Internat. J. Math. & Math. Sci.
VOL. 21 NO. 1 (1998) 107-116

INTEGERS REPRESENTABLE BY $(x+y+z)^3/xyz$

SHARON A. BRUEGGEMAN

Department of Mathematics
University of Illinois
1409 W. Green Street
Urbana, IL 61801

(Received February 21, 1996)

ABSTRACT. In [1], A. Bremner and R. K. Guy discuss the problem of finding integers which may be represented by $(x+y+z)^3/xyz$ where x, y, z are integers. To this end, they present tables of solutions for integers n in the range $-200 \le n \le 200$ and offer several parametric solutions which involve both positive and negative integers. We present four infinite families of solutions which involve only positive integers. Furthermore, these families contain sequences that are generated by linearly recursive relations.

AMS SUBJECT CLASSIFICATION CODE. 11D85.

1. INTRODUCTION

We make the following definitions.

$$v_2(x,y,z) = \frac{(x+y+z)^2}{xyz} \quad (A) \qquad \text{and} \qquad v_3(x,y,z) = \frac{(x+y+z)^3}{xyz} \quad (B)$$

where x, y, z are positive integers.

We say that an integer n is representable by v_2 (resp. v_3) if there exists a triple of integers (x,y,z) such that $v_2(x,y,z) = n$ (resp. $v_3(x,y,z) = n$). On the other hand, we say that a triple of integers (x,y,z) is a solution to (A) (resp. (B)) if there exists an integer n such that $v_2(x,y,z) = n$ (resp. $v_3(x,y,z) = n$). Note that each solution to (A) is also a solution to (B).

In 1993, R. K. Guy [2] proposed the problem of finding all integers n representable by v_3. The following year [3], he submitted the problem of finding all integers n representable by v_2. We attack the former problem from the opposite direction - to find triples of integers which are solutions to (B). To do this we find all triples of positive integers which are solutions to (A) and lift them to (B).

The solutions to (A) are found by constructing trees of triples under an algebraic rule. Then we show that we have found all such trees. So we have actually found all positive solutions to (A). Then the lifting produces infinitely many solutions to (B). Moreover, we show that the algebraic rules used to construct the trees are linearly recursive. Hence we get recursive sequences of integers n representable by v_3.

2. CONSTRUCTION OF THE SOLUTION TREES

First we demonstrate that there is no loss of generality in assuming that x, y, and z have no common nontrivial factor.

Proposition 2.1. *If (x,y,z) is a solution to (A), then there are only finitely many positive integers k such that (kx,ky,kz) is a solution to (A). These k are precisely the divisors of $v_2(x,y,z)$.*

Proof.
$$v_2(kx,ky,kz)=\frac{(kx+ky+kz)^2}{kxkykz}=\frac{k^2}{k^3}\cdot\frac{(x+y+z)^2}{xyz}=\frac{v_2(x,y,z)}{k}.$$
□

Now we may assume that x, y, and z have no common nontrivial factor. This is equivalent to x, y, and z being pairwise relatively prime because if a prime number p divides any two of x, y, or z and $v_2(x,y,z)$ is an integer then p divides the third.

Next we need to find an effective way of identifying triples which are solutions to (A) by using only properties of x, y, z. In fact, three symmetric divisibility conditions are sufficient.

Proposition 2.2. *Suppose x, y, z are pairwise relatively prime. Then x divides $(y+z)^2$, y divides $(x+z)^2$, and z divides $(x+y)^2$, if and only if, (x,y,z) is a solution to (A).*

Proof. Since x, y, z are pairwise relatively prime, it suffices to check that x, y, and z divide the numerator of $v_2(x,y,z)$.

$$\begin{aligned}(x+y+z)^2 &\equiv (y+z)^2 \equiv 0 \text{ (modulo } x)\\ (x+y+z)^2 &\equiv (x+z)^2 \equiv 0 \text{ (modulo } y)\\ (x+y+z)^2 &\equiv (x+y)^2 \equiv 0 \text{ (modulo } z)\end{aligned}$$

The converse is clear. □

By the converse to Proposition 2.2, finding all triples with this property, will yield all the solutions to (A). Now suppose we are given one triple that is a solution to (A). We want to find other solutions using the given one.

Proposition 2.3. *Suppose (x,y,z) is a pairwise relatively prime solution to (A). Then*

$$\left.\begin{aligned}&(a,y,z) \text{ where } a=\frac{(y+z)^2}{x}\\ &(x,b,z) \text{ where } b=\frac{(x+z)^2}{y}\\ &(x,y,c) \text{ where } c=\frac{(x+y)^2}{z}\end{aligned}\right. \quad \text{are also pairwise relatively prime solutions to } (A).$$

Furthermore, $v_2(x,y,z)=v_2(a,y,z)=v_2(x,b,z)=v_2(x,y,c)$.

Proof. By Proposition 2.2, a, b, and c are integers. The new triples are pairwise relatively prime since the two fixed entries still have no common factor. By symmetry it suffices to prove the last statement for the (a,y,z) case.

$$v_2(a,y,z)=\frac{\left(\frac{(y+z)^2}{x}+y+z\right)^2}{\frac{(y+z)^2}{x}yz}=\frac{\left((y+z)^2+x(y+z)\right)^2}{(y+z)^2xyz}=\frac{(x+y+z)^2}{xyz}=v_2(x,y,z).$$
□

We denote the three transformations of triples in Proposition 2.3 as follows:

$$\phi_1(x,y,z) = (a,y,z), \qquad \phi_2(x,y,z) = (x,b,z), \qquad \phi_3(x,y,z) = (x,y,c).$$

We say that two triples belong to the same family if there is a sequence of ϕ_i which takes one triple to a permutation of the other. Next we define a partial order on triples in the same family. Since any two consecutive (under ϕ_i) triples share two entries, we may define an ordering on triples in the following way. We say that $(x,y,w) \le (x,y,z)$ if $w \le z$. The complete ordering follows by transitivity. If $(u,v,w) \le (x,y,z)$, we call (u,v,w) a predecessor of (x,y,z), and we call (x,y,z) a successor of (u,v,w).

We will want to iterate these three transformations to generate more triples in the family of solutions. First we see how a single transformation works in the context of the partial order.

Proposition 2.4. *Suppose x,y,z,a,b,c satisfy Proposition 2.3 and $x \le y \le z$, then $y \le z \le a$ and $x \le z \le b$. That is, the triples (a,y,z) and (x,b,z) are greater than the original triple (x,y,z). We have no information about the ordering with c.*

Proof. This is clear from the definitions of a and b. □

By iterating these transformations, we obtain a tree-like family of solutions to (A), since each triple yields two succeeding triples. See Figure 2.1. We say that an ordered triple (x,y,z) is the root of a family if it does not have a predecessor; that is, $c \ge z$.

Proposition 2.5. *Each family of ordered triples forms a tree relative to the partial order on triples.*

Proof. Note that the construction above allows for at most one predecessor for any ordered triple (x,y,z), namely (x,y,c). Hence each triple has one unique root. There are no cycles for the same reason. □

Next we consider the ordering of $c = (x+y)^2/z$ where $x \le y \le z$. There are three possible situations.

$c > z$ if and only if $x+y > z$ if and only if $(x,y,z) = (1,1,1)$.
$c = z$ if and only if $x+y = z$ if and only if $(x,y,z) = (1,1,2), (1,2,3)$, or $(1,4,5)$.
$c < z$ if and only if $x+y < z$ otherwise, that is, the triple (x,y,z) is not a root.

It remains to show that there are only four possible roots where the entries are pairwise relatively prime.

Proposition 2.6. *There are precisely four root triples (x,y,z) where $x \le y \le z$ and x,y,z are pairwise relatively prime. They are: $(1,1,1)$, $(1,1,2)$, $(1,2,3)$, and $(1,4,5)$.*

Proof. It suffices to find all x,y,z such that $x+y \ge z$ and $v_2(x,y,z)$ is an integer. Suppose that $(a,b,b+d)$ is a solution to (A) where $0 \le d \le a \le b$. We divide the proof into three steps.
Step 1. If $d=0$ or $d=a$, then we obtain all four ordered root triples. Therefore we may assume that $0 < d < a$.
Step 2. Determine which pairs (a,b) yield $v_2(a,b,b+d) < 1$ for all d. There can be no roots for these pairs (a,b) since $v_2(a,b,b+d)$ is not an integer.
Step 3. Eliminate all other $(a,b,b+d)$ either by computer test or algebraic methods.

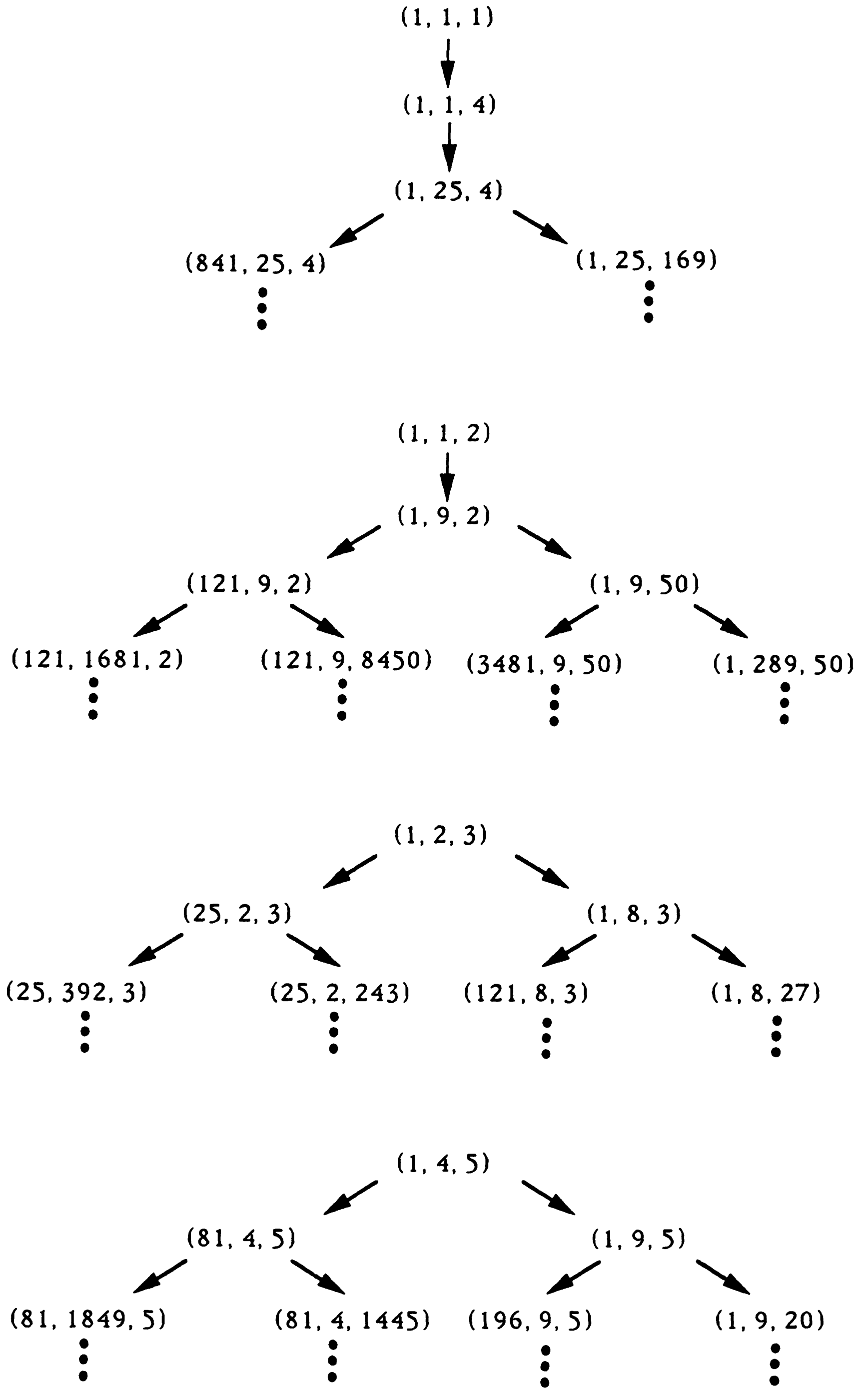

Figure 2.1. The first generations of the four families of solutions to (A).

(Step 1): Set $d=0$. Then $(a,b,b+d)=(a,b,b)$. By the relatively prime condition, $b=1$. Since $a \le b$, $a=1$ also. Note $v_2(1,1,1)=9$.

Set $d=a$. Then

$$v_2(a,b,b+d)=v_2(a,b,b+a)=\frac{(a+b+b+a)^2}{a\cdot b\cdot(b+a)}=\frac{4(a+b)^2}{ab(a+b)}=\frac{4(a+b)}{ab}.$$

Since $(a,b)=1$, $(a,a+b)=(b,a+b)=1$. Hence ab divides 2^2. By the relatively prime condition, there are three possibilities:

$a=1$	$b=1$	$v_2(1,1,2)=8$,
$a=1$	$b=2$	$v_2(1,2,3)=6$,
$a=1$	$b=4$	$v_2(1,4,5)=5$.

(Step 2): Now consider $0<d<a$. Note $a \neq 1$ so $a<b$.

$$v_2(a,b,b+d)=\frac{(a+b+b+d)^2}{a\cdot b\cdot(b+d)}<\frac{(a+b+b+a)^2}{a\cdot b\cdot(b+0)}=\frac{4(a+b)^2}{ab^2}=\frac{4a}{b^2}+\frac{8}{b}+\frac{4}{a}$$

We solve the inequality, $\frac{4a}{b^2}+\frac{8}{b}+\frac{4}{a}<1$, for a,b positive integers. The following table lists values of a and b that do not yield integer-valued v_2.

$a=5$	$b\ge 43$	$a=11$	$b\ge 17$
$a=6$	$b\ge 27$	$a=12$	$b\ge 17$
$a=7$	$b\ge 22$	$a=13$	$b\ge 17$
$a=8$	$b\ge 20$	$a=14$	$b\ge 17$
$a=9$	$b\ge 19$	$a=15$	$b\ge 17$
$a=10$	$b\ge 18$	$a\ge 16$	

(Step 3): Now that we have eliminated the cases in Step 2, it suffices to show that none of the remaining cases yield roots.

Consider the above table. A finite computer check, with a varying from 5 to 15 and b varying from $a+1$ to 42, determines that there are no roots with $a>4$. The algorithm is simple. If (a,b,c) solved (A), then c would divide $(a+b)^2$ by Proposition 2.2. So, for each pair (a,b), it suffices to test $v_2(a,b,c)$ for each divisor c of $(a+b)^2$. But each such $v_2(a,b,c)$ is not an integer.

It remains to show that there are no triples which are solutions to (A) with the property that $a=1,2,3,4$ and $0<d<a<b$. We have the following cases:

$a=1$, no d are possible.

$a=2, d=1$:

$\frac{(2+b+b+1)^2}{2b(b+1)}=\frac{(2b+3)^2}{2b(b+1)}$ is not an integer since 2 does not divide $2b+3$.

$a=3, d=1$:

$\frac{(3+b+b+1)^2}{3b(b+1)}=\frac{4(b+2)^2}{3b(b+1)}$ is not an integer since $(b+1,b+2)=1$ and $b+1>4$.

$a=3, d=2$:

$\frac{(3+b+b+2)^2}{3b(b+2)}=\frac{(2b+5)^2}{3b(b+2)}$ is not an integer since $b+2$ does not divide $(2(b+2)+1)^2$.

$a = 4, d = 1$:
$$\frac{(4+b+b+1)^2}{4b(b+1)} = \frac{(2b+5)^2}{4b(b+1)} \text{ is not an integer since 2 does not divide } 2b+5.$$

$a = 4, d = 2$:
$$\frac{(4+b+b+2)^2}{4b(b+2)} = \frac{(b+3)^2}{b(b+2)} \text{ is not an integer since } b+2 \text{ does not divide } (b+3)^2.$$

$a = 4, d = 3$:
$$\frac{(4+b+b+3)^2}{4b(b+3)} = \frac{(2b+7)^2}{4b(b+3)} \text{ is not an integer since 2 does not divide } 2b+7.$$

This completes the proof of Proposition 2.6. □

Finally we remove the relatively prime condition. We get 8 possible values for v_2 and 13 trees of solutions. This result is a subcase of a problem [2] in the Amer. Math. Monthly. I state it here for completeness.

Proposition 2.7. *The positive integers* 1, 2, 3, 4, 5, 6, 8, *and* 9 *are the only positive integers that are representable as*
$$\frac{(x+y+z)^2}{xyz} \text{ where } x, y, z \text{ are positive integers.}$$

Proof. We have already seen that each triple (x,y,z) in tree $(1,1,1)$ yields $v_2(x,y,z) = 9$. The other trees yield $v_2(x,y,z) = 8$, 6, and 5 respectively. Then the appropriate multiples of the above trees yield $v_2(x,y,z) = 1$, 2, 3, and 4. See Proposition 2.1. □

For example, the tree with root $(1,2,3)$ may be multiplied by 1,2,3 or 6.

$$v_2(1,2,3) = 6; \quad v_2(2,4,6) = 3; \quad v_2(3,6,9) = 2; \quad v_2(6,12,18) = 1.$$

3. LIFTING FROM (A) TO (B)

As stated earlier, each solution to (A) is also a solution to (B). But since $v_3(x,y,z) = (x+y+z) \cdot v_2(x,y,z)$ the function value statements need to be altered. So we repeat the previous propositions with appropriate changes.

Proposition 3.1. *If (x,y,z) is a solution to (B), then $v_3(kx,ky,kz) = v_3(x,y,z)$ for each integer k. Hence we may assume that x,y,z are pairwise relatively prime.*

Proof.
$$v_3(kx,ky,kz) = \frac{(kx+ky+kz)^3}{kxkykz} = \frac{k^3}{k^3} \cdot \frac{(x+y+z)^3}{xyz} = v_3(x,y,z).$$
□

Proposition 3.2. *Suppose x,y,z are pairwise relatively prime. If x divides $(y+z)^2$, y divides $(x+z)^2$, and z divides $(x+y)^2$, then (x,y,z) is a solution to (B).*

Note that the converse to Proposition 3.2 is not true since we are only checking the squares, not the cubes.

Proposition 3.3. *If (x,y,z) is a pairwise relatively prime solution to (A) and (B), then*

$$\begin{aligned} &(a,y,z) \text{ where } a = \frac{(y+z)^2}{x} \\ &(x,b,z) \text{ where } b = \frac{(x+z)^2}{y} \\ &(x,y,c) \text{ where } c = \frac{(x+y)^2}{z} \end{aligned} \qquad \text{are pairwise relatively prime solutions to (A) and (B).}$$

Furthermore,

$$v_3(a,y,z) = \frac{y+z}{x} \cdot v_3(x,y,z), \quad v_3(x,b,z) = \frac{x+z}{y} \cdot v_3(x,y,z),$$

$$\text{and} \qquad v_3(x,y,c) = \frac{x+y}{z} \cdot v_3(x,y,z).$$

Proof.

$$v_3(a,y,z) = \frac{\left(\frac{(y+z)^2}{x} + y + z\right)^3}{\frac{(y+z)^2}{x} yz} = \frac{\left((y+z)^2 + x(y+z)\right)^3}{(y+z)^2 x^2 yz} = \frac{y+z}{x} \cdot v_3(x,y,z)$$

$v_3(x,b,z)$ and $v_3(x,y,c)$ follow similarly. □

We denote the three transformations of triples in Proposition 3.2 as follows:

$$\phi_1(x,y,z) = (a,y,z), \qquad \phi_2(x,y,z) = (x,b,z), \qquad \phi_3(x,y,z) = (x,y,c).$$

Proposition 3.4. *Suppose x,y,z,a,b,c satisfy Proposition 3.3 and $x \le y \le z$, then $y \le z \le a$ and $x \le z \le b$. That is, the triples (a,y,z) and (x,b,z) are greater than the original triple (x,y,z). We have no information about the ordering with c.*

Note that, in this situation, the partial order has a double meaning. Consider the above relation that $(x,y,z) < (a,y,z)$. Since $x \le y \le z$, we also have that $\frac{y+z}{x} > 1$. So that $v_3(x,y,z) < v_3(a,y,z)$. Hence we may regard the partial order as being defined by either the differing entry or by the resulting values of v_3.

Proposition 3.5. *Each family of ordered triples forms a tree relative to the partial order on triples.*

Proposition 3.6. *There are precisely four root triples (x,y,z) where $x \le y \le z$ and x,y,z are pairwise relatively prime. They are: $(1,1,1)$, $(1,1,2)$, $(1,2,3)$, and $(1,4,5)$.*

Now we have four infinite two-leaf trees of solutions to (B). We can list these special families up to any finite point. But we do not have an effective algorithm to determine if a given positive integer occurs in the v_3 values of these families. These trees however do give us infinite sequences of integers representable by v_3 based on linearly recursive relations.

4. INFINITE SEQUENCES

We construct these sequences as follows. Take any triple in a tree and write it in the form (a, bx_0^2, cy_0^2) where b and c are squarefree. Applying the maps ϕ_2 and ϕ_3 following Proposition 3.3 in alternating fashion, we obtain a sequence of triples of the form $(a, bx_0^2, cy_0^2), (a, bx_1^2, cy_0^2)$, $(a, bx_1^2, cy_1^2), \ldots$ where

$$bx_{n+1}^2 = \frac{(a + cy_n^2)^2}{bx_n^2} \quad \text{or} \quad bx_n x_{n+1} = a + cy_n^2 \tag{1}$$

$$cy_{n+1}^2 = \frac{(a + bx_{n+1}^2)^2}{cy_n^2} \quad \text{or} \quad cy_n y_{n+1} = a + bx_{n+1}^2. \tag{2}$$

The forms bx_n^2 and cy_n^2 are okay since b and c are squarefree.

We seek a simplified value formula for v_3 in terms of x_n and y_n. Define

$$A = \frac{(a + bx_0^2 + cy_0^2)}{x_0 y_0}.$$

Proposition 4.1. *Using the notation above,*

$$v_3(a, bx_n^2, cy_n^2) = \frac{A^3}{abc} x_n y_n \tag{3}$$

$$\text{and} \quad v_3(a, bx_{n+1}^2, cy_n^2) = \frac{A^3}{abc} x_{n+1} y_n. \tag{4}$$

Proof. By definition of A,

$$v_3(a, bx_0^2, cy_0^2) = \frac{(a + bx_0^2 + cy_0^2)^3}{abx_0^2 cy_0^2} = \frac{A^3}{abc} x_0 y_0.$$

Apply Proposition 3.3.

$$v_3(a, bx_1^2, cy_0^2) = \frac{A^3}{abc} x_0 y_0 \cdot \frac{a + cy_0^2}{bx_0^2} = \frac{A^3}{abc} y_0 \cdot \frac{a + cy_0^2}{bx_0} = \frac{A^3}{abc} x_1 y_0 \quad \text{by (1)}$$

$$\text{and} \quad v_3(a, bx_1^2, cy_1^2) = \frac{A^3}{abc} x_1 y_0 \cdot \frac{a + bx_1^2}{cy_0^2} = \frac{A^3}{abc} x_1 \cdot \frac{a + bx_1^2}{cy_0} = \frac{A^3}{abc} x_1 y_1 \quad \text{by (2)}.$$

The proposition follows by induction on the subscripts. □

Finally we determine the linearly recursive relation on x_n and y_n. First we need a lemma. Rearrange (3) and (4) as follows:

$$a + bx_n^2 + cy_n^2 = Ax_n y_n \tag{3'}$$

$$a + bx_{n+1}^2 + cy_n^2 = Ax_{n+1} y_n. \tag{4'}$$

Lemma 4.2. *Using the notation above,*

$$c(y_{n+1} + y_n)^2 = (A^2/c) x_{n+1}^2 \tag{5}$$

$$\text{and} \quad b(x_{n+2} + x_{n+1})^2 = (A^2/b) y_{n+1}^2. \tag{6}$$

Proof. Consider equations (3′) and (4′). To show (5), first subtract $a + bx_{n+1}^2 + cy_n^2 = Ax_{n+1}y_n$ from $a + bx_{n+1}^2 + cy_{n+1}^2 = Ax_{n+1}y_{n+1}$. Then remove the factor $y_{n+1} - y_n$ and square, yielding $c^2(y_{n+1} + y_n)^2 = A^2 x_{n+1}^2$. Similarly, to show (6), subtract $a + bx_{n+1}^2 + cy_{n+1}^2 = Ax_{n+1}y_{n+1}$ from $a + bx_{n+2}^2 + cy_{n+1}^2 = Ax_{n+1}y_{n+2}$. Then remove the factor $x_{n+2} - x_{n+1}$ and square, yielding $b^2(x_{n+2} + x_{n+1})^2 = A^2 y_{n+1}^2$.

□

Proposition 4.3. *Using the notation above,*

$$x_{n+2} = (A^2/bc - 2)x_{n+1} - x_n \tag{7}$$

$$\text{and} \quad y_{n+2} = (A^2/bc - 2)y_{n+1} - y_n \tag{8}$$

$$\text{where} \quad A^2/bc - 2 \quad \text{is an integer.}$$

Proof. First note $A^2/bc = a \cdot v_2(a, bx_0^2, cy_0^2)$ is an integer. Consider $bx_{n+1}x_{n+2} = a + cy_{n+1}^2$. By adding $0 = a - a + 2bx_{n+1}^2 - 2bx_{n+1}^2 + cy_n^2 - cy_n^2$, we find that

$$bx_{n+1}x_{n+2} = cy_{n+1}^2 + 2(a + bx_{n+1}^2) + cy_n^2 - 2bx_{n+1}^2 - (a + cy_n^2).$$

By (2) and (1), we find that

$$bx_{n+1}x_{n+2} = c(y_{n+1} + y_n)^2 - 2bx_{n+1}^2 - bx_nx_{n+1}.$$

Now by (5), it is easy to see that

$$x_{n+2} = (A^2/bc)x_{n+1} - 2x_{n+1} - x_n = (A^2/bc - 2)x_{n+1} - x_n.$$

A similar argument works for $y_{n+2} = (A^2/bc - 2)y_{n+1} - y_n$. Consider $cy_{n+1}y_{n+2} = a + bx_{n+2}^2$. Add $0 = a - a + 2cy_{n+1}^2 - 2cy_{n+1}^2 + bx_{n+1}^2 - bx_{n+1}^2$. Apply (6) instead of (5). □

Since these sequences are linearly recursive, it is easy to find formulas in terms of n to satisfy each sequence. Note also that the order of the entries is irrelevant as long as we do not start the recursion by replacing the largest one. So we get 4 increasing sequences starting from a triple (a, b, c) with $a \le b \le c$. They are achieved as follows: (1) Fix c and start by replacing a. Think (c, a, b). (2) Fix c and start by replacing b. Think (c, b, a). (3) Fix b and start by replacing a. Think (b, a, c). (4) Fix a and start by replacing b. Think (a, b, c). Note that sequences (2) and (4) agree at the second triple as do sequences (1) and (3) although they all differ after that. If we traverse by replacing the largest entry first, the sequence decreases initially until it reaches the "oldest ancestor" containing the fixed entry and then it increases. If that ancestor is the root, the sequence will turn around at the root and will give no new values as it will eventually repeat one of the increasing sequences above.

ACKNOWLEDGEMENTS. I would like to thank Nigel Boston for suggesting this problem and Ken Ono for his great suggestions on improving the manuscript.

REFERENCES

1. BREMNER, A. and GUY, R.K. Two More Representation Problems, Edin. Math. Soc. (to appear).
2. GUY, R.K. Problem 93:31, Western Number Theory Problems 93 : 12 : 21 (1993).
3. GUY, R.K. Problem 10382, Amer. Math. Monthly 101 (1994).