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LOCAL PROPERTIES OF FOURIER SERIES

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ABSTRACT. A theorem on local property of $|\bar{N},p_n|_k$ summability of factored Fourier series, which generalizes some known results, and also a general theorem concerning the $|\bar{N},p_n|_k$ summability factors of Fourier series have been proved.

Keywords and phrases. Absolute summability, Fourier series, local property.

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1. Introduction. Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \longrightarrow \infty \quad \text{as } n \longrightarrow \infty, \ (P_{-i} = p_{-i} = 0, \ i \geq 1). \tag{1.1}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.2}$$

defines the sequence (t_n) of the $(\bar{N},p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [8]).

The series $\sum a_n$ is said to be summable $|\bar{N},p_n|_k$, $k \geq 1$, if (see [4])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty. \tag{1.3}$$

In the special case when $p_n = 1$ for all values of n (resp., $k = 1$), $|\bar{N},p_n|_k$ summability is the same as $|C,1|$ (resp., $|\bar{N},p_n|$) summability. Also if we take $k = 1$ and $p_n = 1/n$ summability $|\bar{N},p_n|_k$, is equivalent to the summability $|R,\log n,1|$. A sequence (λ_n) is said to be convex if $\Delta^2\lambda_n \geq 0$ for every positive integer n, where $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

Let $f(t)$ be a periodic function with period 2π, and integrable (L) over $(-\pi,\pi)$. Without loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)\,dt = 0 \tag{1.4}$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t) \tag{1.5}$$

It is familiar that the convergence of the Fourier series at $t = x$ is a local property of f (i.e., it depends only on the behaviour of f in an arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of f. The local property problem of the factored Fourier series have been studied by several authors (see [1, 2, 5, 6, 7, 9]). Few of them are given below.

2. Mohanty [13] has demonstrated that the $|R,\log n,1|$ summability of the factored Fourier series

$$\sum \frac{A_n(t)}{\log(n+1)} \tag{2.1}$$

at $t = x$, is a local property of the generating function of f, whereas the $|C,1|$ summability of this series is not. Later on, Matsumoto [10] improved this result by replacing the series (2.1) by

$$\sum \frac{A_n(t)}{[\log\log(n+1)]^{\delta}}, \quad \delta > 1. \tag{2.2}$$

Generalizing the above result Bhatt [3] proved the following theorem.

Theorem 2.1. *If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then the summability $|R,\log n,1|$ of the series $\sum A_n(t)\lambda_n \log n$ at a point can be ensured by a local property.*

Mishra [12] has proved the following theorem by replacing the factor $(\lambda_n \log n)$ in the most general form.

Theorem 2.2. *Let the sequence (p_n) be such that*

$$P_n = O(np_n), \qquad P_n\Delta p_n = O(p_n p_{n+1}). \tag{2.3}$$

Then the summability $|\bar{N},p_n|$ of the series

$$\sum_{n=1}^{\infty} \frac{A_n(t)\lambda_n P_n}{np_n}, \tag{2.4}$$

at a point can be ensured by local property, where (λ_n) is as in Theorem 2.1.

But this theorem does not directly generalize any of the above mentioned results involving $|R,\log n,1|$ summability since order relations are not satisfied by $p_n = 1/n$.

3. The aim of this paper is to prove a more general theorem which includes some of the above mentioned results as special cases.

Now, we shall prove the following theorem.

Theorem 3.1. *Let $k \geq 1$. If (λ_n) is a convex sequence such that $\sum p_n\lambda_n$ is convergent, then the summability $|\bar{N},p_n|_k$ of the series $\sum A_n(t)\lambda_n P_n$ at a point can be ensured by a local property.*

We need the following lemmas for the proof of our theorem.

Lemma 3.2 [11]. *If (λ_n) is a convex sequence such that $\sum p_n\lambda_n$ is convergent, where (p_n) is a sequence of positive numbers such that $P_n \to \infty$ as $n \to \infty$, then (λ_n) is a nonnegative monotonic decreasing sequence tending to zero, $P_n\lambda_n = o(1)$ as $n \to \infty$ and $\sum P_n\Delta\lambda_n < \infty$.*

Lemma 3.3. *Let $k \geq 1$. If (λ_n) is a convex sequence such that $\sum p_n\lambda_n$ is convergent and (s_n) is bounded, then the series $\sum a_n\lambda_n P_n$ is summable $|\bar{N}, p_n|_k$.*

Proof. Let (T_n) be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum a_n\lambda_n P_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r\lambda_r P_r = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v\lambda_v P_v. \tag{3.1}$$

Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}P_v a_v\lambda_v. \tag{3.2}$$

By Abel's transformation, we have

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_vP_v s_v\Delta\lambda_v - \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_v s_v p_v\lambda_v \\ &\quad - \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_v p_{v+1} s_v\lambda_{v+1} + s_n p_n\lambda_n \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}. \end{aligned} \tag{3.3}$$

By Minkowski's inequality for $k > 1$, to complete the proof of Lemma 3.3, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,r}|^k < \infty, \quad \text{for } r = 1,2,3,4. \tag{3.4}$$

Now, applying Hölder's inequality with indices k and k', where $1/k + 1/k' = 1$, we get that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,1}|^k \leq \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{v=1}^{n-1}|s_v|^k P_vP_v\Delta\lambda_v\right\}\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_vP_v\Delta\lambda_v\right\}^{k-1}. \tag{3.5}$$

Since

$$\sum_{v=1}^{n-1} P_vP_v\Delta\lambda_v \leq P_{n-1}\sum_{v=1}^{n-1} P_v\Delta\lambda_v, \tag{3.6}$$

it follows by Lemma 3.2 that

$$\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_vP_v\Delta\lambda_v \leq \sum_{v=1}^{n-1} P_v\Delta\lambda_v = O(1) \quad \text{as } m \longrightarrow \infty. \tag{3.7}$$

Therefore

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,1}|^k &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}|s_v|^kP_vP_v\Delta\lambda_v\\ &= O(1)\sum_{v=1}^{m}|s_v|^kP_vP_v\Delta\lambda_v\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\ &= O(1)\sum_{v=1}^{m}P_v\Delta\lambda_v = O(1)\quad \text{as } m\longrightarrow\infty,\end{aligned}\tag{3.8}$$

by virtue of the hypotheses and Lemma 3.2. Again

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,2}|^k &\le \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{v=1}^{n-1}|s_v|^k(P_v\lambda_v)^kp_v\right\}\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}.\\ &= O(1)\sum_{v=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}|s_v|^k(P_v\lambda_v)^kp_v\\ &= O(1)\sum_{v=1}^{m}|s_v|^k(P_v\lambda_v)^kp_v\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\ &= O(1)\sum_{v=1}^{m}|s_v|^k(P_v\lambda_v)^k\frac{p_v}{P_v}\\ &= O(1)\sum_{v=1}^{m}|s_v|^k(P_v\lambda_v)^{k-1}p_v\lambda_v\\ &= O(1)\sum_{v=1}^{m}p_v\lambda_v = O(1)\quad \text{as } m\longrightarrow\infty,\end{aligned}\tag{3.9}$$

by virtue of the hypotheses and Lemma 3.2. Using the fact that $P_v < P_{v+1}$, similarly we have that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,3}|^k = O(1)\sum_{v=1}^{m}p_{v+1}\lambda_{v+1} = O(1)\quad \text{as } m\longrightarrow\infty,\tag{3.10}$$

Finally, we have that

$$\begin{aligned}\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,4}|^k &= \sum_{n=1}^{m}|s_n|^k(P_n\lambda_n)^{k-1}p_n\lambda_n\\ &= O(1)\sum_{n=1}^{m}p_n\lambda_n = O(1)\quad \text{as } m\longrightarrow\infty,\end{aligned}\tag{3.11}$$

by virtue of the hypotheses and Lemma 3.2. Therefore, we get that

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,r}|^k = O(1)\quad \text{as } m\longrightarrow\infty,\ \text{for } r=1,2,3,4.\tag{3.12}$$

This completes the proof of Lemma 3.3. □

In the particular case if we take $p_n = 1$ for all values of n in this lemma, then we get the following corollary.

Corollary 3.4. *Let $k \geq 1$. If (λ_n) is a convex sequence such that $\sum \lambda_n$ is convergent and (s_n) is bounded, then the series $\sum n a_n \lambda_n$ is summable $|C,1|_k$.*

Proof of Theorem 3.1. Since the behaviour of the Fourier series, as far as convergence is concerned, for a particular value of x depends on the behaviour of the function in the immediate neighbourhood of this point only, hence the truth of the theorem is a necessary consequence of Lemma 3.3. If we take $p_n = 1$ for all values of n in this theorem, then we get a local property result concerning the $|C,1|_k$ summability. □

Now we shall prove the following theorem for $|\bar{N}, p_n|_k$ summability factors of Fourier series.

Theorem 3.5. *Let $k \geq 1$ and let (λ_n) be a convex sequence such that $\sum p_n \lambda_n < \infty$, where (p_n) is a sequence of positive numbers such that $P_n \to \infty$. If $\sum_{v=1}^{n} P_v A_v(t) = O(P_n)$, then the series $\sum A_n(t) P_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, where $A_v(t)$ is as in (1.5).*

Proof. Let $T_n(t)$ denotes the $(\bar{N}, p_n)$ mean of the series $\sum A_n(t) P_n \lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=0}^{v} A_r(t) P_r \lambda_r = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) A_v(t) \lambda_v P_v. \tag{3.13}$$

Then, for $n \geq 1$, we have

$$T_n(t) - T_{n-1}(t) = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} P_v A_v(t) \lambda_v. \tag{3.14}$$

By Abel's transformation, we have

$$\begin{aligned} T_n(t) - T_{n-1}(t) &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta(P_{v-1}\lambda_v) \sum_{r=1}^{v} P_r A_r(t) + \frac{p_n}{P_n} \lambda_n \sum_{v=1}^{n} P_v A_v(t) \\ &= O(1) \left\{ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} (P_v \lambda_v - p_v \lambda_v - P_v \lambda_{v+1}) P_v \right\} + O(1) p_n \lambda_n \\ &= O(1) \left\{ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \lambda_v - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v p_v \lambda_v + p_n \lambda_n \right\} \\ &= O(1) \{ T_{n,1}(t) + T_{n,2}(t) + T_{n,3}(t) \}. \end{aligned} \tag{3.15}$$

Since

$$|T_{n,1}(t) + T_{n,2}(t) + T_{n,3}(t)|^k \leq 3^k \{ |T_{n,1}(t)|^k + |T_{n,2}(t)|^k + |T_{n,3}(t)|^k \}, \tag{3.16}$$

to complete the proof of Theorem 3.5, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,r}(t)|^k < \infty, \quad \text{for } r = 1, 2, 3. \tag{3.17}$$

Now, applying Hölder's inequality with indices k and k', where $1/k+1/k'=1$ and by using (3.7), we get that

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,1}(t)|^k &\leq \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{v=1}^{n-1}P_vP_v\Delta\lambda_v\right\}\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_vP_v\Delta\lambda_v\right\}^{k-1}.\\ &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_vP_v\Delta\lambda_v\\ &= O(1)\sum_{v=1}^{m}P_vP_v\Delta\lambda_v\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\ &= O(1)\sum_{v=1}^{m}P_v\Delta\lambda_v=O(1)\quad\text{as } m\longrightarrow\infty,\end{aligned}\tag{3.18}$$

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,2}(t)|^k &\leq \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{v=1}^{n-1}(P_v\lambda_v)^kp_v\right\}\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}.\\ &= O(1)\sum_{v=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}(P_v\lambda_v)^kp_v\\ &= O(1)\sum_{v=1}^{m}(P_v\lambda_v)^kp_v\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\ &= O(1)\sum_{v=1}^{m}(P_v\lambda_v)^k\frac{p_v}{P_v}\\ &= O(1)\sum_{v=1}^{m}(P_v\lambda_v)^{k-1}p_v\lambda_v\\ &= O(1)\sum_{v=1}^{m}p_v\lambda_v=O(1)\quad\text{as } m\longrightarrow\infty,\end{aligned}\tag{3.19}$$

by virtue of the hypotheses and Lemma 3.2. Finally, as in $T_{n,1}(t)$, we have that

$$\begin{aligned}\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,3}(t)|^k &= \sum_{n=1}^{m}(P_n\lambda_n)^{k-1}p_n\lambda_n\\ &= O(1)\sum_{n=1}^{m}p_n\lambda_n=O(1)\quad\text{as } m\longrightarrow\infty,\end{aligned}\tag{3.20}$$

Therefore, we get that

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,r}(t)|^k=O(1)\quad\text{as } m\longrightarrow\infty,\ \text{for } r=1,2,3.\tag{3.21}$$

This completes the proof of Theorem 3.5. □

As a special case the following results can be obtained from Theorem 3.5.

(1) If we take $p_n=1$ for all values of n, then we get a result concerning the $|C,1|_k$ summability factors of Fourier series.

(2) If we take $k = 1$ and $p_n = 1/n$, then we get another new result related to $|R, \log n, 1|$ summability factors of Fourier series.

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