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ON S-CLUSTER SETS AND S-CLOSED SPACES

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ABSTRACT. A new type of cluster sets, called S-cluster sets, of functions and multifunctions between topological spaces is introduced, thereby providing a new technique for studying S-closed spaces. The deliberation includes an explicit expression of S-cluster set of a function. As an application, characterizations of Hausdorff and S-closed topological spaces are achieved via such cluster sets.

Keywords and phrases. S-cluster set, S-closedness, θ_s-closure, semi-open sets.

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1. Introduction. The theory of cluster sets was developed long ago, and was initially aimed at the investigations of real and complex function theory (see [15]). A comprehensive collection of works in this direction can be found in the classical book of Collingwood and Lohwater [1]. Weston [14] was the first to initiate the corresponding theory for functions between topological spaces basically for studying compactness. Many others (e.g., see [3, 4, 6]) followed suit with cluster sets, θ-cluster sets and δ-cluster sets of functions and multifunctions, ultimately implicating different covering properties, among other things.

The present paper is intended for the introduction of a new type of cluster sets, called S-cluster sets, which provides a new technique for the study of S-closedness of topological spaces. It is shown that such cluster sets of suitable function can characterize Hausdorffness. Finally, we achieve, as our prime motivation, certain characterizations of an S-closed space.

In what follows, X and Y denote topological spaces, and $f : X \to Y$ is a function from X into Y. By a multifunction $F : X \to Y$ we mean, as usual, a function mapping points of X into the nonempty subsets of Y. The set of all open sets of (X,τ), each containing a given point x of X, is denoted by $\tau(x)$. A set $A\,(\subseteq X)$ is called semi-open [5] if for some open set U, $U \subseteq A \subseteq \operatorname{cl} U$, where $\operatorname{cl} U$ denotes the closure of U in X. The set of all semi-open sets of X, each containing a given subset A of X, is denoted by $\operatorname{SO}(A)$, in particular, if $A = \{x\}$, we write $\operatorname{SO}(x)$ instead of $\operatorname{SO}(\{x\})$. The complements of semi-open sets are called semi-closed. For any subset A of X, the θ-closure [13] (θ-semiclosure [8]) of A, denoted by $\theta\text{-}\operatorname{cl} A$ (respectively, $\theta_s\text{-}\operatorname{cl} A$), is the set of all points x of X such that for every $U \in \tau(x)$ (respectively, $U \in \operatorname{SO}(x)$), $\operatorname{cl} U \cap A \neq \emptyset$. The set A is called θ-closed [13] (θ-semiclosed [8]) if $A = \theta\text{-}\operatorname{cl} A$ (respectively, $A = \theta_s\text{-}\operatorname{cl} A$). It is known [9] that $\theta\text{-}\operatorname{cl} A$ need not be θ-closed, but it is so if A is open. A nonvoid collection Ω of nonempty subsets of a space X is called a grill [12] if

(i) $A \in \Omega$ and $B \supseteq A \Rightarrow B \in \Omega$,

(ii) $A \cup B \in \Omega \Rightarrow A \in \Omega$ or $B \in \Omega$.

A filterbase $\mathcal{F}$ on a space X is said to θ_S-adhere [11] at a point x of X, denoted as $x \in \theta_S\text{-ad}\,\mathcal{F}$, if $x \in \cap\{\theta_s\text{-cl}\,F : F \in \mathcal{F}\}$. A grill Ω on X is said to θ_s-converge to a point x of X, if to each $U \in \mathrm{SO}(x)$, there corresponds some $G \in \Omega$ with $G \subseteq \mathrm{cl}\,U$. A set A in a space X is said to be S-closed relative to X [7] if for every cover $\mathcal{U}$ of A by semi-open sets of X, there exists a finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \subseteq \cup\{\mathrm{cl}\,U : U \in \mathcal{U}_0\}$. If, in addition, $A = X$, then X is called an S-closed space [11].

2. Main theorem and associated results.

We begin by introducing S-cluster set of a function and of a multifunction between two topological spaces.

Definition 2.1. Let $F : X \to Y$ be a multifunction and $x \in X$. Then the S-cluster set of F at x, denoted by $S(F,x)$, is defined to be the set $\cap\{\theta\text{-cl}\,F(\mathrm{cl}\,U) : U \in \mathrm{SO}(x)\}$. Similarly, for any function $f : X \to Y$, the S-cluster set $S(f,x)$ of f at x is given by $\cap\{\theta\text{-cl}\,f(\mathrm{cl}\,U) : U \in \mathrm{SO}(x)\}$.

In the next theorem, we characterize the S-cluster sets of functions between topological spaces.

Theorem 2.2. *For any function $f : X \to Y$, the following statements are equivalent.*
(a) $y \in S(f,x)$.
(b) *The filterbase $f^{-1}(\mathrm{cl}\,\tau(y))\,\theta_S$-adheres at x.*
(c) *There is a grill Ω on X such that $\Omega\,\theta_s$-converges to x and $y \in \cap\{\theta\text{-cl}\,f(G) : G \in \Omega\}$.*

Proof. (a)$\Longrightarrow$(b). $y \in S(f,x) \Rightarrow$ for each $W \in \mathrm{SO}(x)$ and each $V \in \tau(y)$, $\mathrm{cl}\,V \cap f(\mathrm{cl}\,W) \neq \varnothing \Rightarrow$ for each $W \in \mathrm{SO}(x)$ and each $V \in \tau(y)$, $f^{-1}(\mathrm{cl}\,V) \cap \mathrm{cl}\,W \neq \varnothing$. This ensures that the collection $\{f^{-1}(\mathrm{cl}\,V) : V \in \tau(y)\}$ (which can easily be seen to be a filterbase on X) θ_S-adheres at x.

(b)$\Longrightarrow$(c). Let $\mathcal{F}$ be the filter generated by the filterbase $f^{-1}(\mathrm{cl}\,\tau(y))$. Then $\Omega = \{G \subseteq X : G \cap F \neq \varnothing$, for each $F \in \mathcal{F}\}$ is a grill on X. By the hypothesis, for each $U \in \mathrm{SO}(x)$ and each $V \in \tau(y)$, $\mathrm{cl}\,U \cap f^{-1}(\mathrm{cl}\,V) \neq \varnothing$. Hence, $F \cap \mathrm{cl}\,U \neq \varnothing$ for each $F \in \mathcal{F}$ and each $U \in \mathrm{SO}(x)$. Consequently, $\mathrm{cl}\,U \in \Omega$ for all $U \in \mathrm{SO}(x)$, which proves that $\Omega\theta_s$-converges to x. Now, the definition of Ω yields that $f(G) \cap \mathrm{cl}\,W \neq \varnothing$ for all $W \in \tau(y)$ and all $G \in \Omega$, i.e., $y \in \theta\text{-cl}\,f(G)$ for all $G \in \Omega$. Hence, $y \in \cap\{\theta\text{-cl}\,f(G) : G \in \Omega\}$.

(c)$\Longrightarrow$(a). Let Ω be a grill on X such that $\Omega\theta_s$-converges to x, and $y \in \cap\{\theta\text{-cl}\,f(G) : G \in \Omega\}$. Then $\{\mathrm{cl}\,U : U \in \mathrm{SO}(x)\} \subseteq \Omega$ and $y \in \theta\text{-cl}\,f(G)$ for each $G \in \Omega$. Hence, in particular, $y \in \theta\text{-cl}\,f(\mathrm{cl}\,U)$ for all $U \in \mathrm{SO}(x)$. So, $y \in \cap\{\theta\text{-cl}\,f(\mathrm{cl}\,U) : U \in \mathrm{SO}(x)\} = S(f,x)$. □

In what follows, we show that S-cluster sets of a function may be used to ascertain the Hausdorffness of the codomain space.

Theorem 2.3. *Let $f : X \to Y$ be a function on a topological space X onto a topological space Y. Then Y is Hausdorff if $S(f,x)$ is degenerate for each $x \in X$.*

Proof. Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. As f is a surjection, there are $x_1, x_2 \in X$ such that $f(x_i) = y_i$ for $i = 1,2$. Now, since $S(f,x)$ is degenerate for each $x \in X$, $y_2 = f(x_2) \notin S(f,x_1)$. Thus, there are $V \in \tau(y_2)$ and $U \in \mathrm{SO}(x_1)$ such that $\mathrm{cl}\,V \cap f(\mathrm{cl}\,U) =$

$\emptyset$, i.e., $f(\text{cl}\,U) \subseteq Y - \text{cl}\,V$. Then the open sets $Y - \text{cl}\,V$ and V strongly separate y_1 and y_2 in Y, which proves that Y is Hausdorff. □

Remark 2.4. We note that the converse of the above theorem is false. For example, consider the identity map $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$, where τ and σ, respectively, denote the cofinite topology and the usual topology on the set $\mathbb{R}$ of real numbers. Then $S(f,x) = \mathbb{R}$, for each $x \in \mathbb{R}$, though $(\mathbb{R}, \sigma)$ is a T_5-space.

In order to obtain the converse, we introduce the following class of functions.

Definition 2.5. A function $f : X \to Y$ is called θ_s-irresolute on X if for each $x \in X$ and each semi-open set V containing $f(x)$, there is a semi-open set U containing x such that $f(\text{cl}\,U) \subseteq V$.

Theorem 2.6. *Let $f : X \to Y$ be a θ_s-irresolute function with Y a Hausdorff space. Then $S(f,x)$ is degenerate for each $x \in X$.*

Proof. Let $x \in X$. As f is θ_s-irresolute on X, for any $V \in \text{SO}(f(x))$, there is $U \in \text{SO}(x)$ such that $f(\text{cl}\,U) \subseteq V$. Then $S(f,x) = \cap\{\theta\text{-cl}\,f(\text{cl}\,U) : U \in \text{SO}(x)\} \subseteq \cap\{\theta\text{-cl}\,V : V \in \text{SO}(f(x))\}$. Let $y \in Y$ with $y \neq f(x)$. As Y is Hausdorff, there are disjoint open sets U, W with $y \in U$, $f(x) \in W$. Obviously, as $U \cap \text{cl}\,W = \emptyset$, $y \notin \text{cl}\,W = \theta\text{-cl}\,W$. As $W \in \tau(f(x)) \subseteq \text{SO}(f(x))$, $y \notin \cap\{\theta\text{-cl}\,V : V \in \text{SO}(f(x))\}$ and hence $y \notin S(f,x)$. Thus, $S(f,x) = \{f(x)\}$. □

Combining the last two results, we get the following characterization for the Hausdorffness of the codomain space of a kind of function in terms of the degeneracy of its S-cluster set.

Corollary 2.7. *Let $f : X \to Y$ be a θ_s-irresolute function on X onto Y. Then the space Y is Hausdorff if and only if $S(f,x)$ is degenerate for each x of X.*

We have just seen that degeneracy of the S-cluster set of an arbitrary function is a sufficient condition for the Hausdorffness of the codomain space. We thus like to examine some other situations when the S-cluster sets are degenerate, thereby ensuring the Hausdorffness of the codomain space of the function concerned. To this end, we recall that a topological space (X,τ) is almost regular [10] if for every regular closed set A in X and for each $x \notin A$, there exist disjoint open sets U and V such that $x \in U$ and $A \subseteq V$. It is known that in an almost regular space X, $\theta\text{-cl}\,A$ is θ-closed for each $A \subseteq X$. A function $f : X \to Y$ carrying θ-closed sets of X into θ-closed sets of Y is called a θ-closed function [2].

Theorem 2.8. *Let $f : X \to Y$ be a θ-closed map from an almost regular space into a space Y. If $f^{-1}(y)$ is θ-closed in X for all $y \in Y$, then $S(f,x)$ is degenerate for each $x \in X$.*

Proof. We have $S(f,x) = \cap\{\theta\text{-cl}\,f(\text{cl}\,U) : U \in \text{SO}(x)\} \subseteq \cap\{\theta\text{-cl}\,f(\theta\text{-cl}\,U) : U \in \text{SO}(x)\}$. As X is almost regular, $\theta\text{-cl}\,U$ is θ-closed for all $U \in \text{SO}(x)$. Now, since f is a θ-closed map, $\theta\text{-cl}\,f(\theta\text{-cl}\,U) = f(\theta\text{-cl}\,U)$ for each $U \in \text{SO}(x)$. Thus, $S(f,x) \subseteq \cap\{f(\theta\text{-cl}\,U) : U \in \text{SO}(x)\}$. Now, let $y \in Y$ such that $y \neq f(x)$. Then since $f^{-1}(y)$ is θ-closed and $x \notin f^{-1}(y)$, there is some $P \in \tau(x)$ such that $\text{cl}\,P \cap f^{-1}(y) = \emptyset$. So,

$y \notin f(\mathrm{cl}\, P) = f(\theta\text{-}\mathrm{cl}\, P)$ (as P is an open set) and, hence, $y \notin \cap\{f(\theta\text{-}\mathrm{cl}\, U) : U \in \mathrm{SO}(x)\}$. In view of what we have deduced above, we conclude that $y \notin S(f,x)$, which proves that $S(f,x)$ is degenerate. □

Theorem 2.9. *Let $f : X \to X$ be a θ-closed injection on an almost regular Hausdorff space X into Y. Then $S(f,x)$ is degenerate for each $x \in X$.*

Proof. As X is almost regular and f is a θ-closed map, we have $\theta\text{-}\mathrm{cl}\, f(\theta\text{-}\mathrm{cl}\, U) = f(\theta\text{-}\mathrm{cl}\, U)$ for any $U \in \mathrm{SO}(x)$ and, hence,

$$\begin{aligned} S(f,x) &= \cap\{\theta\text{-}\mathrm{cl}\, f(\mathrm{cl}\, U) : U \in \mathrm{SO}(x)\} \subseteq \cap\{\theta\text{-}\mathrm{cl}\, f(\theta\text{-}\mathrm{cl}\, U) : U \in \mathrm{SO}(x)\} \\ &= \cap\{f(\theta\text{-}\mathrm{cl}\, U) : U \in \mathrm{SO}(x)\}. \end{aligned} \tag{2.1}$$

For x, $x_1 \in X$ with $x \neq x_1$, $f(x) \neq f(x_1)$ as f is injective. By the Hausdorffness of X, there are disjoint open sets U, V in X with $x \in U$, $x_1 \in V$. Obviously, $U \cap \mathrm{cl}\, V = \emptyset$. SO, $x_1 \notin \theta\text{-}\mathrm{cl}\, U$ and hence $f(x_1) \notin f(\theta\text{-}\mathrm{cl}\, U)$. Since $U \in \tau(x) \subseteq \mathrm{SO}(x)$, equation (2.1) yields $f(x_1) \notin S(f,x)$. Thus, $S(f,x)$ is degenerate for each $x \in X$. □

The above theorem is equivalent to the following apparently weaker result when X is regular.

Theorem 2.10. *If $f : X \to Y$ is a θ-closed injection on a T_3 space X into a space Y, then $S(f,x)$ is degenerate for each $x \in X$.*

Proof. It is known that in a regular space $X, \theta\text{-}\mathrm{cl}\, U = \mathrm{cl}\, U$ for any $U \subseteq X$. Since X is T_3 and f is a θ-closed injection, $\{f(x)\} \subseteq S(f,x) = \cap\{f(\mathrm{cl}\, U) : U \in \mathrm{SO}(x)\} \subseteq \cap\{f(\mathrm{cl}\, U) : U \in \tau(x)\} = \{f(x)\}$. □

Note that the above result is indeed equivalent to that of Theorem 2.9 follows from the following considerations: for any subset A of a topological space (X,τ), θ-closure of A in (X,τ) is the same as that in (X,τ_s), where (X,τ_s) denotes the semiregularization space [9] of (X,τ). Moreover, it is known [9] that (X,τ) is Hausdorff (almost regular) if and only if (X,τ_s) is Hausdorff (respectively, regular). Now, since $\mathrm{SO}(X,\tau_s) \subseteq \mathrm{SO}(X,\tau)$, it follows that $S(f,x) = S(f : (X,\tau) \to Y, x) \subseteq S(f : (X,\tau_s) \to Y, x)$. So, $S(f,x)$ is degenerate for each $x \in X$ if (X,τ) is an almost regular Hausdorff space and $f : X \to Y$ is a θ-closed injection.

A sort of degeneracy condition for the S-cluster set of a multifunction with θ-closed graph is now obtained.

Theorem 2.11. *For a multifunction $F : X \to Y$, if F has a θ-closed graph, then $S(F,x) = F(x)$.*

Proof. For any $y \in S(F,x)$, $\mathrm{cl}\, W \cap F(\mathrm{cl}\, U) \neq \emptyset$ and hence $F^-(\mathrm{cl}\, W) \cap \mathrm{cl}\, U \neq \emptyset$ for each $U \in \mathrm{SO}(x)$ and each $W \in \tau(y)$, where, as usual, $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$ for any subset B of Y. Then for any basic open set $M \times N$ in $X \times Y$ containing (x,y), $F^-(\mathrm{cl}\, N) \cap \mathrm{cl}\, M \neq \emptyset$. So, $(\mathrm{cl}\, M \times \mathrm{cl}\, N) \cap G(F) \neq \emptyset$ and hence $\mathrm{cl}\,(M \times N) \cap G(F) \neq \emptyset$, where $G(F) = \{(x,y) \in X \times Y : y \in F(x)\}$ denotes the graph of F. So, $(x,y) \in \theta\text{-}\mathrm{cl}\, G(F) = G(F)$ (as $G(F)$ is θ-closed). Hence, $(x,y) \in [G(F) \cap (\{x\} \times Y)]$ so that $y \in p_2[(\{x\} \times Y) \cap G(F)] = F(x)$, where $p_2 : X \times Y \to Y$ is the second projection map.

It is obvious that $F(x) \subseteq S(F,x)$ for each $x \in X$. Hence, $S(F,x) = F(x)$ holds for all $x \in X$. □

The next result serves as a partial converse of the above one.

Theorem 2.12. *For a multifunction $F : X \to Y$, if $S(F,x) = F(x)$ for each $x \in X$, then the graph $G(F)$ of F is θ-semiclosed (and hence semi-closed).*

Proof. Let $(x,y) \in X \times Y - G(F)$. Now, $y \notin F(x) = S(F,x) \Rightarrow$ there exist some $W \in \mathrm{SO}(x)$ and some $V \in \tau(y)$ such that $\mathrm{cl}\, V \cap F(\mathrm{cl}\, W) = \varnothing \Rightarrow (\mathrm{cl}\, W \times \mathrm{cl}\, V) \cap G(F) = \varnothing \Rightarrow \mathrm{cl}\,(W \times V) \cap G(F) = \varnothing$. As $W \times V$ is a semi-open set in $X \times Y$ containing (x,y), $(x,y) \notin \theta_s\text{-}\mathrm{cl}\, G(F)$. Hence, $G(F)$ is θ-semi-closed. □

We now turn our attention to the characterizations of S-closedness via S-cluster sets. We need the following lemmas for this purpose.

Lemma 2.13. *A set A in a topological space X is an S-closed set relative to X if and only if for every filterbase $\mathcal{F}$ on X with $F \cap C \neq \varnothing$ for all $F \in \mathcal{F}$ and for all $C \in \mathrm{SO}(A)$, $A \cap \theta_S$-ad $\mathcal{F} \neq \varnothing$.*

Proof. Let A be an S-closed set relative to X and let $\mathcal{F}$ be a filterbase on X with the stated property. If possible, suppose that $A \cap \theta_S$-ad $\mathcal{F} = \varnothing$. Then for each $x \in A$, there is a semi-open set $V(x)$ in X containing x such that $\mathrm{cl}\,(V(x)) \cap F(x) = \varnothing$ for some $F(x) \in \mathcal{F}$. Now, $\{V(x) : x \in A\}$ is a cover of A by semi-open sets of X. By the S-closedness of A relative to X, there is a finite subset A^* of A such that $A \subseteq \cup\{\mathrm{cl}\, V(x) : x \in A^*\}$. Choose $F^* \in \mathcal{F}$ such that $F^* \subseteq \cap\{F(x) : x \in A^*\}$. Then $F^* \cap (\cup\{\mathrm{cl}\, V(x) : x \in A^*\}) = \varnothing$, i.e., $F^* \cap \mathrm{cl}\,(\cup\{V(x) : x \in A^*\}) = \varnothing$. Now, as $\cup\{V(x) : x \in A^*\}$ is a semi-open set in X,$\cup\{\mathrm{cl}\, V(x) : x \in A^*\} \in \mathrm{SO}\,(A)$, a contradiction.

Conversely, assume that A is not S-closed relative to X. Then for some cover $\{U_\alpha : \alpha \in \Lambda\}$ of A by semi-open sets of X, $A \not\subset \bigcup_{\alpha\in\Lambda_0} \mathrm{cl}\, U_\alpha$ for each finite subset Λ_0 of Λ. So, $\mathcal{F} = \{A - \bigcup_{\alpha\in\Lambda_0} \mathrm{cl}\, U_\alpha : \Lambda_0$ is a finite subset of $\Lambda\}$ is filterbase on X, with $F \cap C \neq \varnothing$, for each $F \in \mathcal{F}$ and each $C \in \mathrm{SO}\,(A)$. But $A \cap \theta_S$-ad $\mathcal{F} = \varnothing$. □

Lemma 2.14 [8, 11]. (a) *A topological space X is S-closed if and only if every filterbase θ_S-adheres in X.*

(b) *Any θ-semiclosed subset of an S-closed space X is S-closed relative to X.*

Definition 2.15. For a function or a multifunction $F : X \to Y$ and a set $A \subseteq X$, the notation $S(F,A)$ stands for the set $\cup\{S(F,x) : x \in A\}$.

Theorem 2.16. *For any topological space X, the following statements are equivalent.*

(a) *X is S-closed.*

(b) $S(F,A) \supseteq \cap\{\theta\text{-}\mathrm{cl} F(U) : U \in \mathrm{SO}\,(A)\}$ *for each θ-semiclosed subset A of X, for each topological space Y and each multifunction $F : X \to Y$.*

(c) $S(F,A) \supseteq \cap\{\theta_s\text{-}\mathrm{cl}\, F(U) : U \in \mathrm{SO}\,(A)\}$ *for each θ-semiclosed subset A of X, for each topological space Y and each multifunction $F : X \to Y$.*

Proof. (a)⟹(b). Let A be any θ-semiclosed subset of X, where X is S-closed. Then by Lemma 2.14(b), A is S-closed relative to X. Now, let $z \in \cap\{\theta\text{-}\mathrm{cl}\, F(W) : W \in \mathrm{SO}\,(A)\}$.

Then for all $W \in \tau(z)$ and for each $U \in \mathrm{SO}(A)$, $\mathrm{cl}\, W \cap F(U) \neq \varnothing$, i.e., $F^{-}(\mathrm{cl}\, W) \cap U \neq \varnothing$. Thus, $\mathcal{F} = \{F^{-}(\mathrm{cl}\, W) : W \in \tau(z)\}$ is clearly a filterbase on X, satisfying the condition of Lemma 2.13. Hence, $x \in A \cap \theta_S\text{-ad}\, \mathcal{F}$. Then $x \in A$, and for all $U \in \mathrm{SO}(x)$ and each $W \in \tau(z)$, $\mathrm{cl}\, U \cap F^{-}(\mathrm{cl}\, W) \neq \varnothing$, i.e., $F(\mathrm{cl}\, U) \cap \mathrm{cl}\, W \neq \varnothing \Rightarrow z \in S(F,x) \subseteq S(F,A)$.

(b)$\Longrightarrow$(c). Obvious.

(c)$\Longrightarrow$(a). In order to show that X is S-closed, it is enough to show, by virtue of Lemma 2.14(a), that every filterbase $\mathcal{F}$ on X θ_S-adheres at some $x \in X$. Let $\mathcal{F}$ be a filterbase on X. Take $y_0 \notin X$, and construct $Y = X \cup \{y_0\}$. Define, $\tau_Y = \{U \subseteq Y : y_0 \notin U\} \cup \{U \subseteq Y : y_0 \in U, F \subseteq U$ for some $F \in \mathcal{F}\}$. Then τ_Y is a topology on Y. Consider the function $\alpha : X \to Y$ by $\alpha(x) = x$. In order to avoid possible confusion, let us denote the closure and θ_s-closure of a set A in $X(Y)$, respectively, by $\mathrm{cl}_X A(\mathrm{cl}_Y A)$ and $\theta_s\text{-}\mathrm{cl}_X A$ (respectively, $\theta_s\text{-}\mathrm{cl}_Y A$). As X is θ-semiclosed in X, by the given condition, $S(\alpha,X) \supseteq \cap\{\theta_s\text{-}\mathrm{cl}_Y\, \alpha(U) : U \in \mathrm{SO}(X)\} = \cap\{\theta_s\text{-}\mathrm{cl}_Y\, U : U \in \mathrm{SO}(X)\} = \theta_s\text{-}\mathrm{cl}_Y X$. We consider $y_0 \in Y$ and $P_0 \in \mathrm{SO}(y_0)$. There is some $W \in \tau_Y$ such that $W \subseteq P_0 \subseteq \mathrm{cl}_Y W$. If $y_0 \notin W$, then $W \subseteq X$ and hence $\mathrm{cl}_Y W \cap X \neq \varnothing$. If on the other hand, $y_0 \in W$, then there is some $F \in \mathcal{F}$ such that $F \subseteq W$, i.e., $\mathrm{cl}_Y F \subseteq \mathrm{cl}_Y W$. So, $X \cap \mathrm{cl}_Y W \neq \varnothing$. So, in any case, $X \cap \mathrm{cl}_Y W \neq \varnothing$ and, consequently, as $\mathrm{cl}_Y W = \mathrm{cl}_Y P_0$, $X \cap \mathrm{cl}_Y P_0 \neq \varnothing$. Thus, $y_0 \in \theta_s\text{-}\mathrm{cl}_Y X$. So, $y_0 \in S(\alpha,x)$ for some $x \in X$. Consider any $V \in \mathrm{SO}(x)$ and $F \in \mathcal{F}$. Then $F \cup \{y_0\} \in \tau_Y$. Again, $Y - (F \cup \{y_0\})$ is a subset of Y not containing y_0. Thus, $Y - (F \cup \{y_0\})$ is open in Y, which proves that $\mathrm{cl}_Y(F \cup \{y_0\}) = F \cup \{y_0\}$. Now, $\mathrm{cl}_X V \cap F = \alpha(\mathrm{cl}_X V) \cap \mathrm{cl}_Y(F \cup \{y_0\}) \neq \varnothing$. Thus, $x \in \theta_S\text{-ad}\, \mathcal{F}$. □

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