^{n+1} x_\alpha = \frac{1}{r_1 \xi} T\left(\frac{1}{\xi^n} T^n r_1 x_\alpha\right) \in \frac{1}{r_1 \xi} T(U) \subset V \, \forall \, \alpha \geq \alpha_1, n \geq n_1.$$

This implies $\frac{1}{\xi^n} T^n \xrightarrow{\Gamma_t} 0$. Then $\gamma_t(T) \leq |\xi|$ and $\gamma_{P_U}(T) \leq \gamma_t(T)$

On the other hand, set $\gamma_t(T) < |\xi|$ and let $\{x_\alpha\}_J$ be a P_U-bounded net, that is, $x_\alpha \in rU$ for all α and some $r > 0$. Then $\left\{\frac{1}{\xi}Tx_\alpha\right\}_J \subset \frac{r}{\xi}T(U)$ which is a t-bounded set. Therefore, $\left\{\frac{1}{\xi}Tx_\alpha\right\}_J \in \Gamma$ Hence, because of $\frac{1}{\xi^n}T^n \xrightarrow{\Gamma_t} 0$, given $\epsilon > 0$ there exists $\alpha_0 \in J$, $n_0 \in \mathbf{N}$ such that

$$\frac{1}{\xi^{n+1}}T^{n+1}x_\alpha = \frac{1}{\xi^n}T^n\left(\frac{1}{\xi}Tx_\alpha\right) \in \epsilon U \,\forall\, \alpha \geq \alpha_0,\, n \geq n_0,$$

that is, $P_U\left(\frac{1}{\xi^{n+1}}T^{n+1}x_\alpha\right) \leq \epsilon$ for those indices. From here $\frac{1}{\xi^n}T^n x_\alpha$ is P_U-convergent to 0, therefore, $\gamma_{P_U}(T) \leq |\xi|$. This implies that $\gamma_t(T) \leq \gamma_{P_U}(T)$.

DEFINITION 14 $N = \{x \in E \mid P_U(x) = 0\}$.

REMARK 15. Since U is a balanced, convex and closed set, for any real $0 \leq r < 1$, $\{x \in E \mid P_U(x) \leq r\} \subset U$. Then $N \subset U$. Since t is not given by a norm, it follows from a Theorem of Kolmogoroff (Theorem 1.39 in Rudin [6]) that $\{0\} \neq N$. On the other hand, if $T \neq 0$ then $N \neq E$.

The following theorem is a generalization of a theorem of Lomonosov [4] about non-trivial invariant subspaces.

THEOREM 16. N is a closed linear subspace of E and $T(x) = 0$ for all $x \in N$. In particular, N is a T-invariant subspace.

PROOF. The first assertion follows from the fact that $P_U(\xi x + y) \leq |\xi|P_U(x) + P_U(y)$. For the second claim let us take $x \in N$, then $mx \in N$ for all $m = 1, 2, ...$, hence, $\{mT(x)\}_{m=1,2,3...} \subset T(N) \subset T(U)$. Since the latter set is bounded, given $V \in \vartheta(t)$ there is a positive real number r such that $\{mT(x)\} \subset rV \Rightarrow T(x) \in \frac{r}{m}V \subset V$ when $m > r$, since V is an arbitrary neighborhood of zero, then $T(x) = 0$.

DEFINITION 17. Let E/N be the quotient linear space and let $\widehat{P}_U$ be defined by $\widehat{P}_U(x + N) = P_U(x)$.

Definition 14 tells us that $\widehat{P}_U$ is well defined. It is easy to show that $\widehat{P}_U$ is a norm in E/N We will denote by $(E/N)[\widehat{P}_U]$ the vector space E/N with the topology given by the norm $\widehat{P}_N$.

DEFINITION 18. Let $\widehat{T} : E/N \to E/N$ be defined by $\widehat{T}(x + N) = T(x) + N$.

By Theorem 16 $\widehat{T}$ is a well defined map.

PROPOSITION 19. $\widehat{T} : (E/N)[\widehat{P}_U] \to (E/N)[\widehat{P}_U]$ is a compact operator.

PROOF. U/N is an open set in $(E/N)[\widehat{P}_U]$ and $\widehat{T}(U/N) = (T(U) + N)/N \subset (\overline{T(U)} + N)/N$ the latter set is $\widehat{P}_U$-compact because the canonical projection from $E[P_U]$ onto $(E/N)[\widehat{P}_U]$ is a continuous map.

REMARK 20. By Remark 2, a compact operator is continuous, hence $\widehat{T}$ is continuous.

PROPOSITION 21 $\gamma_{\widehat{P}_U}(\widehat{T}) = \gamma_{P_U}(T)$.

PROOF. The proof follows immediately from the definition of $\widehat{P}_U$.

PROPOSITION 22. $\xi \in \mathbf{C}\backslash\{0\}$ is an eigenvalue of T if and only if it is an eigenvalue of $\widehat{T}$.

PROOF. Let $\xi \neq 0$ in $\mathbf{C}$ be an eigenvalue of $\widehat{T}$. Hence there exists $x + N \neq N$ such that $\widehat{T}(x + N) = \xi(x + N) \Rightarrow T(x) + N = (\xi x) + N \Rightarrow \xi x - T(x) \in N \Rightarrow \exists\, y \in N$ such that $(\xi I - T)(x) = \xi y$. By Theorem 16 $T(y) = 0$, hence $T(x - y) = T(x) = \xi(x - y)$ where $x - y \neq 0$ because $x \notin N$. Therefore ξ is an eigenvalue of T.

The other contention is trivial.

DEFINITION 23. $(\widetilde{E/N})\,[\widetilde{P_U}]$ will denote the completion (as normed space) of $(E/N)[\widehat{P_U}]$ and $\tilde{T}$ will denote the natural extension of $\widehat{T}$ to $(\widetilde{E/N})\,[\widetilde{P_U}]$.

REMARK 24. $(\widetilde{E/N})\,[\widetilde{P_U}]$ is a Banach space (see Cotlar [1]) Besides, since $\widehat{T}$ is a compact operator, $\tilde{T} : (\widetilde{E/N})\,[\widetilde{P_U}] \to (\widetilde{E/N})\,[\widetilde{P_U}]$ is a compact operator (see Kreyszig [3]). The espectrum $\sigma_{\widetilde{P_U}}$ (see definition 3) is described in Theorem 9.

THEOREM 25. $\xi \in \mathrm{C}\backslash\{0\}$ is an eigenvalue of $\widehat{T}$ if and only if it is an eigenvalue of $\tilde{T}$.

PROOF. Let $q \in \widetilde{E/N}$, $q \neq 0$ be such that $\tilde{T}(q) = \xi q$ Let $\{z_n + N\} \subset E/N$ be such that it converges to q Hence $\{z_n + N\}$ is a $\widehat{P_U}$-bounded set and therefore, because $\widehat{T}$ is a compact operator, $\{\widehat{T}(z_n + N)\}$ has a convergent subsequence. Without loss of generality let us suppose that $\widehat{T}(z_n + N) \to z + N \in E/N$, then $\xi q = \widehat{T}(q) = z + N$ which implies that $q = \frac{1}{\xi}(z + N) \in E/N$, hence ξ is an eigenvalue of $\widehat{T}$.

The proof of the second part of this theorem is trivial since $\widehat{T}$ is a restriction of $\tilde{T}$

PROPOSITION 26. $sr_t(T) = sr_{\widehat{P_U}}(\widehat{T}) = sr_{\widetilde{P_U}}(\tilde{T})$.

PROOF. From Theorem 9, Proposition 22 and Theorem 25 we get

$$\sigma_t(T)\backslash\{0\} = \sigma_{\widehat{P_U}}(\widehat{T})\backslash\{0\} = \sigma_{\widetilde{P_U}}(\tilde{T})\backslash\{0\}$$

and from these equalities the proposition follows.

LEMMA 27. $\gamma_{\widehat{P_U}}(\widehat{T}) \leq \gamma_{\widetilde{P_U}}(\tilde{T})$

The lemma follows from the definitions of $\gamma_{\widehat{P_U}}(\widehat{T})$ and $\gamma_{\widetilde{P_U}}(\tilde{T})$ and the fact that $\widehat{T}$ is the restriction of $\tilde{T}$ to E/N.

THEOREM 28. $\gamma_t(T) = sr_t(T)$.

PROOF. By Remark 8(b), it suffices to show that $sr_t(T) \geq \gamma_t(T)$. Also by Remark 8(c)

$$\gamma_{\widetilde{P_U}}(\tilde{T}) = sr_{\widetilde{P_U}}(\tilde{T}) \tag{2 1}$$

because $(\widetilde{E/N})[\widetilde{P_U}]$ is a Banach space. From Proposition 13 and Proposition 21 we obtain

$$\gamma_t(T) = \gamma_{\widehat{P_U}}(\widehat{T}) \tag{2.2}$$

from Lemma 27 and Equation (2.1) it follows that

$$\gamma_{\widehat{P_U}}(\widehat{T}) \leq sr_{\widetilde{P_U}}(\tilde{T}) \tag{2.3}$$

from Equation (2.2), Equation (2.3) and Proposition 26 we finally get

$$sr_t(T) \geq \gamma_t(T).$$

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