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SEQUENTIAL RISK-EFFICIENT ESTIMATION OF THE PARAMETER IN THE UNIFORM DENSITY

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ABSTRACT. We develop a risk-efficient sequential procedure for estimating the parameter θ of the uniform density on $(0,\theta)$. We give explicit expressions for the distribution of the stopping time and derive its expectation and variance. We also tabulate the values of the expected stopping time and its standard deviation for some selected values of the parameter. Asymptotic properties such as efficiency and risk-efficiency are established.

Keywords and phrases. Uniform density parameter, risk-efficient estimation, sequential procedure.

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1. Introduction. The problem of obtaining a confidence interval having a specified width for the parameter in the density that is uniform on $(0,\theta)(\theta > 0)$ or on $(\xi,1)(\xi < 1)$ has been considered by Govindarajulu and others. For earlier references on this problem (see [2]). In this paper, we provide a risk-efficient sequential procedure for estimating θ or ξ.

2. Notation and the sequential procedure. Let X be distributed uniformly on $(0,\theta)$, where $\theta > 0$ and let $X_1, X_2, \dots, X_n$ be a random sample from that distribution. It is well known that $X_{nn} = \max(X_1,\dots,X_n)$ is sufficient for θ. Consider

$$\hat{\theta} = bX_{nn} \tag{2.1}$$

and minimize $E(bx_{nn} - \theta)^2$ with respect to b. We find the optimal value of b to be $(n+2)/(n+1)$. Then, the mean squared error of $(n+2)Y_{nn}/(n+1)$ is

$$E\left(\frac{n+2}{n+1}X_{nn} - \theta\right)^2 = \frac{\theta^2}{(n+1)^2}. \tag{2.2}$$

Next consider

$$R(\theta) = E\left(\frac{n+2}{n+1}X_{nn} - \theta\right)^2 + cn = \theta^2(n+1)^{-2} + cn, \tag{2.3}$$

where c is proportional to the cost of a single observation. We can write

$$R(\theta) = \theta^2 n'^{-2} + cn' - c, \quad \text{where } n' = n+1. \tag{2.4}$$

The value of n' which minimizes R can be obtained by solving $\partial R/\partial n' = 0$. This gives

$$n' = \left(\frac{2\theta^2}{c}\right)^{1/3}. \tag{2.5}$$

Hence

$$\min R(\theta) = 3\left(\frac{c\theta}{2}\right)^{2/3} + o(c^{2/3}). \tag{2.6}$$

However, since θ is unknown, we cannot compute the optimal n' given by (2.4). Hence, we resort to the following adaptive sequential rule. Stop at N, where

$$\begin{aligned} N &= \inf\left\{n \ge 1 : (n+1)^3 \ge \frac{2\hat{\theta}^2}{c}\right\} \\ &= \inf\left\{n \ge 1 : X_{nn} \le \left(\frac{c}{2}\right)^{1/2}(n+1)^{3/2}\right\}. \end{aligned} \tag{2.7}$$

3. Properties of the stopping time. Consider

$$P(N = \infty) = \lim_{n\to\infty} P(N > n), \tag{3.1}$$

where

$$\begin{aligned} P(N > n) &= P\left(X_{kk} > \left(\frac{c}{2}\right)^{1/2}(k+1)^{3/2},\ k = 1,\dots,n\right) \\ &\le P\left(X_{nn} > \left(\frac{c}{2}\right)^{1/2}(n+1)^{3/2}\right), \end{aligned} \tag{3.2}$$

which tends to zero as $n \to \infty$ since X_{nn} converges to θ almost surely (a.s.). Thus, the sequential procedure terminates finitely with probability 1. Now, let $a = (c/2)^{1/2}$. $N = n$ implies that $X_{nn} \le a(n+1)^{3/2}$ and $X_{n-1,n-1} > an^{3/2}$. Hence,

$$\left(\frac{X_{nn}}{a}\right)^{2/3} - 1 \le n < \left(\frac{X_{n-1,n-1}}{a}\right)^{2/3} \le \left(\frac{\theta}{a}\right)^{2/3}. \tag{3.3}$$

Next, we explicitly evaluate the first two moments of N. Recall that

$$\begin{aligned} P(N > n) &= P\left(U_{kk} > \frac{a(k+1)^{3/2}}{\theta},\ k = 1,\dots,n\right) \\ &= 1 - P(U_{k=1}^{n} A_k), \end{aligned} \tag{3.4}$$

where

$$A_k = \left\{U_{kk} \le \frac{a(k+1)^{3/2}}{\theta}\right\}, \quad U_{kk} = \max(U_1,\dots,U_k). \tag{3.5}$$

The sample $(U_1,\dots,U_k)$ is a random sample of size k from the standard uniform distribution. Also note that $P(N > n) = 0$ whenever $n > (\theta/a)^{2/3} - 1$ (because, then $U_{nn} > 1$). In the following lemma, we obtain a recurrence relation for evaluating the values of

$$P_n = P(U_{k=1}^{n} A_k). \tag{3.6}$$

Lemma 3.1. *Let α_k $(0<\alpha_k<1)$ be an increasing sequence of real numbers, i.e., $0<\alpha_1<\alpha_2<\cdots<1$. Then, for all n*

$$P_n - P_{n-1} = \alpha_n^n - (P_1-P_0)\alpha_n^{n-1} - (P_2-P_1)\alpha_n^{n-2} - \cdots - (P_{n-1}-P_{n-2})\alpha_n, \tag{3.7}$$

where $P_0 = 0$.

Proof. From the addition law, we have

$$\begin{aligned} P_n = P\big(U_1^n A_i\big) &= \sum_{1\le i\le n} P(A_i) - \sum_{1\le i<j\le n} P(A_iA_j) \\ &\quad + \sum_{1\le i<j<k\le n} P(A_iA_jA_k) - \cdots + (-1)^{n-1}P(A_1A_2\dots A_n). \end{aligned} \tag{3.8}$$

We also note that

$$\begin{aligned} P(A_i) &= P\big(U_{ii}\le\alpha_i\big) = P\big(U_j\le\alpha_i,\ j=1,\dots,i\big) = \alpha_i^i; \\ P(A_iA_j) &= P\big(U_{ii}\le\alpha_i, U_{jj}\le\alpha_j\big) = \alpha_i^i\alpha_j^{j-i}, \quad i<j; \\ P(A_iA_jA_k) &= \alpha_i^i\alpha_j^{j-i}\alpha_k^{k-j}, \quad i<j<k; \\ &\vdots \\ P(A_1A_2\cdots A_n) &= \alpha_1\alpha_2\cdots\alpha_n. \end{aligned} \tag{3.9}$$

Next, we write

$$\begin{aligned} \sum_{1\le i\le n} P(A_i) &= \sum_{1\le i\le n-1} P(A_i) + \alpha_n^n, \\ \sum_{1\le i<j\le n} P(A_iA_j) &= \sum_{1\le i<j\le n-1} P(A_iA_j) + \sum_{1\le i\le n-1} \alpha_i^i\alpha_n^{n-i}, \\ \sum_{1\le i<j<k\le n} P(A_iA_jA_k) &= \sum_{1\le i<j<k\le n-1} P(A_iA_jA_k) + \sum_{1\le i<j\le n-1} \alpha_i^i\alpha_j^{j-i}\alpha_n^{n-j}, \text{ etc.} \end{aligned} \tag{3.10}$$

Then, we can express $P_n - P_{n-1}$ as

$$\begin{aligned} P_n - P_{n-1} = \alpha_n^n &- \sum_{1\le i\le n-1} \alpha_i^i\alpha_n^{n-i} + \sum_{1\le i<j\le n-1} \alpha_i^i\alpha_j^{j-i}\alpha_n^{n-j} \\ &- \cdots + (-1)^{n-1}\alpha_1\alpha_2\cdots\alpha_n. \end{aligned} \tag{3.11}$$

Further expressing the right-hand side of (3.11) as a polynomial in α_n, we obtain (3.7). This completes the proof of Lemma 3.1. □

Now, using (3.7) recurrently, we can evaluate the values of the P_i's in terms of the α_i's. In particular, we have

$$\begin{aligned} &P_1 = \alpha_1, \qquad P_2 = P_1 + \alpha_2^2 - \alpha_1\alpha_2, \\ &P_3 = P_2 + \alpha_3^3 - \alpha_1\alpha_3^2 - (\alpha_2^2 - \alpha_1\alpha_2)\alpha_3, \text{ etc.} \end{aligned} \tag{3.12}$$

Thus

$$E(N) = \sum_{0\le n\le J} P(N>n) = J + 1 - \sum_{1\le n\le J} P_n, \tag{3.13}$$

where

$$J = [\eta^{2/3}] - 1, \quad \eta = \frac{\theta}{a}. \tag{3.14}$$

Writing $P(N = n) = P(N > n-1) - P(N > n)$, we can obtain

$$E(N^2) = \sum_{0 \le n \le J} n^2 P(N = n) = 2 \sum_{0 \le j \le J} jP(N > j) + EN, \tag{3.15}$$

since $P(N > J) = 0$. Consequently,

$$\text{var}\, N = J(J+1) - 2 \sum_{1 \le j \le J} jP_j + EN - (EN)^2. \tag{3.16}$$

In Table 3.1, we provide numerical values of EN and σ_N for certain selected values of η. From Table 3.1, we can see that the expected stopping time is increasing in η, whereas its standard deviation is tending to 1 as η becomes large (or as c tends to 0).

TABLE 3.1. Exact values of EN and σ_N for selected values of η.

$\eta = \theta\sqrt{2/c}$	8	27	64	125	216	343	512	729
EN	2.10	6.69	13.95	23.12	34.22	47.30	62.35	79.39
σ_N	0.89	2.28	3.01	3.50	3.92	4.30	4.64	4.95

REMARK 3.2. It should be noted that EN, when rounded upward to the nearest integer, coincides with $J = [\eta^{2/3}] - 1$.

4. Asymptotic consideration.

In this section, we derive some of the asymptotic properties of the sequential procedure which are given in the following theorem.

THEOREM 4.1. *For the sequential procedure given by (2.7).*

(i) $N/n^* \to 1$, a.s. *as* $c \to 0$,

(ii) $EN/n^* \to 1$ *as* $c \to 0$ *(asymptotic efficiency),*

(iii) $R_N(\theta)/R_{n^*}(\theta) \to 1$ *as* $c \to 0$ *(Risk-efficiency).*

PROOF. Statement (i) follows from the almost sure convergence of X_{nn} to θ.

Statement (ii) follows from the boundedness of X_{nn} by θ and Lemma 2 in [1].

Now we prove (iii). Note that from (2.6), $R_{n^*}(\theta) \approx 3(c\theta/2)^{2/3}$. Since $cEN \approx 2(c\theta/2)^{2/3}$, it suffices to show that

$$E\left(\frac{N+2}{N+1}X_{NN} - \theta\right)^2 \approx \left(\frac{c\theta}{2}\right)^{2/3} \tag{4.1}$$

or

$$\theta^2 E(U_{NN} - 1)^2 \approx \left(\frac{c\theta}{2}\right)^{2/3}, \tag{4.2}$$

where U_{NN} denotes the largest order statistic in a random sample of size N drawn from the standard uniform distribution. Now, equation (4.2) follows from Theorem 1 in reference [3] provided the following conditions are satisfied.

Let the stopping rule be

$$N = \inf\left\{n \geq m_c : X_{nn} \leq \left(\frac{c}{2}\right)^{1/2}(n+1)^{3/2}\right\} \tag{4.3}$$

with

$$c^{-\alpha} \leq m_c \leq o(c^{1/2}), \quad \text{where } 0 < \alpha < \frac{1}{2}. \tag{4.4}$$

(i) Let $S_n = n(\hat{\theta}_n - \theta)$ and

$$S_{k,n} = S_{k+n} - S_n = (n+k)(X_{k+n,k+n} - \theta) - k(X_{kn} - \theta). \tag{4.5}$$

Then there exists a $p > 2$, $B > 1$ and $k_0 \geq 1$ such that $E|S_{k_0}|^p < \infty$ and

$$E|S_{kn}|^p \leq Bn^{p/2} \quad \forall k \geq k_0, n \geq 1. \tag{4.6}$$

(ii) Let $n^2E(\hat{\theta}_n - \theta)^2 = n^2/(n+1)^2 \to \theta^2 = \zeta$. Then, for every $\varepsilon > 0$, other exists a $b_\varepsilon > 1$ such that

$$P\left(\max_{m\leq n\leq b_\varepsilon m}(\zeta - \hat{\zeta}_n) > \varepsilon\right) = o(m^{-r}) \tag{4.7}$$

for some $r > 1$ with $r > p(1-2\alpha)/(2\alpha(p-2))$, where α and p are given by (4.4) and (4.7), respectively, and $\hat{\zeta}_n$ is an estimate of ζ. In the following, we establish (4.6) and (4.7).

Lemma 4.2. *For any $\varepsilon > 0$ and $b_\varepsilon > 1$,*

$$P\left(\max_{m\leq n\leq b_\varepsilon m}(\zeta - \hat{\zeta}_n) > \varepsilon\right) \leq e^{-m\varepsilon'}, \quad \textit{where } \varepsilon' = \frac{\varepsilon}{2\theta^2}. \tag{4.8}$$

Proof. We have

$$\begin{aligned}P\left(\max_{m\leq n\leq b_\varepsilon m}(\zeta - \hat{\zeta}_n) > \varepsilon\right) &= P\left(\max_{m\leq n\leq b_\varepsilon m}(\theta^2 - \hat{\theta}_n^2) > \varepsilon\right)\\ &= P\left(\max_{m\leq n\leq b_\varepsilon m}(1 - U_{nn}^2) > \frac{\varepsilon}{\theta^2}\right)\\ &\leq P\left(\max_{m\leq n\leq b_\varepsilon m}(1 - U_{nn}) > \varepsilon'\right), \quad \varepsilon' = \frac{\varepsilon}{2\theta^2}.\end{aligned} \tag{4.9}$$

Since U_{nn} is equivalent in distribution to $e^{-\delta/n}$, where δ has the standard exponential distribution

$$\begin{aligned}P\left(\max_{m\leq n\leq b_\varepsilon m}(1 - U_{nn}) > \varepsilon'\right) &= P\left(\max_{m\leq n\leq b_\varepsilon m}(1 - e^{-\delta/n}) > \varepsilon'\right)\\ &\leq P(1 - e^{-\delta/m} > \varepsilon')\\ &= P\left(\frac{\delta}{m} > -\ln(1-\varepsilon')\right)\\ &\leq P\left(\frac{\delta}{m} > \varepsilon'\right) \leq e^{-m\varepsilon'}.\end{aligned} \tag{4.10}$$

□

Lemma 4.3. *Let $S_{k,n}$ be defined by (i). Then, for $p > 1$,*

$$E|S_{k,n}|^p = O(1). \tag{4.11}$$

Proof. We have

$$\begin{aligned} E|S_{kn}|^p &= \theta^p E\left|(U_{k+n,k+n}-1)(n+k)-k(U_{kk}-1)\right|^p \\ &\le C\theta\left[E\left((n+k)(1-U_{n+k,n+k})\right)^p + E\left(k(1-U_{kk})\right)^p\right], \end{aligned} \tag{4.12}$$

where $C = \max(2^{p-1}, 1)$ by the c_r inequality (cf. [4, page 155]). □

Now, we can write

$$E\left|n(1-U_{nn})\right|^p = p\int_0^\infty x^{p-1}P(n(1-U_{nn}) > x)\,dx, \tag{4.13}$$

where

$$P(n(1-U_{nn}) \ge x) = p\left(U_{nn} \le 1-\frac{x}{n}\right) = \left(1-\frac{x}{n}\right)^n \le e^{-x}. \tag{4.14}$$

Consequently,

$$E(n(1-U_{nn}))^p \le P\int_0^\infty x^{p-1}e^{-x}dx = p\Gamma(p) = \Gamma(p+1). \tag{4.15}$$

Thus, it readily follows that

$$E|S_{k,n}|^p = O(1) \tag{4.16}$$

as n becomes large.

The proof for (iii) of Theorem 4.1 is now complete by using Lemmas 4.2 and 4.3. □

Next, we present a lemma giving the exact distribution of S_{kn} which might be of interest elsewhere and from which we can also assert (4.6).

Lemma 4.4. *We have*

$$\begin{aligned} F_{S_{k,n}}(y) &= P(S_{k,n} \le y) \\ &= \begin{cases} k\displaystyle\int_0^{1+y/n\theta} u^{k-1}\left(1+\frac{k(u-1)+y/\theta}{n+k}\right)^n du, & -n\theta < y < 0, \\ 1-\left(1-\dfrac{y}{k\theta}\right)^k + k\displaystyle\int_0^{1-y/k\theta}\left(1+\frac{k(u-1)+y/\theta}{n+k}\right)^n u^{k-1}\,du, & 0 < y < k\theta. \end{cases} \end{aligned} \tag{4.17}$$

Proof. Note that we can write $S_{k,n}$ as

$$S_{k,n} = \theta\left[(n+k)\max\left(U_{kk}, \tilde{U}_{nn}\right) - kU_{kk} - n\right], \tag{4.18}$$

where $U_{kk} = \max(U_1, \dots, U_k)$ and $\tilde{U}_{nn} = \max(U_{k+1}, \dots, U_{k+n})$. Thus

$$
\begin{aligned}
P\left(\frac{S_{k,n}}{\theta} \le \frac{y}{\theta}\right) &= P\left(\frac{S_{k,n}}{\theta} \le \frac{y}{\theta}, U_{kk} \ge \tilde{U}_{nn}\right) + P\left(\frac{S_{k,n}}{\theta} \le \frac{y}{\theta}, U_{kk} < \tilde{U}_{nn}\right) \\
&= P\left(n(U_{kk}-1) \le \frac{y}{\theta}, U_{kk} < \tilde{U}_{nn}\right) \\
&\quad + P\left(n(\tilde{U}_{nn}-1) + k(\tilde{U}_{nn}-U_{kk}) \le \frac{y}{\theta}, U_{kk} < \tilde{U}_{nn}\right) \\
&= k\int_0^1 u^{k-1} P\left(n(\tilde{U}_{nn}-1) \le \frac{y}{\theta}, \tilde{U}_{nn} \le u\right) du \\
&\quad + k\int_0^1 u^{k-1} P\left(n(\tilde{U}_{nn}-1) + k(\tilde{U}_{nn}-u) \le \frac{y}{\theta}, \tilde{U}_{nn} > u\right) du \\
&= k\int_0^1 u^{k+n-1} I\left(u \le 1 + \frac{y}{n\theta}\right) du \\
&\quad + k\int_0^1 u^{k-1} P\left(u < \tilde{U}_{nn} \le \frac{ku+n+y/\theta}{n+k}\right) du \\
&= \frac{k}{n+k}\left[\min\left(1, 1+\frac{y}{n\theta}\right)\right]^{k+n} \\
&\quad + k\int_0^{\min(1,1+y/n\theta)} \left\{\left[\min\left(1, \frac{ku+n+y/\theta}{n+k}\right)\right]^n - u^n\right\} u^{k-1} du \\
&= k\int_0^{\min(1,1+y/n\theta)} \left[\min\left(1, \frac{ku+n+y/\theta}{n+k}\right)\right]^n u^{k-1} du.
\end{aligned}
\tag{4.19}
$$

Now

$$
\begin{aligned}
\frac{ku+n+y/\theta}{n+k} &> 1 \quad \text{if } u > 1 - \frac{y}{k\theta}, \\
&< 1 \quad \text{otherwise.}
\end{aligned}
\tag{4.20}
$$

Next we consider the cases $y > 0$ and $y < 0$.

Case 1. Let $y > 0$. Then

$$
\begin{aligned}
P(S_{k,n} \le y) &= k\int_0^{1-y/k\theta} u^{k-1}\left(\frac{ku+n+y/\theta}{n+k}\right)^n du + k\int_{1-y/k\theta}^1 u^{k-1} du \\
&= 1 - \left(1 - \frac{y}{k\theta}\right)^k + k\int_0^{1-y/k\theta}\left(1 + \frac{k(u-1)+y/\theta}{n+k}\right)^n u^{k-1} du.
\end{aligned}
\tag{4.21}
$$

Case 2. Let $y < 0$. Then

$$
P(S_{k,n} \le y) = k\int_0^{1+y/n\theta} u^{k-1}\left(1 + \frac{k(u-1)+y/\theta}{n+k}\right)^n du. \tag{4.22}
$$

This completes the proof of Lemma 4.4. □

Corollary 4.5. *The probability density function of $S_{k,n}$ is given by*

$$
f_{S_{k,n}}(y) = \frac{k}{n\theta}\left(1 + \frac{y}{n\theta}\right)^{n+k-1} + \frac{kn}{(n+k)\theta}\int_0^{1+y/n\theta} u^{k-1}\left(1 + \frac{k(u-1)+y/\theta}{n+k}\right)^{n-1} du
\tag{4.23}
$$

if $y < 0$ *and by*

$$f_{S_{k,n}}(y) = \frac{kn}{(n+k)\theta}\int_0^{1-y/k\theta}\left(1+\frac{k(u-1)+y/\theta}{n+k}\right)^{n-1}u^{k-1}du \tag{4.24}$$

if $y > 0$.

Corollary 4.6. $E|S_{k,n}|^p = O(1)$ *for all* p.

Proof. We can write

$$\begin{aligned} E\left|\frac{S_{k,n}}{\theta}\right|^p &= \frac{k}{n}\int_{-n}^0 |v|^p\left(1+\frac{v}{n}\right)^{n+k-1}dv \\ &\quad + \frac{kn}{n+k}\int_{-n}^0 |v|^p\left[\int_0^{1+v/n} u^{k-1}\left(1+\frac{k(u-1)+v}{n+k}\right)^{n-1}du\right]dv \\ &\quad + \frac{kn}{n+k}\int_0^k v^p\left[\int_0^{1-v/k}\left(1+\frac{k(u-1)+v}{n+k}\right)^{n-1}u^{k-1}du\right]dv \\ &= I_1+I_2+I_3, \end{aligned} \tag{4.25}$$

where I_1, I_2, I_3 denote the three terms in the above equation. Let $v/n = -s$,

$$\begin{aligned} I_1 &= kn^p\int_0^1 s^p(1-s)^{n+k-1}ds \\ &= kn^p\frac{\Gamma(p+1)\Gamma(n+k)}{\Gamma(n+k+p+1)} \sim kn^p\Gamma(p+1)(n+k)^{-p-1} \end{aligned} \tag{4.26}$$

and

$$I_2 = \frac{kn^{p+1}}{n+k}\int_0^1 s^p\left[\int_0^{1-s}u^{k-1}\left(1+\frac{k(u-1)-ns}{n+k}\right)^{n-1}du\right]ds. \tag{4.27}$$

Now since $[k(u-1)-ns]/(n+k) \le 1-s$,

$$I_2 \le \frac{kn^{p+1}}{n+k}\int_0^1 s^p\frac{(1-s)^{n-1+k}}{k}ds = \frac{n^{p+1}\Gamma(p+1)\Gamma(n+k)}{(n+k)\Gamma(n+k+p+1)} = O\big((n+k)^{-1}\big). \tag{4.28}$$

Next since $1+[k(u-1)+v](n+k)^{-1} \le 1$,

$$I_3 \le \frac{n}{n+k}\int_0^k v^p\left(1-\frac{v}{k}\right)^k dv = \frac{n}{n+k}k^{p+1}\int_0^1 s^p(1-s)^k ds = O(1). \tag{4.29}$$

Thus, $E|S_{k,n}/\theta|^p = O(1)$. □

Concluding remark. In order to solve the dual problem of estimating ξ when X is distributed uniformly on $(\xi,1)$, change θ to $1-\xi$ and X_{nn} to $1-X_{1n}$, where $X_{1n} = \min(X_1,\dots,X_n)$.

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