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AN APPLICATION OF ALMOST INCREASING SEQUENCES

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Abstract. We extended a theorem of Mishra and Srivastava (1983–1984) on $|C,1|_k$ summability factors, using almost increasing sequences, to $|\bar{N},p_n|_k$ summability under weaker conditions.

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Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by z_n the nth $(C,1)$ mean of the sequence (s_n). The series $\sum a_n$ is said to be summable $|C,1|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} n^{k-1} \left| z_n - z_{n-1} \right|^k < \infty. \tag{1}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty, \ (P_{-i} = p_{-i} = 0, \ i \geq 1). \tag{2}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{3}$$

defines the sequence (t_n) of the $(\bar{N},p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [3]). The series $\sum a_n$ is said to be summable $|\bar{N},p_n|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} \left| t_n - t_{n-1} \right|^k < \infty. \tag{4}$$

In the special case when $p_n = 1$ for all values of n (resp., $p_n = 1/(n+1)$), $|\bar{N},p_n|_k$ summability is the same as $|C,1|_k$ (resp., $|\bar{N},1/(n+1)|_k$) summability.

Concerning the $|C,1|_k$ summability factors the following theorem is known.

Theorem 1 (see [4]). *Let (X_n) be a positive nondecreasing sequence and let (β_n) and (λ_n) be sequences such that*

$$|\Delta\lambda_n| \leq \beta_n, \tag{5}$$

$$\beta_n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{6}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty, \tag{7}$$

$$|\lambda_n|X_n = O(1), \quad as\ n \longrightarrow \infty. \tag{8}$$

If

$$\sum_{n=1}^{m} \frac{1}{n}|s_n|^k = O(X_m), \quad as\ m \longrightarrow \infty, \tag{9}$$

then the series $\sum a_n\lambda_n$ *is summable* $|C,1|_k$, $k \geq 1$.

The aim of this paper is to extend Theorem 1 to $|\bar{N},p_n|_k$ summability under weaker conditions. Thus we need the concept of almost increasing sequence. A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$. Obviously every increasing sequence is almost increasing but the converse need not be true as can be seen from the example $b_n = ne^{(-1)^n}$.

Now, we shall prove the following theorem.

Theorem 2. *Let* (X_n) *be an almost increasing sequence and let the condition (9) of Theorem 1 be satisfied. If the sequences* (β_n) *and* (λ_n) *such that conditions (5), (6), (7), and (8) of Theorem 1 are satisfied. If* (p_n) *is a sequence such that*

$$\sum_{n=1}^{m} \frac{p_n}{P_n}|s_n|^k = O(X_m), \quad as\ m \longrightarrow \infty, \tag{10}$$

then the series $\sum a_n\lambda_n$ *is summable* $|\bar{N},p_n|_k$, $k \geq 1$.

We need the following lemma for the proof of our theorem.

Lemma 3. *Under the conditions on* (X_n), (β_n), *and* (λ_n) *as taken in the statement of the theorem, the following conditions hold, when (7) is satisfied,*

$$n\beta_n X_n = O(1), \quad as\ n \longrightarrow \infty, \tag{11}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{12}$$

Proof. Let $Ac_n \leq X_n \leq Bc_n$, where (c_n) is an increasing sequence. In this case,

$$\begin{aligned} nX_n\beta_n &\leq nBc_n \left| \sum_{v=n}^{\infty} \Delta\beta_v \right| \leq nBc_n \sum_{v=n}^{\infty} |\Delta\beta_v| \\ &\leq B \sum_{v=n}^{\infty} vc_v |\Delta\beta_v| \leq \frac{B}{A} \sum_{v=n}^{\infty} v|\Delta\beta_v|X_v. \end{aligned} \tag{13}$$

Hence $n\beta_n X_n = O(1)$ as $n \to \infty$. Again

$$\begin{aligned}\sum_{n=1}^{\infty} X_n\beta_n \le B\sum_{n=1}^{\infty} c_n\beta_n &= B\sum_{n=1}^{\infty} c_n \left|\sum_{v=n}^{\infty}\Delta\beta_v\right| \\ &\le B\sum_{n=1}^{\infty} c_n \sum_{v=n}^{\infty} |\Delta\beta_v| = B\sum_{v=1}^{\infty} |\Delta\beta_v| \sum_{n=1}^{v} c_n \\ &\le B\sum_{v=1}^{\infty} v c_v |\Delta\beta_v| \le \frac{B}{A}\sum_{v=1}^{\infty} v X_v |\Delta\beta_v| < \infty.\end{aligned} \tag{14}$$

Hence $\sum_{n=1}^{\infty} X_n\beta_n < \infty$. □

Proof of the theorem. Let (T_n) be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum a_n\lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{i=0}^{v} a_i\lambda_i = \frac{1}{P_n}\sum_{v=0}^{n} (P_n - P_{v-1}) a_v\lambda_v. \tag{15}$$

Then, for $n \ge 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v. \tag{16}$$

By Abel's transformation, we have

$$\begin{aligned} T_n - T_{n-1} &= -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta(P_{v-1}\lambda_v) s_v + \frac{p_n s_n \lambda_n}{P_n} \\ &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \Delta\lambda_v + \frac{p_n s_n \lambda_n}{P_n} \\ &= T_{n,1} + T_{n,2} + T_{n,3}, \end{aligned} \tag{17}$$

let us denote the three terms in (17) by $T_{n,1}$, $T_{n,2}$, and $T_{n,3}$.

Since

$$|T_{n,1} + T_{n,2} + T_{n,3}|^k \le 3^k \left(|T_{n,1}|^k + |T_{n,2}|^k + |T_{n,3}|^k\right), \tag{18}$$

to complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1,2,3. \tag{19}$$

Since $|\lambda_n| = O(1/X_n) = O(1)$, by (8), applying Hölder's inequality with indices k and k', where $(1/k) + (1/k') = 1$ and $k > 1$, we get

$$
\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n,1}\right|^k &\le \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{v=1}^{n-1}p_v\left|\lambda_v\right|\left|s_v\right|\right)^k \\
&\le \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left(\sum_{v=1}^{n-1}p_v\left|\lambda_v\right|^k\left|s_v\right|^k\right)\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}p_v\left|\lambda_v\right|^k\left|s_v\right|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}} \\
&= O(1)\sum_{v=1}^{m}\frac{p_v}{P_v}\left|\lambda_v\right|^k\left|s_v\right|^k \\
&= O(1)\sum_{v=1}^{m}\frac{p_v}{P_v}\left|\lambda_v\right|\left|\lambda_v\right|^{k-1}\left|s_v\right|^k \qquad (20)\\
&= O(1)\sum_{v=1}^{m}\frac{p_v}{P_v}\left|\lambda_v\right|\left|s_v\right|^k \\
&= O(1)\sum_{v=1}^{m-1}\Delta\left|\lambda_v\right|\sum_{i=1}^{v}\frac{p_i}{P_i}\left|s_i\right|^k+O(1)\left|\lambda_m\right|\sum_{v=1}^{m}\frac{p_v}{P_v}\left|s_v\right|^k \\
&= O(1)\sum_{v=1}^{m-1}\left|\Delta\lambda_v\right|X_v+O(1)\left|\lambda_m\right|X_m \\
&= O(1), \quad \text{as } m\longrightarrow\infty,
\end{aligned}
$$

by virtue of (5), (8), (10), and (12). Again applying Hölder's inequality, as in $T_{n,1}$, we have

$$
\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n,2}\right|^k &\le \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{v=1}^{n-1}P_v\left|\Delta\lambda_v\right|\left|s_v\right|\right)^k \\
&\le \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{v=1}^{n-1}P_v\left|s_v\right|\beta_v\right)^k \\
&\le \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v\left|s_v\right|^k\beta_v\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_v\beta_v\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v\left|s_v\right|^k\beta_v \\
&= O(1)\sum_{v=1}^{m}P_v\left|s_v\right|^k\beta_v\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}} \\
&= O(1)\sum_{v=1}^{m}\left|s_v\right|^k\beta_v \\
&= O(1)\sum_{v=1}^{m}v\beta_v\frac{1}{v}\left|s_v\right|^k \\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{u=1}^{v}\frac{1}{u}\left|s_u\right|^k+O(1)m\beta_m\sum_{v=1}^{m}\frac{1}{v}\left|s_v\right|^k
\end{aligned}
$$

$$\begin{aligned}
&= O(1)\sum_{v=1}^{m-1} |\Delta(v\beta_v)| X_v + O(1) m\beta_m X_m \\
&= O(1)\sum_{v=1}^{m-1} v|\Delta\beta_v| X_v + O(1)\sum_{v=1}^{m-1}\beta_{v+1}X_v + O(1) m\beta_m X_m \\
&= O(1), \quad \text{as } m \longrightarrow \infty, \qquad (21)
\end{aligned}$$

by virtue of (5), (7), (9), (11), and (12). Finally, as in $T_{n,1}$, we have that

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,3}|^k = O(1)\sum_{n=1}^{m}\frac{p_n}{P_n}|\lambda_n||s_n|^k = O(1), \quad \text{as } m \longrightarrow \infty. \qquad (22)$$

Therefore we get (19). This completes the proof of the theorem. □

It should be noted that if we take (X_n) is a positive nondecreasing sequence and $p_n = 1$ for all values of n in this theorem, then we get Theorem 1. In this case the condition (10) reduces to the condition (9). Also, if we take $p_n = 1/(n+1)$ in this theorem, then we get a result concerning the $|\bar{N}, 1/(n+1)|_k$ summability factors.

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