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ON THE EXISTENCE OF A PERIODIC SOLUTION OF A NONLINEAR ORDINARY DIFFERENTIAL EQUATION

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ABSTRACT. Consider a planar forced system of the following form

$$\begin{cases} \dfrac{dx}{dt} = \mu(x,y) + h(t) \\ \dfrac{dy}{dt} = -\nu(x,y) + g(t), \end{cases}$$

where $h(t)$ and $g(t)$ are 2π-periodic continuous functions, $t \in (-\infty,\infty)$ and $\mu(x,y)$ and $\nu(x,y)$ are continuous and satisfy local Lipschitz conditions. In this paper, by using the Poincáre's operator we show that if we assume the conditions, (C_1), (C_2) and (C_3) (see Section 2), then there is at least one 2π-periodic solution. In conclusion, we provide an explicit example which is not in any class of known results.

KEY WORDS AND PHRASES: Poincáre mapping, Poincáre index, Nonlinear jumping differential equation.
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1. INTRODUCTION

We consider the following planar forced system of ordinary differential equations:

$$\begin{cases} \dfrac{dx}{dt} = \mu(x,y) + h(t) \\ \dfrac{dy}{dt} = -\nu(x,y) + g(t), \end{cases} \tag{1.1}$$

where $h(t)$ and $g(t)$ are 2π-periodic continuous functions, $t \in (-\infty,\infty)$ and $\mu(x,y)$ and $\nu(x,y)$ are continuous and satisfy local Lipschitz conditions. Our work is inspired by previous work on the existence of periodic solutions of nonlinear ordinary differential equations by Aguinaldo and Schmitt [1], Cesari [4], Ding

[6-8], Ding [9], Lazer and Leach [12], Leach [13], Loud [14], Mawhin [15], [16], Mawhin and Ward [17], Mawhin et al. [18], among others. In this paper, we show that if we assume the conditions, (C1), (C2) and (C3) (see Section 2), then there is at least one 2π-periodic solution. In conclusion, we provide an explicit example which is not in any class of known results.

The authors wish to take this opportunity to thank the referee for pointing out the wonderful paper Capietto et al. [2], which has results related to Theorem 2.1 as well as Lemma 2.1, and which both of us had overlooked.

2. MAIN RESULTS

Consider the system (1.1). Let $x = \rho\cos\theta$ and $y = \rho\sin\theta$. Expressing (1.1) in polar coordinates form, we have

$$\begin{cases} \dfrac{d\rho}{dt} = \lambda(\rho,\theta) + \tilde{\tilde{h}}(\theta,t) \\ \dfrac{d\theta}{dt} = -\omega(\rho,\theta) - \dfrac{\tilde{\tilde{g}}(\theta,t)}{\rho} \end{cases} \tag{2.1}$$

where

$$\lambda(\rho,\theta) = \mu(\rho\cos\theta,\rho\sin\theta)\cos\theta - \nu(\rho\cos\theta,\rho\sin\theta)\sin\theta.$$

$$\omega(\rho,\theta) = \frac{1}{\rho}(\mu(\rho\cos\theta,\rho\sin\theta)\sin\theta + \nu(\rho\cos\theta,\rho\sin\theta)\cos\theta),$$

$$\tilde{\tilde{h}}(\theta,t) = h(t)\cos\theta + g(t)\sin\theta, \text{ and}$$

$$\tilde{\tilde{g}}(\theta,t) = h(t)\sin\theta - g(t)\cos\theta.$$

We now shall establish the following theorem.

THEOREM 2.1. Suppose the system of differential equations (1.1) or (2.1) satisfies the conditions:

(C1) $|\lambda(\rho,\theta)| < k(\rho)$, where $k(\rho)$ is a continuous function and satisfies a local Lipschitz condition.

(C2) (a) $\dfrac{2\pi+\delta}{n+1} < \underline{\lim}_{\rho\to+\infty} \displaystyle\int_0^{2\pi} \frac{d\theta}{\omega(\rho,\theta)} \le \overline{\lim}_{\rho\to+\infty} \int_0^{2\pi} \frac{d\theta}{\omega(\rho,\theta)} < \frac{2\pi-\delta}{n}$ for some $n \in Z^+$ and some δ, $0 < \delta < \min\{2\pi/(2n+1),1\}$ or

(b) $2\pi + \delta < \underline{\lim}_{\rho\to+\infty} \displaystyle\int_0^{2\pi} \frac{d\theta}{\omega(\rho,\theta)} \le \overline{\lim}_{\rho\to+\infty} \int_0^{2\pi} \frac{d\theta}{\omega(\rho,\theta)} < \frac{1}{\delta}$ for some δ, $0 < \delta < (\sqrt{\pi^2+1}-\pi)$

(C3) $\underline{\lim}_{\rho\to\infty} \min_{0\le\theta<2\pi} \omega(\rho,\theta)\rho^\alpha > 0$ for some α in $(0,1)$.

Then the system has at least one 2π-periodic solution. Take note that the assumptions of this theorem are comparable with the assumptions of Theorem 3 in [2, p.367].

We first establish two lemmas.

LEMMA 2.1. If the system (1.1) or (2.1) satisfies (C1), then for each positive number M_1, there exists a positive number M, with $M \ge M_1$ so that for each solution $(\rho(t),\theta(t))$ of (2.1), with an initial condition $\rho(t_0) > M$ for $t_0 \in [0,2\pi)$, we have $\rho(t) > M_1$ for $0 \le t \le 2\pi$.

This lemma is very close to the "elastic property" in [2,p.351].

PROOF. From (2.1) and (C1), we have

$$-c_1 - k(\rho) < \frac{d\rho}{dt} < k(\rho) + c_1$$

where

$$c_1 = \max_{\substack{0 \le \theta \le 2\pi \\ 0 \le t \le 2\pi}} |\tilde{\tilde{h}}(\theta,t)| + \varepsilon_0, \quad \text{where} \quad \varepsilon_0 > 0.$$

Let $\rho^+(t)$ and $\rho^-(t)$ be solutions of the equations

$$\frac{d\rho}{dt} = k(\rho) + c_1 \qquad (+)$$

and

$$\frac{d\rho}{dt} = -k(\rho) - c_1 \qquad (-)$$

with the initial conditions $\rho^+(t_0) = M_1$ for (+) and $\rho^-(t_0) = \rho^+(2\pi)$ for (-). Observe that if $\rho^+(t)$ is a solution of (+) then $\rho^+(2\pi+t_0-t)$ will be a solution of (-). Thus $\rho^-(t) = \rho^+(2\pi+t_0-t)$ with $\rho^-(t_0) = \rho^+(2\pi)$. Let $M = \rho^+(2\pi)$. Because $-k(\rho) - c_1 < 0$, from (-), we have $\rho^-(2\pi) \le \rho^-(t)$ for all $0 \le t \le 2\pi$. Let $\rho(t)$ be a solution of $\frac{d\rho}{dt} = \lambda(\rho,\theta) + \tilde{\tilde{h}}(\theta,t)$ with an initial condition $\rho(t_0) > M$. Then, according to the comparison theorem (e.g., see [10,p.26]) of ordinary differential equations, we have $\rho^-(t) < \rho(t)$, as $0 \le t \le 2\pi$. Together, we have $M_1 = \rho^-(2\pi) \le \rho^-(t) < \rho(t)$ for all t, $0 \le t \le 2\pi$. Observe that $M = \rho^-(t_0) \ge \rho(2\pi) = M_1 > 0$, from the above inequality. The lemma is now proved.

LEMMA 2.2. If the system (2.1) satisfies (C1), (C2), (a) or (b), and (C3) then there exists a constant $M > 0$ such that for every solution $(\rho(t),\theta(t))$ of (2.1) with an initial condition $\rho(t_0) > M$, for $t_0 \in [0,2\pi]$, the inequality:

$$2n\pi < \theta(0) - \theta(2\pi) < 2(n+1)\pi, \quad \text{when} \quad n \in Z^+, \tag{2.2a}$$

holds if (C2)(a) is assumed, and the inequality

$$0 < \theta(0) - \theta(2\pi) < 2\pi \tag{2.2b}$$

holds if (C2)(b) is assumed.

PROOF. By (C3) and (C2), (a) or (b), there is a positive constant M_2 such that for all $\rho > M_2$, we have $\omega(\rho,\theta) > 0$,

$$\omega(\rho,\theta)\rho^\alpha > \frac{c}{2}, \tag{2.3}$$

for any c, $\underline{\lim}_{p\to\infty} \min_{0 \le \theta < 2\pi} \omega(p,\theta)p^\alpha > c > 0$, and

$$\frac{2\pi+(\delta/2)}{n+1} < \int_0^{2\pi} \frac{d\theta}{\omega(\rho,\theta)} < \frac{2\pi-(\delta/2)}{n} \tag{2.4a}$$

for some $n \in Z^+$ if (C2)(a) is assumed, or

$$2\pi + \frac{\delta}{2} < \int_0^{2\pi} \frac{d\theta}{\omega(\rho,\theta)} < \frac{1}{\delta} \tag{2.4b}$$

if (C2)(b) is assumed. Let

$$c_2 = \max_{\substack{0 \le \theta \le 2\pi \\ 0 \le t \le 2\pi}} |\tilde{\tilde{g}}(\theta,t)|$$

and

$$M_3 = \max\left\{ M_2, \left(\frac{8\pi c_2}{c\delta}\right)^{\frac{1}{1-\alpha}} \right\}. \tag{2.5}$$

By Lemma 2.1, there exists $M > 0$ such that for every solution $(\rho(t,\theta(t))$ of (2.1) with $\rho(t_0) > M$ we have $\rho(t) > M_3$ for $0 \le t \le 2\pi$. Divide the second equation of (2.1) by $-\omega(\rho,\theta)$ and integrate between $\theta(0)$ and $\theta(2\pi)$:

$$-\int_{\theta(0)}^{\theta(2\pi)} \frac{d\theta}{\omega(\rho,\theta)} = \int_0^{2\pi} dt + \int_0^{2\pi} \frac{\tilde{\tilde{g}}(\theta,t)}{\omega(\rho,\theta)\rho}\, dt \tag{2.6}$$

Assume (2.2a (or (2.2b)) is false. Then, say $\theta(0) - \theta(2\pi) \ge 2(n+1)\pi$, or $\theta(0) - 2(n+1)\pi \ge \theta(2\pi)$, where $n \in Z^+$, or $n = 0$. By using (2.3), (2.4a) or (2.4b), (2.5), (2.6), we have:

$$2\pi + \frac{\delta}{2} < (n+1)\int_{\theta(0)-2\pi}^{\theta(0)} \frac{d\theta}{\omega(\rho,\theta)} = \int_{\theta(0)-2(n+1)\pi}^{\theta(0)} \frac{d\theta}{\omega(\rho,\theta)} < \int_{\theta(2\pi)}^{\theta(0)} \frac{d\theta}{\omega(\rho,\theta)}.$$

However,

$$\int_{\theta(2\pi)}^{\theta(0)} \frac{d\theta}{\omega(\rho,\theta)} = -\int_{\theta(0)}^{\theta(2\pi)} \frac{d\theta}{\omega(\rho,\theta)} \le 2\pi + \int_0^{2\pi} \left|\frac{\tilde{g}(\theta,t)}{\omega(\rho,\theta)\rho}\right| dt < 2\pi + \frac{\delta}{2}.$$

This is a contradiction. We must have $\theta(0) - \theta(2\pi) < 2(n+1)\pi$, where $n \in Z^+$ (or $\theta(0) - \theta(2\pi) < 2\pi$). Similarly we have $\omega(0) - \theta(2\pi) > 2n\pi$, where $n \in Z^+$ (or $\theta(0) - \theta(2\pi) > 0$). The lemma is now proved.

As a direct consequence of Lemma 2.2, we have the following Corollary.

COROLLARY 2.1. If, in addition to the conditions of Lemma 2.2, we assume $g(t) = h(t) = 0$, then all 2π-periodic solutions of (2.1) are uniformly bounded, that is, there is a positive number M such that $\rho(t) \le M$ for every 2π-periodic solution $(\rho(t),\theta(t))$ of (2.1).

PROOF OF THEOREM 2.1. Let us denote $(x(t;x_0,y_0),\ y(t;x_0,y_0))$ the solution of (1.1) with the initial condition $(x(0;x_0,y_0),\ y(0;x_0,y_0)) = (x_0,y_0)$. For every point (x_0,y_0), define the Poincáre mapping $P(x_0,y_0) = (x(2\pi;x_0,y_0),\ y(2\pi;x_0,y_0))$. Since $u(x,y)$, $v(x,y)$, $h(t)$ and $g(t)$ of (1.1) are continuous, the Poincáre mapping $P(x_0,y_0)$ of (1.1) is continuous. Let $u(x_0,y_0) = x(2\pi;x_0,y_0) - x_0$ and $v(x_0,y_0) = y(2\pi;x_0,y_0) - y_0$. The pair (u,v) is a continuous vector field in the plane.

Let M be the positive constant guaranteed by Lemma 2.2. Choose a positive constant M' such that $M' > M$. According to the Lemma 2.2, for all (x_0,y_0) such that $x_0^2 + y_0^2 = (M')^2$, we have $2n\pi < \theta(0) - \theta(2\pi) < 2(n+1)\pi$ for some nonnegative integer n. Thus $(u(x_0,y_0),\ v(x_0,y_0)) \ne \alpha(x_0,y_0)$ for any real constant α and for all (x_0,y_0) such that $x_0^2 + y_0^2 = (M')^2$. As (x_0,y_0) moves along the circle $x_0^2 + y_0^2 = (M')^2$ in counter clockwise direction, the vector $u(x_0,y_0)$, $v(x_0,y_0)$ rotates through an angle of 2π. The sum of the Poincáre index of a singular point of the continuous vector field $(u(x_0,y_0),\ v(x_0,y_0))$ in the compact domain $= \{(x_0,y_0) | x_0^2 +$

$y^2 \le (M')^2\}$ is equal to 1. There must exist at least one singular point of the vector field in this domain. We denote it by $(\tilde{\tilde{x}},\tilde{\tilde{y}})$, $P(\tilde{\tilde{x}},\tilde{\tilde{y}}) = (\tilde{\tilde{x}},\tilde{\tilde{y}})$. It is clear that $\tilde{\tilde{x}}^2 + \tilde{\tilde{y}}^2 \le (M')^2$. Hence $(x(2\pi;\tilde{\tilde{x}},\tilde{\tilde{y}}), y(2\pi;\tilde{\tilde{x}},\tilde{\tilde{y}})) = (\tilde{\tilde{x}},\tilde{\tilde{y}})$. This is a 2π-periodic solution.

The following two theorems (2.2 and 2.3) are almost the same as two well-known theorems. However, these particular two theorems require slightly different assumptions than the two well-known theorems. Furthermore, these two theorems can be proved using Theorem 2.1.

THEOREM 2.2. Consider the following nonlinear ordinary differential equation:

$$\frac{d^2x}{dt^2} + (\omega_+)^2x^+ - (\omega_-)^2x^- = H(t), \tag{2.7}$$

where ω_+ and ω_- are positive constants, $x^+ = \max\{x,0\}$, $x^- = \max\{-x,0\}$ and $H(t)$ is a continuous 2π-periodic function. If

$$\frac{2}{n+1} < \frac{1}{\omega_+} + \frac{1}{\omega_-} < \frac{2}{n} \quad \text{for some} \quad n \in Z^+ \tag{2.8a}$$

or

$$2 < \frac{1}{\omega_+} + \frac{1}{\omega_-} < \infty , \tag{2.8b}$$

then the equation (2.7) has a 2π-periodic solution.

PROOF. Let us define

$$u(x) = \begin{cases} (\omega_+)^2 & \text{if } x \ge 0 \\ (\omega_-)^2 & \text{if } x < 0. \end{cases}$$

Then

$$u(x)\cdot x = (\omega_+)^2x^+ - (\omega_-)^2x^- = \begin{cases} (\omega_+)^2x & \text{if } x \ge 0 \\ (\omega_-)^2x & \text{if } x < 0. \end{cases}$$

Clearly, $u(x)\cdot x$ is a continuous function of x.

Let $\frac{dx}{dt} = y$. Then the equation (2.7) is equivalent to:

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dx}{dt} = -u(x)\cdot x + H(t) \end{cases} \tag{2.9}$$

which is of the same type as (1.1). It satisfies a local Lipschitz condition. The Lipschitz constant is equal to $\text{Max}\,\{(\omega_+)^2,(\omega_-)^2,1\}$. Expressing (2.9) in the polar coordinates, we have

$$\begin{cases} \frac{d\rho}{dt} = \lambda(\rho,\theta) + H(t)\sin\theta \\ \frac{d\theta}{dt} = -\omega(\rho,\theta) - \frac{1}{\rho}(-H(t)\cos\,\theta), \end{cases} \tag{2.10}$$

when $\lambda(\rho,\theta) = \rho\,\sin\theta\,\cos\theta - u(x)\cdot x\,\sin\theta = \rho\,\sin\theta\,\cos\theta(1-u(x))$

$$= \begin{cases} \rho\sin\theta\cos\theta(1-\omega_+)^2) & \text{if } \frac{-\pi}{2} \le \theta < \frac{\pi}{2} \\ \rho\sin\theta\cos\theta(1-\omega_-)^2) & \text{if } \frac{\pi}{2} \le \theta < \frac{3\pi}{2} \end{cases}$$

and

$$\omega(\rho,\theta) = \sin^2\theta + \frac{1}{\rho}\mu(x)\cdot x\cos\theta$$

$$= \sin^2\theta + \mu(x)\cos^2\theta = \begin{cases} \sin^2\theta + \omega_+^2\cos^2\theta, & \text{if } \frac{-\pi}{2} \le \theta < \frac{\pi}{2} \\ \sin^2\theta + \omega_-^2\cos^2\theta, & \text{if } \frac{\pi}{2} \le \theta < \frac{3\pi}{2}. \end{cases}$$

Now,

$$\int_0^{2\pi} \frac{d\theta}{\omega(\rho,\theta)} = \pi\left[\frac{1}{\omega_+} + \frac{1}{\omega_-}\right]$$

It is easy to verify that the equation (2.7) satisfies the conditions (C1), (C2) and (C3).

THEOREM 2.3. Consider

$$\frac{d^2x}{dx^2} + G(x) = H(t) \tag{2.11}$$

where $G(x)$ is Lipschitz continuous with respect to x, and $H(t)$ is 2π-periodic and continuous with respect to t. Suppose (2.11) satisfies the following condition:

$$(n+\delta)^2 < \varliminf_{|x|\to\infty} \frac{G(x)}{x} \le \varlimsup_{|x|\to\infty} \frac{G(x)}{x} < (n+1-\delta)^2, \tag{2.12}$$

for the some non-negative integer n and some δ,

$$0 < \delta < \min\left\{\frac{2\pi}{2n+1}, \frac{1}{4\pi}, \sqrt{\pi^2+1}-\pi\right\} = \min\left\{\frac{2\pi}{2n+1}, \frac{1}{4\pi}\right\}.$$

Then the equation (2.11) has a 2π-periodic solution.

PROOF. Let $\frac{dx}{dt} = y$, then the equation (2.11) is equivalent to the system

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -G(x) + H(t). \end{cases} \tag{2.13}$$

Observe that the system (2.13) is a special case of the system (1.1), where $\mu(x,y) = y$, $\nu(x,y) = G(x)$, $h(t) = 0$ and $g(t) = H(t)$. Expressing (2.11) in the polar coordinates form (2.1), we have: $\lambda(\rho,\theta) = \rho\cos\theta - G(\rho\cos\theta)\sin\theta$, $\omega(\rho,\theta) = \sin^2\theta + \frac{1}{\rho}G(\rho\cos\theta)\cos\theta$, $\tilde{h}(\theta,t) = H(t)\sin\theta$ and $\tilde{\tilde{g}}(\theta,t) = -H(t)\cos\theta$.

Now, one can easily verify that the equation (2.13) satisfies the conditions (C1), (C2), and (C3).

3. EXAMPLE

Consider the system

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -f(x,y) + H(t) \end{cases} \tag{3.1}$$

where

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{k^2x^3}{x^2+y^2}, & \text{otherwise,} \end{cases}$$

$k > \frac{1}{2}$ and H(t) is a continuous 2π-periodic function. Then (3.1) has a 2π-periodic solution.

PROOF. Observe that the function $f(x,y)$ is continuous, and $\frac{\delta f}{\delta x}$ and $\frac{\delta f}{\delta y}$ exist, for all x,y. We now show that (3.1) satisfies the conditions (C1), (C2), and (C3).

Express (3.1), in polar coordinate form. Then

$$\begin{cases} \frac{d\rho}{dt} = \rho \sin\theta \cos\theta(1-k^2\cos^2\theta) + H(t)\sin\theta \\ \frac{d\theta}{dt} = (\sin^2\theta + k^2\cos^4\theta) + H(t)\cos\theta \end{cases} \tag{3.2}$$

with $\lambda(\rho,\theta) = \rho \sin\theta \cos\theta\, \theta(1 - k^2\cos^2\theta)$ and $\omega(\rho,\theta) = \sin^2\theta + k^2\cos^4\theta$.

Using the Residue Theorem of complex analysis we find

$$\int_0^{2\pi} \frac{d\theta}{\omega(\rho,\theta)} = \int_0^{2\pi} \frac{d\theta}{\sin^2_\theta + k^2\cos^2\theta} = 2\int_{-\pi/2}^{\pi/2} \frac{d\tan\theta}{\tan^2\theta + \dfrac{k^2}{1+\tan^2\theta}} .$$

Let $\tan\theta = x$ then

$$2\int_{-\pi/2}^{\pi/2} \frac{d\tan\theta}{\tan^2\theta + \dfrac{k^2}{1+\tan^2\theta}} = 2\int_{-\infty}^{\infty} \frac{(1+x^2)dx}{x^4+x^2+k^2} = 2\int_{-\infty}^{\infty} \frac{(1+x^2)dx}{(x^2+\sqrt{2k-1}\,x+k)(x^2-\sqrt{2k-1}\,x+k)}.$$

There are two complex roots:

$$x_1 = \frac{-\sqrt{2k-1} + i\sqrt{2k+1}}{2}$$

and

$$x_2 = \frac{\sqrt{2k-1} + i\sqrt{2k+1}}{2}$$

in the upper-half complex plane. Thus the above integral is equal to

$$4\pi i\left[\frac{1 + x_1^2}{(x_1-\bar{x}_1)(x_1^2-\sqrt{2k-1}\,x_1+k)} + \frac{1 + x_1^2}{(x_2-\bar{x}_2)(x_2^2+\sqrt{2k-1}\cdot x_2+k)}\right]$$

$$= 4\pi i\left[\frac{4(2k-1)(k+1)}{2i\sqrt{2k+1}\,(4k(2k-1))}\right] = \frac{2\pi(k+1)}{k\sqrt{2k+1}} .$$

Since k and $k + 1$ are relatively prime and $k + 1 > \sqrt{2k+1}$ as $k \geq \frac{1}{2}$, it follows that $\frac{k\sqrt{2k+1}}{k+1}$ cannot be an integer, and thus there exists a non-negative integer n such that $n < \frac{k\sqrt{2k+1}}{k+1} < n + 1$. The condition (C2), (a) or (b), is satisfied.

It is evident that (C1) and (C3) are true. In fact:

$$|\lambda(\rho,\theta)| = \rho|\sin\theta \cos\theta(1-k^2\cos^2\theta)| < \rho(1 + k^2) = k(p)$$

and

$$\lim_{\rho\to\infty} \min_{0\le\theta<2\pi} \omega(\rho,\theta)\rho^\alpha = \lim_{\rho\to\infty}\left(1-\frac{1}{4k^2}\right)\rho^\alpha = \infty.$$

Hence, by Theorem 2.1, there is a 2π-periodic solution.

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