# THE DIMENSION OF THE BOUNDARY OF THE LÉVY DRAGON

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**ABSTRACT.** In this paper we describe the computations done by the authors in determining the dimension of the boundary of the Lévy Dragon. A general theory was developed for calculating the dimension of a self-similar tile and the theory was applied to this particular set. The computations were challenging It seemed that a matrix which was $2^{15} \times 2^{15}$ would have to be analyzed. It was possible to reduce the analysis to a $752 \times 752$ matrix. At last it was seen that if $\lambda$ was the largest eigenvalue of a certain $734 \times 734$ matrix, then $\dim_H(K) = \frac{\ln(\lambda)}{\ln(\sqrt{(2)})}$. Perron-Frobenius theory played an important role in analyzing this matrix.

**KEY WORDS AND PHRASES:** Hausdorff dimension, iterated function systems, attractors, fractal geometry
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## 1. INTRODUCTION

Self-similar sets have become an important area of research. They are not only interesting geometric figures, but they have also proved to be useful in image compression and storage [1], [2], and [3]. It is possible that those self-similar sets that are tiles might play an important role in image compression in the future. So, these sets deserve to be studied in some detail. The work summarized here is an attempt to extend what is known about these sets.



Figure 1. The Lévy Dragon

In [4], P Lévy studied the complex curve which has come to be known as the Lévy Dragon  He showed that the Dragon is self-similar, and argued that the plane can be tiled by copies of it  Lévy's paper is a *tour de force* on this subject  It is amazing that he was able to determine so many properties of this and other curves in [4] without the use of modern computers

Although Lévy found many properties of the Levy Dragon, he did not compute the Hausdorff dimension of its boundary. In [5, page 236] G Edgar asks whether the dimension of the boundary of the Dragon is greater than one  In a previous paper [6] we answered that question in the affirmative and in fact determined a precise value for this dimension.  In the process of doing this we developed a general theory for computing the Hausdorff dimension of the boundary of self-similar fractal tiles

The Lévy Dragon is the attractor for the $IFS\,F = \{f_1, f_2\}$, where $f_1$ and $f_2$ are similitudes of Euclidean space $\mathbf{R}^2$ given by

$$f_1(x,y) = \left(\frac{x-y}{2}, \frac{x+y}{2}\right),$$
$$f_2(x,y) = \left(\frac{x+y+1}{2}, \frac{y-x+1}{2}\right).$$

Much that we discovered about the Dragon holds in a broader setting  So, it is appropriate to have fairly general notational conventions.  The Hausdorff dimension of a set $E$ is denoted by $\dim_H E$, and the lower and upper box-counting dimensions of $E$ are denoted by $\underline{\dim}_B E$ and $\overline{\dim}_B E$.  Definitions of these dimension functions can be found in [7], [8] and [9].

Let $\Omega$ denote the collection of sequences $I = \{i_1, i_2, \cdots\}$ with $i_k \in \{1, 2, \cdots, n\}$.  $\Omega_k$ will denote the set of sequences of length $k$ with entries from $\{1, 2, \cdots, n\}$.  $\Omega$ has the natural metric given by

$$d(I,J) = \begin{cases} 1, & \text{if } i_1 \neq j_1 \\ c_{i_1} c_{i_2} \cdots c_{i_m}, & \text{if } i_k = j_k \text{ for } k \leq m \text{ and } i_{m+1} \neq j_{m+1}. \end{cases}$$

For $I \in \Omega$, we let $I_k$ denote the finite sequence $\{i_1, i_2, \cdots, i_k\}$, and will often use the shorthand $f_{I_k}$ for the composition $f_{i_1} f_{i_2} \cdots f_{i_k}$ and $c_{I_k}$ for the product $c_{i_1} c_{i_2} \cdots c_{i_k}$  We will also need the maps $\sigma : \Omega \to \Omega$ and $g : \Omega \to K$ given by $\sigma(i_1, i_2, \cdots) = \{i_2, i_3, \cdots\}$ and $g(I) = \lim_{k \to \infty} f_{i_1} f_{i_2} \cdots f_{i_k}(x)$  It is not difficult to see that $g(I)$ is independent of the choice of $x \in X$.  The map $\sigma$ is the well-known shift map on the sequence space

## 2.  SUB-SELF-SIMILAR SETS

In [7], Falconer gave an (abstract) way to determine the dimension of sub-self-similar sets  Given a set $F = \{f_1, f_2, \cdots, f_n\}$ of contracting similitudes on $\mathbf{R}^l$, the closed set $E$ is said to be *sub-self-similar* for $F$ if $E \subset \bigcup_{i=1}^n f_i(E)$.  It follows that $E \subset K$, where $K$ is the attractor for $F$.  Note in particular that the topological boundary $\partial K$ of $K$ is s.s.s., since the interior int $K$ of $K$ is mapped into itself by each of the open mappings $f_i$.  The first crucial observation about s.s.s. sets is

**PROPOSITION 2.1.** Let $E$ be a closed set.  Then $E$ is s.s.s. for $F$ if and only if $E = g(A)$ for some compact set $A \subset \Omega$ such that $\sigma(A) \subset A$.  Furthermore, if $E$ is s.s.s., such a set $A$ is given by $A = \{I \in \Omega | g(\sigma^k(I)) \in E \text{ for all } k \geq 0\}$.

The set $A$ in Proposition 2.1 is fundamental in the calculation of $\dim_H E$.  For $k > 0$, let $A_k$ be the set of finite sequences obtained by truncating elements of $A$ after $k$ terms.  For $s \geq 0$ definite

$$\tau(s) = \lim_{k \to \infty} \left(\sum_{I_k \in A_k} c_{I_k}^s\right)^{1/k}.$$

Recall that $F$ satisfies the *open set condition* [10] if there is a bounded nonempty open set $U$ such that

$$\bigcup_{i=1}^{n} f_i(U) \subset U, \quad \text{and}$$

$$f_i(U) \cap f_j(U) = \emptyset, \quad \text{if } i \neq j.$$

**THEOREM 2.2.** Let $E$ be a closed set which is s.s.s. with respect to the family $\{f_1, f_2, \cdots, f_n\}$ of contracting similitudes which satisfies the open set condition. Let $s$ be the number satisfying $\tau(s) = 1$ Then $s = \dim_H E = \underline{\dim}_B E = \overline{\dim}_B E$.

## 3. BOUNDARIES OF ATTRACTORS

One obvious difficulty in applying Theorem 2.2 is that one has to sum over finite sequences determined by an infinite condition   It was important to show that membership in $A_n$ can be determined from finite data

**PROPOSITION 3.1.** $A_k = \{I \in \Omega_k | f_{I_k}(K) \cap \partial K \neq \emptyset\}$

Once we had Proposition 3.1 we had a strategy for computing $\dim_H \partial K$. Suppose for convenience that the $c_i$ all have the same value $c$, and suppose that we can determine $\alpha = \lim_{k \to \infty} |A_k|^{1/k}$, where $|S|$ denotes the cardinality of the set $S$. Then

$$\tau(s) = \lim_{k \to \infty} \left( \sum_{I_k \in A_k} c_{I_k}^s \right)^{1/k} = \lim_{k \to \infty} \left( |A_k| c^{ks} \right)^{1/k} = \alpha c^s.$$

Thus $\dim_H \partial K$ is the solution to $\alpha c^s = 1$, or

$$\dim_H \partial K = -\frac{\ln(\alpha)}{\ln(c)}.$$

## 4. THE LÉVY DRAGON

We now revert to letting $K$ denote the Lévy Dragon with its $IFS\{f_1, f_2\}$   Lévy [4] showed, among other things, that $K$ tiles the plane in the sense that the plane can be written as the union of congruent copies of $K$ that meet only in their boundaries.

Let $S$ be the unit square with vertices $\{(0,0), (0,1), (1,1), (1,0)\}$ and let $T_0 \subset S$ be the triangle with vertices $\{(0,0), (1/2, 1/2), (1,0)\}$. Lévy viewed $K$ as $\lim_{k \to \infty} F^k(T_0)$.
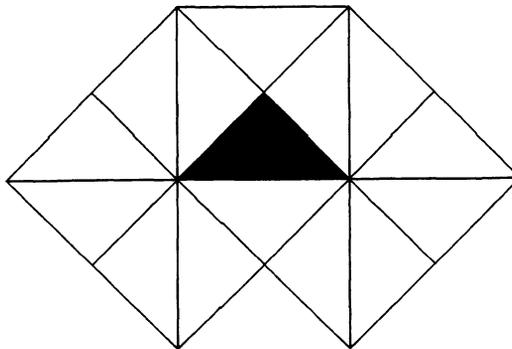


Figure 2.  $T_0$ and its Neighborhood

$S$ is the union of four triangles in $S$ with hypotenuse an edge of $S$ and third vertex at $(1/2, 1/2)$
We denote the triangulation of $\mathbf{R}^2$ consisting of these triangles and their integral translates by $\mathcal{T}_0$  $\mathcal{T}_0$ and
its neighborhood in $\mathcal{T}_0$ are shown in Figure 2  If $T$ is a right triangle in the plane, we refer to the vertex
opposite the hypotenuse as the *top vertex* of $T$, and the other vertices as *left* and *right* so that we
encounter the vertices in the order left, top, and right as we traverse the boundary in clockwise order
The *left (right) edge* of $T$ is the edge determined by the left (right) and top vertices of $T$  The
triangulation $\mathcal{T}_0$ has a subdivision $\mathcal{T}_1$ obtained by subdividing each triangle $T$ into the two triangles
determined by the left (right) edge of $T$ and the midpoint of the hypotenuse of $T$  Continuing in the
obvious way gives a sequence $\mathcal{T}_0 \succ \mathcal{T}_1 \succ \cdots \succ \mathcal{T}_k \succ \cdots$ of subdivisions of $R^2$ into right isoceles
triangles.  Each triangle in $\mathcal{T}_k$ has diameter $\left(\frac{1}{\sqrt{2}}\right)^k$  For each $T \in \mathcal{T}_k$, let $T^L$ be the triangle in $\mathcal{T}_{k+1}$
whose hypotenuse is the left edge of $T$ and which has a vertex outside of $T$.  Define $T^R$ similarly.  One
can easily see that $F(T_0) = T_0^L \bigcup T_0^R$.  In fact, the following proposition is immediate

**PROPOSITION 4.1.**  For each $k > 0$, $F^k(T_0)$ is a union of $2^k$ triangles in $\mathcal{T}_k$.  $F^{k+1}(T_0)$ is
obtained from $F^k(T_0)$ by replacing each $T$ in $F^k(T_0)$ by $T^L$ and $T^R$    $\square$

We used the formula at the end of Section 3 to compute the Hausdorff dimension of $\partial K$  The
difficulty that we encountered was that $K$ and $\partial K$ are very complicated objects, and we can only see
finite approximations to them, so it was not clear how to compute $A_k$  What we were able to work with
' was

$$B_k = \left\{ I_k \in \Omega_k | f_{i_k}(T_0) \cap \partial F^k(T_0) \neq \emptyset \right\}.$$

Then we were able to determine the relationship between $A_k$ and $B_k$ to complete the calculation
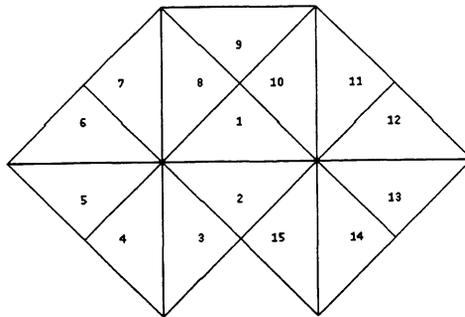


Figure 3  The Neighborhood Ordering

One of the difficulties in dealing with $K$ is that it is not easy to locate points in the interior of $K$ by
looking at $F^k(T_0)$.  One needs to find objects of positive area which persist from iteration to the next, but
as soon as a triangle $T$ appears at one level, its interior is discarded at the next level and is replaced by
two triangles exterior to $T$.  It is not until the iterations have become sufficiently intertwined to create
covered triangles that the discarded mass is filled back in by adjacent triangles  Through computer
experimentation we discovered that covered triangles do not exist until the 14th iteration, and only 8 of
the $2^{14}$ triangles that comprise $F^{14}(T_0)$ are covered!  However, if a triangle $T \in \mathcal{T}_k$ is covered, then $T$ is
the union of two triangles in $\mathcal{T}_{k+1}$ which are also covered.  It follows that $\text{int} \, T \subset F^{k+m}(T_0)$ for all
$m > 0$, so that $\text{int} \, T \subset \text{int} \, K$.
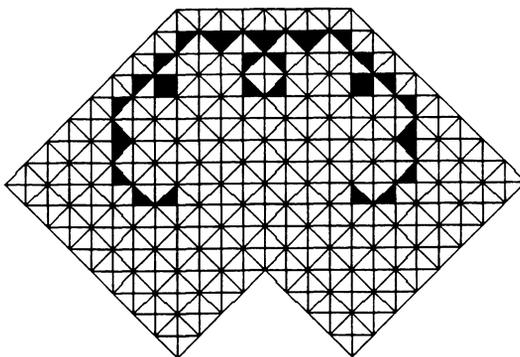
Figure 4. $N_5$ and $F^5 T_0$

**PROPOSITION 4.2.** $B_k \subset A_k$

It follows that $|B_k| \leq |A_k|$  We also used the fact that at each stage $F^k(I)$ is a tiling of $\mathbf{R}^2$  So, in using $B_k$ two ways, we were able to determine lower and upper bounds for $\dim_H \partial K$ which were the same number thus giving us a complete determination of this dimension  Let $\beta = \lim_{k \to \infty} |B_k|^{1/k}$

**THEOREM 4.3.**

$$\dim_H \partial K = \frac{\ln(\beta)}{\ln(\sqrt{2})}.$$

## 5.   COMPUTATIONAL ISSUES

In the light of Theorem 4.3, our goal was to understand the asymptotic behavior of the number $|B_k|$ of triangles in $F^k(T_0)$ which are not covered.  To this end, we needed to study the neighborhood structures introduced in Section 4.  In order to keep track of the number of neighborhood types at each stage in $F^k(T_0)$ we needed a matrix which was $2^{15} \times 2^{15}$  However, by experimenting with this matrix, we found that we could reduce it to a smaller $734 \times 734$ matrix with nonnegative integer entries which was irreducible  Perron-Frobenius theory was used to determine that there was a unique largest eigenvalue $\lambda$ to this matrix.  This eigenvalue determined the cardinality of $|B_k|$ to be $\lambda^k$ asymptotically This allowed us to compute the dimension of the boundary of the Lévy Dragon  We were able to determine precise bounds for $\lambda$ using integer arithmetic and also determine as estimate for $\lambda$ to high precision using the power method.

By Section 4,

$$\dim_H \partial K = \frac{\ln(\lambda)}{\ln(\sqrt{2})}.$$

Floating point computations using the power method give the estimate $\lambda \approx 1.934007183$  The rigorous estimate that we obtained using integer arithmetic was $1.824190 < \lambda < 1.974189$.  This last estimate proved that $\lambda$ was greater than $\sqrt{2}$, and that the Hausdorff dimension of $\partial K$ is greater than 1 Thus, the boundary of the Dragon has Hausdorff dimension greater than 1, and Edgar's question was answered in the affirmative.

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