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REMARKS ON μ''-MEASURABLE SETS: REGULARITY, σ-SMOOTHNESS, AND MEASURABILITY

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ABSTRACT. Let $\mathbf{X}$ be an arbitrary nonempty set and $\mathcal{L}$ a lattice of subsets of $\mathbf{X}$ such that $\phi, \mathbf{X} \in \mathcal{L}$. $\mathcal{A}(\mathcal{L})$ is the algebra generated by $\mathcal{L}$ and $\mathcal{M}(\mathcal{L})$ denotes those nonnegative, finite, finitely additive measures μ on $\mathcal{A}(\mathcal{L})$. $I(\mathcal{L})$ denotes the subset of $\mathcal{M}(\mathcal{L})$ of nontrivial zero-one valued measures. Associated with $\mu \in I(\mathcal{L})$ (or $I_\sigma(\mathcal{L})$) are the outer measures μ' and μ'' considered in detail. In addition, measurability conditions and regularity conditions are investigated and specific characteristics are given for $\mathcal{S}_{\mu''}$, the set of μ''-measurable sets. Notions of strongly σ-smooth and vaguely regular measures are also discussed. Relationships between regularity, σ-smoothness and measurability are investigated for zero-one valued measures and certain results are extended to the case of a pair of lattices $\mathcal{L}_1, \mathcal{L}_2$ where $\mathcal{L}_1 \subset \mathcal{L}_2$.

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1. Introduction. Let $\mathbf{X}$ be an arbitrary nonempty set and $\mathcal{L}$ a lattice of subsets of $\mathbf{X}$ such that $\phi, \mathbf{X} \in \mathcal{L}$. $\mathcal{A}(\mathcal{L})$ is the algebra generated by $\mathcal{L}$ and $\mathcal{M}(\mathcal{L})$ denotes those nonnegative, finite, finitely additive measures on $\mathcal{A}(\mathcal{L})$. Associated with a $\mu \in \mathcal{M}(\mathcal{L})$, there is a finitely subadditive outer measure μ'(see below for definitions) whose properties, especially pertaining to measurability have been investigated (see [9, 8]).

For a measure $\mu \in \mathcal{M}_\sigma(\mathcal{L})$, the elements of $\mathcal{M}(\mathcal{L})$ which are σ-smooth on $\mathcal{L}$, we associate an outer measure μ''. If μ is also $\mathcal{L}$-regular, then μ'' coincides with the usual induced outer measure μ^*. The more general case is investigated here. The results so obtained extend the results of [4] obtained only for zero-one valued measures. In particular, we investigate $\mathcal{S}_{\mu''}$ the μ''-measurable sets. Restrictions on μ yield still stronger results, for example, if we assume that μ is strongly σ-smooth on $\mathcal{L}$ or if μ is vaguely regular.

It is well known (see [6]) that, for a $\mu \in \mathcal{M}(\mathcal{L})$, there exists an $\mathcal{L}$-regular measure ν such that $\mu \le \nu(\mathcal{L})$ (i.e., $\mu(\mathrm{L}) \le \nu(\mathcal{L})$ for all $\mathrm{L} \in \mathcal{L}$) and $\mu(\mathbf{X}) = \nu(\mathbf{X})$. If $\mu \in \mathcal{M}_\sigma(\mathcal{L})$, we would like ν to be not only regular but also σ-smooth on $\mathcal{L}$ and, hence, countably additive. This will always be the case under certain strong lattice demands (such as normal and countably paracompact, see [5]). Here, we investigate conditions in terms of μ'' for such results to hold.

We finally extend a number of these results to the case of a pair of lattices $\mathcal{L}_1$ and $\mathcal{L}_2$, where $\mathcal{L}_1 \subset \mathcal{L}_2$. We begin with a brief review of some lattice definitions and notations which are used throughout. We adhere to standard notation. See, for example, [1, 3, 2, 5, 8, 10].

2. Background and notations. We begin with some standard background material for the reader's convenience. Let $\mathbf{X}$ be an abstract set and $\mathcal{L}$ a lattice of subsets of $\mathbf{X}$. It is assumed that $\phi, \mathbf{X} \in \mathcal{L}$. $\mathcal{A}(\mathcal{L})$ denotes the algebra generated by $\mathcal{L}$ and $\sigma(\mathcal{L})$ the σ-algebra generated by $\mathcal{L}$. The lattice $\mathcal{L}$ is called *normal* if, for any $\mathrm{L}_1, \mathrm{L}_2 \in \mathcal{L}$ with $\mathrm{L}_1 \cap \mathrm{L}_2 = \phi$, there exist $\mathrm{L}_3, \mathrm{L}_4 \in \mathcal{L}$ with $\mathrm{L}_1 \subset \mathrm{L}_3', \mathrm{L}_2 \subset \mathrm{L}_4'$ and $\mathrm{L}_3' \cap \mathrm{L}_4' = \phi$ (where prime denotes complement).

We give now some measure terminology. $\mathcal{M}(\mathcal{L})$ denotes the set of finite valued, nonnegative finitely additive measures on $\mathcal{A}(\mathcal{L})$. A measure $\mu \in \mathcal{M}(\mathcal{L})$ is called

σ-smooth on $\mathcal{L}$ if, for all sequences $\{\mathrm{L}_n\}$ of sets of $\mathcal{L}$ with $\mathrm{L}_n \downarrow \phi$, $\mu(\mathrm{L}_n) \to 0$.

σ-smooth on $\mathcal{A}(\mathcal{L})$ if, for all sequences $\{A_n\}$ of sets of $\mathcal{A}(\mathcal{L})$ with $A_n \downarrow \phi$, $\mu(A_n) \to 0$, i.e., countably additive.

$\mathcal{L}$-regular if, for any $A \in \mathcal{A}(\mathcal{L})$,

$$\mu(A) = \sup\{\mu(L) \mid \mathrm{L} \subset A, \mathrm{L} \in \mathcal{L}\}. \tag{2.1}$$

We denote by $\mathcal{M}_R(\mathcal{L})$ the set of $\mathcal{L}$-regular measures of $\mathcal{M}(\mathcal{L})$; $\mathcal{M}_\sigma(\mathcal{L})$ the set of σ-smooth measures on $\mathcal{L}$, of $\mathcal{M}(\mathcal{L})$; $\mathcal{M}^\sigma(\mathcal{L})$ the set of σ-smooth measures on $\mathcal{A}(\mathcal{L})$ of $\mathcal{M}(\mathcal{L})$; $\mathcal{M}_R^\sigma(\mathcal{L})$ the set of $\mathcal{L}$-regular measures of $\mathcal{M}^\sigma(\mathcal{L})$. In addition, $I(\mathcal{L})$, $I_R(\mathcal{L})$, $I_\sigma(\mathcal{L})$, $I^\sigma(\mathcal{L})$, and $I_R^\sigma(\mathcal{L})$ are the subsets of the corresponding $\mathcal{M}$'s which consist of the nontrivial zero-one valued measures.

3. Finitely subadditive and cover regular outer measures. In this section, we define finitely subadditive outer measures (f.s.a.) and (cover) regular outer measures in contrast to an ordinary outer measure which is countably subadditive. Associated with $\mu \in \mathcal{M}(\mathcal{L})$ and $\mu \in \mathcal{M}_\sigma(\mathcal{L})$ are the outer measures μ' and μ'' which were investigated in [9]. We first review some of the basic properties of these outer measures and then develop new results.

Definition 3.1. A nonnegative μ defined on $\mathcal{P}(\mathbf{X})$ is a f.s.a. outer measure if

(a) μ is nondecreasing.

(b) $\mu(\bigcup_{i=1}^n E_I) \le \sum_{i=1}^n \mu(E_i)$ for any $E_1, E_2, \dots, E_n \subset \mathbf{X}$.

(c) $\mu(\phi) = 0$.

Definition 3.2. Let ν be a f.s.a. outer measure. We say that a set E is measurable with respect to ν if, for any $A \subset \mathbf{X}$,

$$\nu(A) = \nu(A \cap E) + \nu(A \cap E'). \tag{3.1}$$

Let $\mathcal{S}_\nu$ be the set of ν-measurable sets, with ν a f.s.a. outer measure. ν is called *(cover) regular* if, for any $S \subset \mathbf{X}$, there exists $E \in \mathcal{S}_\nu$ such that $S \subset E$ and $\nu(S) = \nu(E)$. It is easy to prove that if ν is a f.s.a. outer (cover) regular measure and $\nu(\mathbf{X})$ is finite, then $E \in \mathcal{S}_\nu$ if and only if $\nu(\mathbf{X}) = \nu(E) + \nu(E')$.

Definition 3.3. Let $\mu \in \mathcal{M}(\mathcal{L})$ and define

$$\mu'(E) = \inf \sum_{i=1}^n \mu(\mathrm{L}_i'), \quad E \subset \bigcup_{i=1}^n L_i', \mathrm{L}_i \in \mathcal{L}, E \subset \mathbf{X}. \tag{3.2}$$

The definition is equivalent to

$$\mu'(E) = \inf \mu(\mathrm{L}'), \quad E \subset \mathrm{L}', \mathrm{L} \in \mathcal{L}. \tag{3.3}$$

Clearly, μ' is a f.s.a. outer measure and $E \in \mathcal{S}_{\mu''}$ if and only if

$$\mu'(A') \geq \mu'(A' \cap E) + \mu'(A' \cap E') \quad \text{for all } A \in \mathcal{L} \quad \text{(see [9]).} \tag{3.4}$$

Definition 3.4. Let $\mu \in \mathcal{M}_\sigma(\mathcal{L})$ and define

$$\mu''(E) = \inf \sum_{i=1}^{\infty} \mu(\mathrm{L}_i'), \quad E \subset \bigcup_{i=1}^{\infty} L_i', \mathrm{L}_i \in \mathcal{L}, E \subset \mathbf{X}. \tag{3.5}$$

Clearly, μ'' is a countably subadditive outer measure and $E \in \mathcal{S}_{\mu''}$ if and only if

$$\mu''(A') \geq \mu''(A' \cap E) + \mu''(A' \cap E') \quad \text{for all } A \in \mathcal{L} \quad \text{(see [9]).} \tag{3.6}$$

Clearly, for $\mu \in I(\mathcal{L})$ (or $I_\sigma(\mathcal{L})$), μ' and μ'' are regular outer measures. In addition, if $\mu \in I(\mathcal{L})$, then

$$\mathcal{S}_{\mu'} = \{E \subset \mathbf{X} \mid E \supset \mathrm{L}, \mathrm{L} \in \mathcal{L}, \mu(\mathrm{L}) = 1 \text{ or } E' \supset \mathrm{L}, \mathrm{L} \in \mathcal{L}, \mu(\mathrm{L}) = 1\}. \tag{3.7}$$

Also, if $\mu \in I_\sigma(\mathcal{L})$, then

$$\mathcal{S}_{\mu''} = \left\{E \subset \mathbf{X} \mid E \supset \bigcap_{n=1}^{\infty} L_n, \mathrm{L}_n \in \mathcal{L}, \mu(\mathrm{L}_n) = 1 \text{ or } E' \supset \bigcap_{n=1}^{\infty} L_n, \mathrm{L}_n \in \mathcal{L}, \mu(\mathrm{L}_n) = 1\right\}. \tag{3.8}$$

Furthermore, if $\mu \in I_\sigma(\mathcal{L})$, then

(a) $\mu \leq \mu'' \leq \mu'(\mathcal{L})$,

(b) $\mu'' \leq \mu' = \mu(\mathcal{L}')$, and

(c) If $\mu \in I_R^\sigma(\mathcal{L})$, then $\mu = \mu'' = \mu'(\mathcal{L})$ and $\mu'' = \mu' = \mu(\mathcal{L}')$.

These results extend readily to the general case of $\mu \in \mathcal{M}_\sigma(\mathcal{L})$.

Theorem 3.1. *Let $\mu \in \mathcal{M}_\sigma(\mathcal{L})$. Then*

(a) $\mu''(\mathbf{X}) = \mu(\mathbf{X})$,

(b) $\mu \leq \mu''(\mathcal{L})$.

Proof. (a) Suppose that $\mu''(\mathbf{X}) \leq \mu(\mathbf{X})$. Then there exists $\mathrm{L}_i \in \mathcal{L}$ such that

$$\mathbf{X} \subset \bigcup_{i=1}^{\infty} L_i' \quad \text{and} \quad \sum_{i=1}^{\infty} \mu(\mathrm{L}_i') < \mu(\mathbf{X}). \tag{3.9}$$

Hence,

$$\mu(\mathbf{X}) > \sum_{i=1}^{\infty} \mu(\mathrm{L}_i') = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(\mathrm{L}_i') \geq \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} L_i'\right) \tag{3.10}$$

with $\bigcup_{i=1}^n L_i' \in \mathcal{L}'$ and $\bigcup_{i=1}^n L_i' \uparrow \bigcup_{i=1}^\infty L_i' = \mathbf{X}$. Since $\mu \in \mathcal{M}_\sigma(\mathcal{L})$, $\lim_{n\to\infty} \mu(\bigcup_{i=1}^n L_i') = \mu(\mathbf{X})$, a contradiction.

(b)

$$\begin{aligned} \mu''(\mathrm{L}) &\geq \mu''(\mathbf{X}) - \mu''(\mathrm{L}') = \mu(\mathbf{X}) - \mu''(\mathrm{L}') \\ &\geq \mu(\mathbf{X}) - \mu'(\mathrm{L}') = \mu(\mathbf{X}) - \mu(\mathrm{L}') = \mu(\mathrm{L}), \end{aligned} \tag{3.11}$$

since $\mu' = \mu(\mathcal{L}')$. □

Definition 3.5. Let $\mu \in \mathcal{M}(\mathcal{L})$ and define

$$\mu_i(E) = \sup\{\mu(\mathrm{L}), \mathrm{L} \subset E, \mathrm{L} \in \mathcal{L}, E \subset \mathbf{X}\}. \tag{3.12}$$

The following statements are easy to prove and they can be found in [9].

(a) $\mu(\mathbf{X}) = \mu_i(\mathrm{L}) + \mu'(\mathrm{L}'), \mathrm{L} \in \mathcal{L}$.

(b) $\mu(\mathbf{X}) = \mu_i(\mathrm{L}') + \mu'(\mathrm{L}), \mathrm{L} \in \mathcal{L}$.

(c) $E \in \mathcal{S}_{\mu'}$ if and only if $\mu_i(E) = \mu'(E)$, $E \subset \mathbf{X}$.

(d) If $\mathcal{L}$ is normal, then μ_i is finitely additive on $\mathcal{L}'$.

Definition 3.6. Let $\mu \in \mathcal{M}_\sigma(\mathcal{L})$ and define, for $E \subset \mathbf{X}$,

$$\mu_j(E) = \mu(\mathbf{X}) - \mu''(E'). \tag{3.13}$$

Theorem 3.2. *Let $\mu \in \mathcal{M}_\sigma(\mathcal{L})$. Then*

(a) $\mu_i \le \mu_j \le \mu'' \le \mu'$,

(b) *If, in addition, μ'' is a (cover) regular outer measure, then*

$$E \in \mathcal{S}_{\mu''} \text{ if and only if } \mu_j(E) = \mu''(E). \tag{3.14}$$

Proof. (a) Clearly, the relation $\mu'' \le \mu'$ always holds. Now, consider

$$\begin{aligned} \mu(\mathbf{X}) - \mu_i(E) &= \mu(\mathbf{X}) - \sup\{\mu(\mathrm{L}), \mathrm{L} \subset E, \mathrm{L} \in \mathcal{L}\} \\ &= \inf\{\mu(\mathbf{X}) - \mu(\mathrm{L}), E' \subset \mathrm{L}', \mathrm{L} \in \mathcal{L}\} \\ &= \inf\{\mu(\mathrm{L}'), E' \subset \mathrm{L}', \mathrm{L} \in \mathcal{L}\} \\ &= \mu'(E'). \end{aligned} \tag{3.15}$$

Hence,

$$\mu_i(E) = \mu(\mathbf{X}) - \mu'(E') \tag{3.16}$$

and since $\mu_j(E) = \mu(\mathbf{X}) - \mu''(E')$, it follows that $\mu_i \le \mu_j$. Finally,

$$\mu_j(E) = \mu''(\mathbf{X}) - \mu''(E') \le \mu''(E) \tag{3.17}$$

since μ'' is an outer measure.

(b) Since μ'' is a regular outer measure, $E \in \mathcal{S}_{\mu''}$ if and only if $\mu''(\mathbf{X}) = \mu''(E) + \mu''(E')$. Suppose that $E \in \mathcal{S}_{\mu''}$. Since

$$\mu \in \mathcal{M}_\sigma(\mathcal{L}), \qquad \mu(\mathbf{X}) = \mu''(\mathbf{X}) = \mu''(E) + \mu''(E'), \tag{3.18}$$

hence, $\mu_j(E) = \mu''(E)$.

Conversely, assume that $\mu_j(E) = \mu''(E)$. We have

$$\mu(\mathbf{X}) = \mu_j(E) + \mu''(E') = \mu''(E) + \mu''(E'), \tag{3.19}$$

hence, $\mu''(\mathbf{X}) = \mu''(E) + \mu''(E')$. □

Definition 3.7. Let $\mu \in \mathcal{M}_\sigma(\mathcal{L})$. We say that μ satisfies

Condition (3.1). If $\mu''(\mathrm{L}') = \sup\{\mu(\tilde{L}), \tilde{L} \subset \mathrm{L}', \tilde{L} \in \mathcal{L}, \mathrm{L} \in \mathcal{L}\}$.

In particular, if $\mu \in I_\sigma(\mathcal{L})$, we say that μ satisfies

Condition (3.2). If $\mu''(\mathrm{L}') = 1$, $\mathrm{L} \in \mathcal{L}$ implies that there exists $\tilde{L} \in \mathcal{L}, \tilde{L} \subset \mathrm{L}'$, and $\mu(\tilde{L}) = 1$.

Theorem 3.3. *Let $\mu \in \mathcal{M}_\sigma(\mathcal{L})$ and suppose that μ'' is a (cover) regular outer measure and that μ satisfies condition (3.1). Then*

(a) $\mathcal{L} \subset \mathcal{S}_{\mu''}$,

(b) $\mu \leq \mu''(\mathcal{L})$, *where* $\mu''|_{\mathcal{A}(\mathcal{L})} \in \mathcal{M}^\sigma_{R(\mathcal{L})}$.

Proof. (a) By Theorem 3.2(a), we have $\mu_i \leq \mu_j \leq \mu'' \leq \mu'$. But since μ satisfies condition (3.1), it follows that $\mu_i(\mathrm{L}') = \mu''(\mathrm{L}'), \mathrm{L} \in \mathcal{L}$. Therefore, $\mu_i = \mu_j = \mu''$ on $\mathcal{L}'$. Hence, by Theorem 3.2(b), $\mathcal{L}' \subset \mathcal{S}_{\mu''}$ which implies that $\mathcal{L} \subset \mathcal{S}_{\mu''}$.

(b) By Theorem 3.1, $\mu \leq \mu''(\mathcal{L})$; $\mathcal{L} \subset \mathcal{S}_{\mu''}$ implies that $\mathcal{A}(\mathcal{L}) \subset \mathcal{S}_{\mu''}$, since $\mathcal{S}_{\mu''}$ is an algebra; μ'' a measure on $\mathcal{S}_{\mu''}$ implies that $\mu''|_{\mathcal{A}(\mathcal{L})}$ is a measure on $\mathcal{A}(\mathcal{L})$. We have

$$\mu''(\mathrm{L}') = \sup\left\{\mu(\tilde{L}), \tilde{L} \subset \mathrm{L}', \tilde{L} \in \mathcal{L}, \mathrm{L} \in \mathcal{L}\right\}. \tag{3.20}$$

Therefore, for given $\varepsilon > 0$, there exists $\tilde{L} \subset \mathrm{L}'$ such that

$$\mu''(\mathrm{L}') < \mu(\tilde{L}) + \varepsilon \leq \mu''(\tilde{L}) + \varepsilon, \tag{3.21}$$

hence, μ'' is $\mathcal{L}$-regular. Finally, $\mu'' \in \mathcal{M}_\sigma(\mathcal{L})$, $\mathcal{M}_{\mathrm{R}}(\mathcal{L})$ implies that $\mu'' \in \mathcal{M}^\sigma_{\mathrm{R}}(\mathcal{L})$. □

As a special case, we obtain the following result (see [4]).

Corollary 3.1. *Let $\mu \in I_\sigma(\mathcal{L})$ and suppose that μ satisfies condition (3.2). Then $\mathcal{L} \subset \mathcal{S}_{\mu''}$ and $\mu \leq \mu''(\mathcal{L})$, where $\mu''|_{\mathcal{A}(\mathcal{L})} \in I^\sigma_R(\mathcal{L})$.*

We next consider a pair of lattices of subsets of $\mathbf{X}$, $\mathcal{L}_1$ and $\mathcal{L}_2$, where $\mathcal{L}_1 \subset \mathcal{L}_2$, and investigate some of the above results.

Definition 3.8. Let $\mathcal{L}_1 \subset \mathcal{L}_2$ be lattices of subsets of $\mathbf{X}$ and let $\mu \in \mathcal{M}_\sigma(\mathcal{L}_1)$. We say that μ satisfies

Condition (3.3). If $\mu''(B') = \sup\{\mu(A), A \subset B', A \in \mathcal{L}_1, B \in \mathcal{L}_2\}$.

In particular, if $\mu \in I_\sigma(\mathcal{L}_1)$, we say that μ satisfies

Condition (3.4). If $\mu''(B') = 1$ for some $B \in \mathcal{L}_2$ implies that there exists $A \in \mathcal{L}_1$, $A \subset B'$, and $\mu(A) = 1$.

Theorem 3.4. *Let $\mathcal{L}_1 \subset \mathcal{L}_2$ be lattices of subsets of $\mathbf{X}$ and let $\mu \in \mathcal{M}_\sigma(\mathcal{L}_1)$. Suppose that μ'' is a (cover) regular outer measure and μ satisfies condition (3.3). Then*

(a) $\mathcal{L}_2 \subset \mathcal{S}_{\mu''}$,

(b) $\mu \leq \mu''|_{\mathcal{A}(\mathcal{L}_1)} \in \mathcal{M}^\sigma_R(\mathcal{L}_1)$ *on* $\mathcal{L}_1$, *and*

(c) $\mu'|_{\mathcal{A}(\mathcal{L}_2)} \in \mathcal{M}^\sigma_R(\mathcal{L}_2)$.

Proof. (a) Let $B \in \mathcal{L}_2$. We have

$$\mu_i(B') = \sup\{\mu(A), A \subset B', A \in \mathcal{L}_1, B \in \mathcal{L}_2\} = \mu''(B'). \tag{3.22}$$

Combining with Theorem 3.2(b), $\mu_i = \mu_j = \mu''(\mathcal{L}_2')$. μ'' regular implies that $\mathcal{L}_2' \subset \mathcal{S}_{\mu''}$, hence, $\mathcal{L}_2 \subset \mathcal{S}_{\mu''}$.

(b) and (c) Now, $\mu \in \mathcal{M}_\sigma(\mathcal{L}_1)$. Therefore, $\mu \leq \mu''(\mathcal{L}_1)$. Since $\mathcal{L}_1 \subset \mathcal{L}_2 \subset \mathcal{S}_{\mu''}$, it follows that $\mathcal{A}(\mathcal{L}_1), \mathcal{A}(\mathcal{L}_2) \subset \mathcal{S}_{\mu''}$ which is a σ-algebra. Also, μ'' being a measure on $\mathcal{S}_{\mu''}$ implies that $\mu''|_{\mathcal{A}(\mathcal{L}_1)}$, is a measure on $\mathcal{A}(\mathcal{L}_1)$ and $\mu''|_{\mathcal{A}(\mathcal{L}_2)}$ is a measure on $\mathcal{A}(\mathcal{L}_2)$. Since μ

satisfies condition (3.3), for given $\varepsilon > 0$, there exists $A \subset B'$ such that

$$\mu''(B') < \mu(A) + \varepsilon \le \mu''(A) + \varepsilon, \tag{3.23}$$

hence, μ'' is $\mathscr{L}_1$-regular and $\mathscr{L}_2$-regular. As in Theorem 3.3, it follows that $\mu''|_{\mathscr{A}(\mathscr{L}_1)} \in \mathcal{M}_R^\sigma(\mathscr{L}_1)$ and $\mu'|_{\mathscr{A}(\mathscr{L}_2)} \in \mathcal{M}_R^\sigma(\mathscr{L}_2)$. □

Corollary 3.2. *Let $\mathscr{L}_1 \subset \mathscr{L}_2$ be lattices of subsets of* $\mathbf{X}$ *and let $\mu \in I_\sigma(\mathscr{L}_1)$. If μ satisfies condition (3.4), then $\mathscr{L}_2 \subset \mathscr{S}_{\mu''}$, $\mu \le \mu''|_{\mathscr{A}(\mathscr{L}_1)} \in I_R^\sigma(\mathscr{L}_1)$ on $\mathscr{L}_1$ and $\mu''|_{\mathscr{A}(\mathscr{L}_2)} \in I_R^\sigma(\mathscr{L}_2)$.*

4. Strongly σ-smooth measures. To continue investigating $\mathscr{S}_{\mu''}$, we consider first the notion of a strongly σ-smooth measure and give some new results and extensions of some of the preceding theorems. Then we consider vaguely regular measures and their relationship with strongly σ-smooth measures.

Definition 4.1. A measure $\mu \in \mathcal{M}(\mathscr{L})$ is *strongly σ-smooth on $\mathscr{L}$* or $\mu \in \mathcal{M}(\sigma, \mathscr{L})$ if and only if for any sequence $\{\mathrm{L}_n \in \mathscr{L}\}\, \mathrm{L}_n \downarrow \mathrm{L}$ where $\mathrm{L} \in \mathscr{L}$, then

$$\mu(\mathrm{L}) = \inf_n \mu(\mathrm{L}_n) = \lim_n \mu(\mathrm{L}_n). \tag{4.1}$$

Correspondingly, for $\mu \in I(\mathscr{L})$, we have $I(\sigma, \mathscr{L})$ the set of strongly σ-smooth zero-one valued measures on $\mathscr{L}$.

Theorem 4.1. (a) *If $\mu \in \mathcal{M}(\sigma, \mathscr{L})$, then $\mu'' = \mu(\mathscr{L}')$ and $\mathscr{S}_{\mu'} \subset \mathscr{S}_{\mu''}$.*

(b) *If $\mu \in \mathcal{M}_\sigma(\mathscr{L})$ and μ'' is a (cover) regular outer measure, then $\mathscr{S}_{\mu'} \subset \mathscr{S}_{\mu''}$.*

(c) *If $\mu \in \mathcal{M}_\sigma(\mathscr{L})$ and μ'' is a (cover) regular outer measure and $\mu'' = \mu(\mathscr{L}')$, then $\mu \in \mathcal{M}(\sigma, \mathscr{L})$.*

Proof. (a) By Theorem 3.1(b), $\mu'' \le \mu(\mathscr{L}')$. Let $\mathrm{L} \in \mathscr{L}$ and $\mathrm{L}_n \in \mathscr{L}$ such that $\mathrm{L}' \subset \bigcup_{\infty}^{n=1} L_n'$. Then $\mathrm{L}' = \bigcup_{n=1}^{\infty} (L_n' \cap \mathrm{L}')$, hence,

$$\mu(\mathrm{L}') \le \sum_{n=1}^{\infty} \mu(\mathrm{L}_n' \cap \mathrm{L}') \le \sum_{n=1}^{\infty} \mu(\mathrm{L}_n'), \tag{4.2}$$

and then

$$\mu(\mathrm{L}') \le \inf\left\{\sum_{n=1}^{\infty} \mu(\mathrm{L}_n'), \mathrm{L}' \subset \bigcup_{n=1}^{\infty} L_n', \mathrm{L}, \mathrm{L}_n \in \mathscr{L}\right\} = \mu''(\mathrm{L}'). \tag{4.3}$$

Thus, $\mu'' = \mu(\mathscr{L}')$. Now, in general , $\mu' = \mu(\mathscr{L}')$, hence, $\mu' = \mu''(\mathscr{L}')$. Let $E \in \mathscr{S}_{\mu'}$ and $A \in \mathscr{L}$.

$$\mu''(A') = \mu'(A') \ge \mu'(A' \cap E) + \mu'(A' \cap E') \ge \mu''(A' \cap E) + \mu''(A' \cap E'), \tag{4.4}$$

since $\mu'' \le \mu'$ in general. Hence, $E \in \mathscr{S}_{\mu''}$ and E arbitrary in $\mathscr{S}_{\mu'}$. Therefore, $\mathscr{S}_{\mu'} \subset \mathscr{S}_{\mu''}$.

(b) By Theorem 3.2(a), $\mu_i \le \mu_j \le \mu'' \le \mu'$. Let $E \in \mathscr{S}_{\mu'}$. Then $\mu_i(E) = \mu'(E)$, hence, $\mu_i(E) = \mu_j(E) = \mu''(E) = \mu'(E)$. By Theorem 3.2(b), it follows that $E \in \mathscr{S}_{\mu''}$ and since E is arbitrary in $\mathscr{S}_{\mu'}$, it follows that $\mathscr{S}_{\mu'} \subset \mathscr{S}_{\mu''}$.

(c) Suppose that $\mu \notin \mathcal{M}(\sigma,\mathcal{L})$. Then there exist $\mathrm{L}_n \downarrow \mathrm{L}$, with $\mathrm{L}_n, \mathrm{L} \in \mathcal{L}$, and $\lim \mu(\mathrm{L}_n) > \mu(\mathrm{L}) + \varepsilon, \varepsilon > 0$. Hence,

$$\lim \mu(\mathrm{L}'_n) < \mu(\mathrm{L}') - \varepsilon. \tag{4.5}$$

But $\bigcap_{n=1}^{\infty} L_n = \mathrm{L}$ implies that $\bigcup_{n=1}^{\infty} L'_n = \mathrm{L}'$ and μ'' is regular and $\mu'' = \mu(\mathcal{L}')$, hence,

$$\lim \mu(\mathrm{L}'_n) = \lim \mu''(\mathrm{L}'_n) = \mu''(\mathrm{L}') = \mu(\mathrm{L}') \geq \lim \mu(\mathrm{L}'_n) + \varepsilon, \tag{4.6}$$

a contradiction. □

Theorem 4.2. *Let $\mu \in \mathcal{M}(\sigma,\mathcal{L})$ and suppose that μ satisfies condition (3.1). Then $\mu \in \mathcal{M}_R^{\sigma}(\mathcal{L})$.*

Proof. $\mu \in \mathcal{M}(\sigma,\mathcal{L})$ implies that $\mu'' = \mu(\mathcal{L}')$, hence,

$$\mu''(\mathrm{L}') = \mu(\mathrm{L}') = \sup\{\mu(\tilde{L}), \tilde{L} \subset \mathrm{L}', \tilde{L} \in \mathcal{L}, \mathrm{L} \in \mathcal{L}\}, \tag{4.7}$$

i.e., $\mu \in \mathcal{M}_{\mathrm{R}}(\mathcal{L})$. Therefore, $\mu \in \mathcal{M}_{\mathrm{R}}^{\sigma}(\mathcal{L})$. □

Corollary 4.1. *Let $\mu \in I(\sigma,\mathcal{L})$ and suppose that μ satisfies condition (3.2). Then $\mu \in I_R^{\sigma}(\mathcal{L})$. See* [4].

Theorem 4.3. (a) *Let $\mu \in \mathcal{M}(\mathcal{L})$. Then*

$$\mathcal{S}_{\mu'} \cap \mathcal{L} = \{\mathrm{L} \in \mathcal{L} \mid \mu(\mathrm{L}) = \mu'(\mathrm{L})\}. \tag{4.8}$$

(b) *Let $\mu \in \mathcal{M}_{\sigma}(\mathcal{L})$ and suppose that μ'' is a (cover) regular outer measure. Then*

$$\mathcal{S}_{\mu''} \cap \mathcal{L} = \{\mathrm{L} \in \mathcal{L} \mid \mu(\mathrm{L}) = \mu''(\mathrm{L})\} \tag{4.9}$$

if and only if $\mu \in \mathcal{M}(\sigma,\mathcal{L})$.

Proof. (a) Let $\mathrm{L} \in \mathcal{S}_{\mu'} \cap \mathcal{L}$. Then $\mu'(\mathrm{L}) = \mu_i(\mathrm{L}) = \mu(\mathrm{L})$ since $\mu_i = \mu(\mathcal{L})$. Conversely, let $\mathrm{L} \in \mathcal{L}$ such that $\mu(\mathrm{L}) = \mu'(\mathrm{L})$. Then $\mu'(\mathrm{L}) = \mu_i(\mathrm{L})$, i.e., $\mathrm{L} \in \mathcal{S}_{\mu'}$.

(b) Suppose that $\mu \in \mathcal{M}(\sigma,\mathcal{L})$ and let $\mathrm{L} \in \mathcal{S}_{\mu''} \cap \mathcal{L}$. By Theorem 4.1(a), $\mu'' = \mu(\mathcal{L}')$ and since $\mu''(\mathbf{X}) = \mu(\mathbf{X})$ and μ'' regular, we get

$$\mu(\mathrm{L}) = \mu''(\mathrm{L}). \tag{4.10}$$

Now, suppose that $\mathrm{L} \in \mathcal{L}$ and $\mu(\mathrm{L}) = \mu''(\mathrm{L})$. Hence,

$$\mu''(\mathbf{X}) = \mu(\mathbf{X}) = \mu(\mathrm{L}) + \mu(\mathrm{L}') = \mu''(\mathrm{L}) + \mu''(\mathrm{L}'), \tag{4.11}$$

i.e., $\mathrm{L} \in \mathcal{S}_{\mu''}$. Conversely, suppose that $\mathcal{S}_{\mu''} \cap \mathcal{L} = \{\mathrm{L} \in \mathcal{L} \mid \mu(\mathrm{L}) = \mu''(\mathrm{L})\}$. Then $\mu'' = \mu(\mathcal{L}')$ and since μ'' regular, it follows by Theorem 4.1(c) that $\mu \in \mathcal{M}(\sigma,\mathcal{L})$. □

Theorem 4.4. *Suppose that $\mathcal{L}$ is normal and let $\mu \in \mathcal{M}(\sigma,\mathcal{L})$. If $A = \bigcap_{n=1}^{\infty} B'_n$, $A \in \mathcal{L}$, $B_n \in \mathcal{L}$ all n, then $A \in \mathcal{S}_{\mu'} \subset S_{\mu''}$*

Proof. By Theorem 4.1(a), $\mathcal{S}_{\mu'} \subset \mathcal{S}_{\mu''}$. By normality, there exist C_n and $D_n \in \mathcal{L}$ such that $A \subset C_n' \subset D_n \subset B_n'$. Therefore,

$$A = \bigcap_{n=1}^{\infty} C_n' = \bigcap_{n=1}^{\infty} D_n = \bigcap_{n=1}^{\infty} B_n', \tag{4.12}$$

and we may assume that $C_n' \downarrow$ and $D_n \downarrow$. Then $\mu(A) = \lim \mu(D_n)$ since $\mu \in \mathcal{M}(\sigma, \mathcal{L})$. Hence,

$$\mu(A) \le \mu(C_n') \le \mu(D_n) \longrightarrow \mu(A) \quad \text{as } n \longrightarrow \infty. \tag{4.13}$$

Clearly,

$$\mu(A) = \lim_{n} \mu(C_n') \ge \mu'(A), \tag{4.14}$$

and so

$$\mu(A) \ge \mu'(A). \tag{4.15}$$

But, in general, $\mu \le \mu'(\mathcal{L})$, hence, $\mu(A) = \mu'(A)$. Now, by Theorem 4.3(a), it follows that $A \in \mathcal{S}_{\mu'} \cap \mathcal{L}$. □

Remark. In the zero-one valued case, we can weaken the hypothesis and the conclusion to obtain that if $\mu \in I_\sigma(\mathcal{L})$ and $\mathcal{L}$ is normal, then, for $A \in \mathcal{L}$ with $A = \bigcap_{n=1}^{\infty} B_n'$, $B_n \in \mathcal{L}$ all n, it follows that $A \in \mathcal{S}_{\mu''}$.

Definition 4.2. We say that $\mu \in \mathcal{M}_\sigma(\mathcal{L})$ satisfies

Condition 4.1. If

$$\mu(\mathrm{L}') = \sup\left\{\mu''\left(\bigcap_{n=1}^{\infty} \mathrm{L}_n\right), \bigcap_{n=1}^{\infty} \mathrm{L}_n \subset \mathrm{L}', \mathrm{L}, \mathrm{L}_n \in \mathcal{L}\right\}. \tag{4.16}$$

We say that $\mu \in \mathcal{M}_\sigma(\mathcal{L})$ satisfies

Condition 4.2. If

$$\mu''(\mathrm{L}') = \sup\left\{\mu''\left(\bigcap_{n=1}^{\infty} \mathrm{L}_n\right), \bigcap_{n=1}^{\infty} \mathrm{L}_n \subset \mathrm{L}', \mathrm{L}, \mathrm{L}_n \in \mathcal{L}\right\}. \tag{4.17}$$

Definition 4.3. Let $\mu \in \mathcal{M}_\sigma(\mathcal{L})$. μ is called *vaguely regular* if

$$\mu(\mathrm{L}') = \sup\{\mu''(\tilde{L}), \tilde{L} \subset \mathrm{L}', \tilde{L} \in \mathcal{L}, \mathrm{L} \in \mathcal{L}\}. \tag{4.18}$$

The set of vaguely regular measures on $\mathcal{L}$ is denoted by $\mathcal{M}_{\mathrm{v}}(\mathcal{L})$.

Theorem 4.5. *Let $\mu \in \mathcal{M}_\sigma(\mathcal{L})$. Then*

(a) *If μ satisfies condition (4.1), then it also satisfies condition (4.2).*

(b) *If $\mu \in \mathcal{M}(\sigma, \mathcal{L})$, then μ satisfies condition (4.1) if and only if μ satisfies condition (4.2).*

(c) *If μ satisfies condition (4.1) and μ'' is (cover) regular, then $\mu \in \mathcal{M}(\sigma, \mathcal{L})$.*

(d) *If $\mu \in \mathcal{M}_\nu(\mathcal{L})$ and μ'' is (cover) regular, then μ satisfies condition (4.1).*

(e) *If $\mu \in \mathcal{M}_\nu(\mathcal{L})$ and μ'' is (cover) regular, then $\mu \in \mathcal{M}(\sigma, \mathcal{L})$.*

Proof. In general, $\mu'' \le \mu(\mathcal{L}')$.

(a) Suppose that μ satisfies condition (4.1). Then for $\mathrm{L} \in \mathcal{L}$

$$\mu''(\mathrm{L}') \leq \mu(\mathrm{L}') = \sup\left\{\mu''\left(\bigcap_{n=1}^{\infty} \mathrm{L}_n\right), \bigcap_{n=1}^{\infty} \mathrm{L}_n \subset \mathrm{L}', \mathrm{L}, \mathrm{L}_n \in \mathcal{L}\right\} \leq \mu''(\mathrm{L}') \leq \mu(\mathrm{L}'). \quad (4.19)$$

Hence,

$$\mu''(\mathrm{L}') = \mu(\mathrm{L}') = \sup\left\{\mu''\left(\bigcap_{n=1}^{\infty} \mathrm{L}_n\right), \bigcap_{n=1}^{\infty} \mathrm{L}_n \subset \mathrm{L}', \mathrm{L}, \mathrm{L}_n \in \mathcal{L}\right\}, \quad (4.20)$$

i.e., μ satisfies condition (4.2).

(b) Suppose that μ satisfies condition (4.2). Since $\mu \in \mathcal{M}(\sigma, \mathcal{L})$, we have $\mu'' = \mu'(\mathcal{L}')$. Hence, $\mu''(\mathrm{L}') = \mu(\mathrm{L}') = \sup\{\mu''(\bigcap_{n=1}^{\infty} \mathrm{L}_n), \bigcap_{n=1}^{\infty} \mathrm{L}_n \subset \mathrm{L}', \mathrm{L}, \mathrm{L}_n \in \mathcal{L}\}$.

(c) Let $\mathrm{L} \in \mathcal{L}$. Since μ satisfies condition (4.1),

$$\mu(\mathrm{L}') = \sup\left\{\mu''\left(\bigcap_{n=1}^{\infty} \mathrm{L}_n\right), \bigcap_{n=1}^{\infty} \mathrm{L}_n \subset \mathrm{L}', \mathrm{L}, \mathrm{L}_n \in \mathcal{L}\right\}. \quad (4.21)$$

If $\mu \notin \mathcal{M}(\sigma, \mathcal{L})$, then there exist $\mathrm{L}_n \downarrow \mathrm{L}$, where $\mathrm{L}, \mathrm{L}_n \in \mathcal{L}$ and $\lim \mu(\mathrm{L}_n) > \mu(\mathrm{L}) + \varepsilon, \varepsilon > 0$. Then $\lim \mu(\mathrm{L}_n') < \mu(\mathrm{L}') - \varepsilon$, $\bigcap_{n=1}^{\infty} \mathrm{L}_n = \mathrm{L}$.

Hence, $\bigcup_{n=1}^{\infty} L_n' = \mathrm{L}'$. By condition (4.1), there exists $A_m \in \mathcal{L}$ such that

$$\bigcap_{n=1}^{\infty} A_m \subset \mathrm{L}' \quad \text{and} \quad \mu''\left(\bigcap_{n=1}^{\infty} A_m\right) > \mu(\mathrm{L}') - \varepsilon > \lim \mu(\mathrm{L}_n'). \quad (4.22)$$

But μ'' is regular, thus

$$\lim \mu''(\mathrm{L}_n') = \mu''(\mathrm{L}') \geq \mu''\left(\bigcap_{n=1}^{\infty} A_m\right) > \lim \mu(\mathrm{L}_n') \geq \lim \mu''(\mathrm{L}_n'), \quad (4.23)$$

a contradiction.

(d)

$$\begin{aligned} \mu''(\mathrm{L}') \leq \mu(\mathrm{L}') &= \sup\left\{\mu''(\tilde{L}), \tilde{L} \subset L', \tilde{L} \in \mathcal{L}, \mathrm{L} \in \mathcal{L}\right\} \\ &\leq \sup\left\{\mu''\left(\bigcap_{n=1}^{\infty} \mathrm{L}_n\right), \bigcap_{n=1}^{\infty} \mathrm{L}_n \subset \mathrm{L}', \mathrm{L}, \mathrm{L}_n \in \mathcal{L}\right\} \\ &\leq \mu''(\mathrm{L}') \leq \mu(\mathrm{L}'). \end{aligned} \quad (4.24)$$

Therefore, $\mu'' = \mu(\mathcal{L}')$ and μ'' is regular. By Theorem 4.3(c), it follows that $\mu \in \mathcal{M}(\sigma, \mathcal{L})$. Hence, $\mu(\mathrm{L}') = \sup\{\mu''(\bigcap_{n=1}^{\infty} \mathrm{L}_n), \bigcap_{n=1}^{\infty} \mathrm{L}_n \subset \mathrm{L}', \mathrm{L}, \mathrm{L}_n \in \mathcal{L}\}$.

(e) See part (d) above. Or, μ satisfies condition (4.1) by (d). Use (c) to obtain that $\mu \in \mathcal{M}(\sigma, \mathcal{L})$. □

Theorem 4.6. *Let $\mu \in \mathcal{M}_\sigma(\mathcal{L})$. If $\mu = \mu''(\mathcal{L})$ and μ'' is a (cover) regular outer measure, then*

(a) $\mathcal{L} \subset \mathcal{S}_{\mu''}$,

(b) $\mu \in \mathcal{M}^\sigma(\mathcal{L})$, *and*

(c) μ *satisfies condition (4.1).*

Proof. (a) We must show that $\mu''(\mathbf{X}) = \mu''(\mathrm{L}) + \mu''(\mathrm{L}')$, for all $\mathrm{L} \in \mathscr{L}$. If we assume that $\mu''(\mathbf{X}) < \mu''(\mathrm{L}) + \mu''(\mathrm{L}')$, we get a contradiction because $\mu'' \leq \mu(\mathscr{L}')$.

(b) Since $\mathscr{S}_{\mu''}$ is a σ-algebra, we have $\mathscr{A}(\mathscr{L}) \subset \mathscr{S}_{\mu''}$ and then $\mu''|_{\mathscr{A}(\mathscr{L})}$ is a measure on $\mathscr{A}(\mathscr{L})$. μ'' countably additive and $\mu = \mu''(\mathscr{L})$ implies that $\mu \in \mathscr{M}^{\sigma}(\mathscr{L})$.

(c) Let $\mathrm{L} \in \mathscr{L}$ and $\varepsilon > 0$. Then there exists $\mathrm{L}_n \in \mathscr{L}$ such that

$$\mathrm{L} \subset \bigcup_{n=1}^{\infty} L'_n \tag{4.25}$$

and

$$\mu(\mathrm{L}) + \varepsilon = \mu''(\mathrm{L}) + \varepsilon > \sum_{n=1}^{\infty} \mu(\mathrm{L}'_n) \geq \sum_{n=1}^{\infty} \mu''(\mathrm{L}'_n) \geq \mu''\left(\bigcup_{n=1}^{\infty} L'_n\right). \tag{4.26}$$

Therefore, since $\mathscr{L} \subset \mathscr{S}_{\mu''}$, we get

$$\mu(\mathrm{L}') - \varepsilon < \mu''\left(\bigcup_{n=1}^{\infty} \mathrm{L}_n\right). \tag{4.27}$$

□

Remark. In the case of zero-one valued measures, the above theorem was investigated in [4].

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