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A NOTE ON COMMUTATIVITY OF NONASSOCIATIVE RINGS

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ABSTRACT. A theorem on commutativity of nonassociate ring is given.

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In 1968, Johnsen, Outcalt, and Yaqub [3] have established that a nonassociative ring R with identity 1 satisfying the relation $(xy)^2 = x^2y^2$ for every x and y in R, is commutative. Gupta [2] has shown that if R is a nonassociative 2-torsion free ring with unity 1 satisfying $(xy)^2 = (yx)^2$ for all x, y in R, then R is commutative. Later, Yuanchun [4] proved that a Baer-semisimple ring R is commutative if and only if $(xy)^2 - xy^2x$ is central. The existence of noncommutative ring R with $R^2 \subseteq Z(R)$, center of R, rules out the possibility that $(xy)^2 - xy^2x \epsilon Z(R)$ might yield commutativity even in associative rings. As an example, consider $A_3 = \{(a_{ij})/a_{ij}$ are integers with $a_{ij} = 0,\ i \geq j\}$. Then A_3 is a noncommutative nilpotent ring of index 3 in which $(xy)^2 - xy^2x$ is central for all x, y in A_3.

This naturally gives rise to the following question: what additional conditions are needed to insure the commutativity of R when R is an arbitrary ring? With this motivation, Ashraf, Quadri, and Zelinsky [1] established the following result.

THEOREM 1. *Let R be an associative ring with unity 1 satisfying $(xy)^2 = yx^2y$ for all x, y in R, then R is commutative.*

They used very complicated combinatorial arguments. In this connection we prove the following results.

THEOREM 2. *Let R be a nonassociative ring with unity 1 satisfying $(xy)^2 = (xy^2)x$ for all x, y in R. Then R is commutative.*

PROOF. Replacing $y+1$ for y in $(xy)^2 = (xy^2)x$, we obtain

$$\big(x(y+1)\big)^2 = \big(x(y+1)^2\big)x, \quad \text{which yields } x(xy) = (xy)x. \tag{1}$$

Repeating this argument for $x+1$ in place of x, equation (1) gives

$$x(xy) + xy = (xy)x + yx. \tag{2}$$

Thus equation (2) together with equation (1), shows that R is commutative.

Similarly, we can prove the following theorem. □

Theorem 3. *Let R be a nonassociative ring with unity 1 satisfying $(xy)^2 = (yx^2)y$ for all x, y in R. Then R is commutative.*

If we drop the restriction of unity 1 in the hypothesis, R may be badly noncommutative.

Example. Let

$$R = \left\{ \alpha I + B \;\middle|\; I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 0 & \beta & \gamma \\ 0 & 0 & \delta \\ 0 & 0 & 0 \end{pmatrix},\ \alpha, \beta, \gamma, \delta \in Z_p \right\}, \tag{3}$$

p is a prime such that p/n if n odd or $2p/n$ if n even, and Z_p is the ring of integers modulo p. Then $B^3 = 0$, for $n \geq 3$ and

$$(\alpha I + B)^n = \alpha^n I + n\alpha^{n-1} B + \frac{n(n-1)}{2!}\alpha^{n-2} B^2 + \cdots = \alpha^n I, \tag{4}$$

because $n = 0$ and $n(n-1)/2! = 0$ in Z_p, where p/n and $2p/n(n-1)$.

However, R need not be commutative.

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