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THE SECOND DUAL SPACES OF THE SETS OF Λ-STRONGLY CONVERGENT AND BOUNDED SEQUENCES

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ABSTRACT. We give the second β-, γ-, and f-duals of the sets $w_0^p(\Lambda)$, $w_\infty^p(\Lambda)$ $(0 < p < \infty)$, $c_0^p(\Lambda)$, $c^p(\Lambda)$, and $c_\infty^p(\Lambda)$ $(0 < p \le 1)$ and the second continuous dual spaces of $w_0^p(\Lambda)$, $c_0^p(\Lambda)$, and $c^p(\Lambda)$ for $0 < p \le 1$. Furthermore, we determine the α-duals of $c_0^p(\Lambda)$, $c^p(\Lambda)$, and $c_\infty^p(\Lambda)$ for $1 < p < \infty$.

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1. Introduction and well-known results. We write ω for the set of all complex sequences $x = (x_k)_{k=0}^\infty$, ϕ, l_∞, c and c_0 for the sets of all finite, bounded, convergent sequences, and sequences convergent to naught, respectively, further cs, bs, and l_1 for the sets of all convergent, bounded, and absolutely convergent series.

By e and $e^{(n)}$ $(n \in \mathbb{N}_0)$, we denote the sequences such that $e_k = 1$ for $k = 0,1,\dots$, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \ne n$. For any sequence $x = (x_k)_{k=0}^\infty$, let $x^{[n]} = \sum_{k=0}^n x_k e^{(k)}$ be its n-section.

Let $X, Y \subset \omega$ and $z \in \omega$. Then we write

$$\begin{aligned} z^{-1} \times X &= \{x \in \omega : xz = (x_k z_k)_{k=0}^\infty \in X\}, \\ M(X,Y) &= \bigcap_{x \in X} x^{-1} \times Y = \{a \in \omega : ax \in Y \ \forall x \in X\} \end{aligned} \tag{1.1}$$

for the *multiplier space of* X *and* Y. The sets $M(X,l_1)$, $M(X,cs)$, and $M(X,bs)$ are called the α-, β-, and γ*-duals of* X.

A Fréchet subspace X of ω is called an FK space if it has continuous coordinates, that is, if convergence in X implies coordinatewise convergence. An FK space $X \supset \phi$ is said to have AK if, for every sequence $x = (x_k)_{k=0}^\infty \in X$, $x^{[n]} \to x$ $(n \to \infty)$; and it is said to have AD if ϕ is dense in X. A BK space is an FK space which is a Banach space.

If X is a p-normed space, then we write X^* for the set of all continuous linear functionals on X, the so-called *continuous dual of* X, with its norm $\|\cdot\|$ is given by

$$\|f\| = \sup\{|f(x)| : \|x\| = 1\} \quad \forall f \in X^*. \tag{1.2}$$

Let $X \supset \phi$ be an FK space. Then the set $X^f = \{(f(e^{(n)}))_{n=0}^\infty : f \in X^*\}$ is called the f*-dual of* X.

Given any infinite matrix $A = (a_{nk})_{n,k=0}^\infty$ of complex numbers and any sequence $x \in \omega$, let $A_n(x) = \sum_{k=0}^\infty a_{nk} x_k$ $(n = 0,1,\dots)$, and let $A(x) = (A_n(x))_{n=0}^\infty$ provided the series converge, and $X_A = \{x \in \omega : A(x) \in X\}$. If $0 < p < \infty$, then we write $|x|^p = (|x_k|^p)_{k=0}^{\infty}$ and $X_{[A]^p} = \{x \in \omega : A(|x|^p) \in X\}$.

Let $0 < p < \infty$ and $\mu = (\mu_n)_{n=0}^{\infty}$ be a nondecreasing sequence of positive integers tending to infinity, throughout. We define the matrices Δ and M by

$$\Delta_{nk} = \begin{cases} 1 & (k = n), \\ -1 & (k = n-1), \\ 0 & (\text{otherwise}), \end{cases}$$
$$M_{nk} = \begin{cases} \dfrac{1}{\mu_n^p} & (0 \le k \le n) \\ 0 & (k > n) \end{cases} \quad (n = 0, 1, \dots) \tag{1.3}$$

and use the convention that any symbol with a negative subscript has the value zero.

The sets

$$w_0^p(\mu) = (c_0)_{[M]^p}, \qquad w_\infty^p(\mu) = (l_\infty)_{[M]^p},$$
$$c_0^p(\mu) = (\mu)^{-1} \times (w_0^p(\mu))_\Delta, \qquad c_\infty^p(\mu) = (\mu)^{-1} \times (w_\infty^p(\mu))_\Delta, \qquad c^p(\mu) = c_0^p(\mu) \oplus e \tag{1.4}$$

were studied in [1], and their first duals were given there. If $p = 1$, then we omit the index p, i.e., we write $w_0(\mu) = w_0^1(\mu)$, etc.

Following the notation introduced in [3], we say that a nondecreasing sequence $\Lambda = (\lambda_n)_{n=0}^{\infty}$ of positive reals tending to infinity is *exponentially bounded* if there are reals s and t with $0 < s \le t < 1$ such that for some subsequence $(\lambda_{n(\nu)})_{\nu=0}^{\infty}$ of Λ, we have

$$s \le \frac{\lambda_{n(\nu)}}{\lambda_{n(\nu+1)}} \le t \quad \forall \nu = 0, 1, \dots; \tag{1.5}$$

such a subsequence $(\lambda_{n(\nu)})_{\nu=0}^{\infty}$ is called an *associated subsequence*.

If $(n(\nu))_{\nu=0}^{\infty}$ is a strictly increasing sequence of nonnegative integers, then we write $K^{\langle\nu\rangle}$ for the set of all integers k with $n(\nu) \le k \le n(\nu+1) - 1$, and $\sum_\nu$ and $\max_\nu$ for the sum and maximum taken over all k in $K^{\langle\nu\rangle}$.

If $X^p(\Lambda)$ denotes any of the sets $w_0^p(\Lambda)$, $w_\infty^p(\Lambda)$, $c_0^p(\Lambda)$, $c^p(\Lambda)$, or $c_\infty^p(\Lambda)$, then we write $\tilde{X}^p(\Lambda)$ for the respective space with the sections $1/\lambda_n^p \sum_{k=0}^{n} \dots$ replaced by the blocks $1/\lambda_{n(\nu+1)}^p \sum_\nu \dots$. Furthermore, we define

$$\|x\|_{w_\infty^p(\Lambda)} = \begin{cases} \sup_n \left(\dfrac{1}{\lambda_n^p} \displaystyle\sum_{k=0}^{n} |x_k|^p \right) & (0 < p \le 1), \\ \sup_n \left(\dfrac{1}{\lambda_n^p} \displaystyle\sum_{k=0}^{n} |x_k|^p \right)^{1/p} & (1 \le p < \infty), \end{cases}$$
$$\|x\|_{\tilde{w}_\infty^p(\Lambda)} = \begin{cases} \sup_\nu \left(\dfrac{1}{\lambda_{n(\nu+1)}^p} \displaystyle\sum_\nu |x_k|^p \right) & (0 < p \le 1), \\ \sup_\nu \left(\dfrac{1}{\lambda_{n(\nu+1)}^p} \displaystyle\sum_\nu |x_k|^p \right)^{1/p} & (1 \le p < \infty), \end{cases} \tag{1.6}$$
$$\|x\|_{c_\infty^p(\Lambda)} = \|\Delta(\Lambda x)\|_{w_\infty^p(\Lambda)}, \qquad \|x\|_{\tilde{c}_\infty^p(\Lambda)} = \|\Delta(\Lambda x)\|_{\tilde{w}_\infty^p(\Lambda)}.$$

2. The second duals of the sets $w_0^p(\Lambda)$ and $w_\infty^p(\Lambda)$ for $0 < p < \infty$. Let $\Lambda = (\lambda_n)_{n=0}^\infty$ be a nondecreasing exponentially bounded sequence of positive reals throughout and let $(\lambda_{n(\nu)})_{\nu=0}^\infty$ be an associated subsequence. We put

$$\mathcal{W}^p(\Lambda) = \begin{cases} \left\{a \in \omega : \sum_{\nu=0}^\infty \lambda_{n(\nu+1)} \max_\nu |a_k| < \infty\right\} & (0 < p \le 1), \\ \left\{a \in \omega : \sum_{\nu=0}^\infty \lambda_{n(\nu+1)} \left(\sum_\nu |a_k|^p\right)^{1/p} < \infty\right\} & \left(1 < p < \infty,\ q = \dfrac{p}{p-1}\right) \end{cases} \tag{2.1}$$

and, on $\mathcal{W}^p(\Lambda)$,

$$\|a\|_{\mathcal{W}^p(\Lambda)} = \begin{cases} \sum_{\nu=0}^\infty \lambda_{n(\nu+1)} \max_\nu |a_k| & (0 < p \le 1), \\ \sum_{\nu=0}^\infty \lambda_{n(\nu+1)} \left(\sum_\nu |a_k|^p\right)^{1/p} & \left(1 < p < \infty,\ q = \dfrac{p}{p-1}\right). \end{cases} \tag{2.2}$$

In [1, Theorem 2], it was shown that if $X^p(\Lambda) = w_0^p(\Lambda)$ or $X^p(\Lambda) = w_\infty^p(\Lambda)$ and $\dagger$ stands for α, β, γ, or f, then $(X^p(\Lambda))^\dagger = \mathcal{W}^p(\Lambda)$, that the continuous dual $(w_0^p(\Lambda))^*$ of $w_0^p(\Lambda)$ is norm isomorphic to $\mathcal{W}^p(\Lambda)$ when $w_0^p(\Lambda)$ has the norm $\|\cdot\|_{\tilde{w}_\infty^p(\Lambda)}$, and finally that $\|a\|^*_{\tilde{w}_\infty^p(\Lambda)} = \|a\|_{\mathcal{W}^p(\Lambda)}$ on $(w_\infty^p(\Lambda))^\beta$. Furthermore, $\mathcal{W}^p(\Lambda)$ is a BK space with AK with $\|\cdot\|_{\mathcal{W}^p(\Lambda)}$ (cf. [2]). Therefore the following result gives the second duals of the sets $w_0^p(\Lambda)$ and $w_\infty^p(\Lambda)$.

Theorem 2.1. *We put $p' = \max\{1,p\}$. If $\dagger$ stands for any of the symbols α, β, γ, or f, then $(\mathcal{W}^p(\Lambda))^\dagger = w_\infty^{p'}(\Lambda)$ for $0 < p < \infty$, and the continuous dual $(\mathcal{W}^p(\Lambda))^*$ of $\mathcal{W}^p(\Lambda)$ is norm isomorphic to $w_\infty^{p'}(\Lambda)$ with $\|\cdot\|_{\tilde{w}_\infty^{p'}}$.*

Proof. The statements of the theorem with the exception of those concerning the γ- and f-duals are well known (cf. [2, Theorems 2, 4, 5, and 6]).
Since $\mathcal{W}^p(\Lambda)$ has AK, it follows that $(\mathcal{W}^p(\Lambda))^\beta = (\mathcal{W}^p(\Lambda))^f$ by [4, Theorem 7.2.7(ii), page 106], and so $(\mathcal{W}^p(\Lambda))^f = w_\infty^{p'}(\Lambda)$. Further $\mathcal{W}^p(\Lambda)$ has AD, since it has AK, and so $(\mathcal{W}^p(\Lambda))^\beta = (\mathcal{W}^p(\Lambda))^\gamma$ by [4, Theorem 7.2.7(iii), page 106], hence $(\mathcal{W}^p(\Lambda))^\gamma = w_\infty^{p'}(\Lambda)$. □

3. The α-duals of the sets $c_0^p(\Lambda)$, $c^p(\Lambda)$, and $c^p(\Lambda)$ for $1 < p < \infty$

Theorem 3.1. *We put*

$$\mathcal{C}_\alpha^p(\Lambda) = \left\{a \in \omega : \sum_{\nu=0}^\infty \lambda_{n(\nu+1)} \left(\sum_\nu \left(\sum_{k=n}^\infty \frac{|a_k|}{\lambda_k}\right)^q\right)^{1/q} < \infty\right\} \quad \left(1 < p < \infty; q = \frac{p}{p-1}\right),$$
$$\|a\|_{\mathcal{C}_\alpha^p(\Lambda)} = \sum_{\nu=0}^\infty \lambda_{n(\nu+1)} \left(\sum_\nu \left(\sum_{k=n}^\infty \frac{|a_k|}{\lambda_k}\right)^q\right)^{1/q}. \tag{3.1}$$

If $X^p(\Lambda)$ denotes any of the sets $c_0^p(\Lambda)$, $c^p(\Lambda)$, and $c_\infty^p(\Lambda)$, then $(X^p(\Lambda))^\alpha = \mathcal{C}_\alpha^p(\Lambda)$. Furthermore, $\mathcal{C}_\alpha^p(\Lambda)$ is a BK space with $\|\cdot\|_{\mathcal{C}_\alpha^p(\Lambda)}$.

Proof. First, we assume $a \in \mathscr{C}_\alpha^p(\Lambda)$, and let $x \in c_\infty^p(\Lambda)$. Then there is a constant M such that

$$\left(\sum_\nu |(\Delta(\Lambda x))_n|^p\right)^{1/p} \le \lambda_{n(\nu+1)} M \quad \forall \nu = 0,1,\dots. \tag{3.2}$$

Putting $R_n = \sum_{k=n}^\infty(|a_k|/\lambda_k)$ $(n = 0,1,\dots)$ and using Hölder's inequality, we obtain

$$\begin{aligned}\sum_{k=0}^\infty |a_k x_k| &\le \sum_{\nu=0}^\infty \frac{|a_k|}{\lambda_k}\sum_{n=0}^k |(\Delta(\Lambda x))_n| = \sum_{n=0}^\infty |(\Delta(\Lambda x))_n| \sum_{k=n}^\infty \frac{|a_k|}{\lambda_k} \\ &= \sum_{\nu=0}^\infty \sum_\nu |(\Delta(\Lambda x))_n| R_n \le M\sum_{\nu=0}^\infty \lambda_{n(\nu+1)}\left(\sum_\nu R_n^q\right)^{1/q}.\end{aligned} \tag{3.3}$$

This shows that $\mathscr{C}_\alpha^p(\Lambda) \subset (c_\infty^p(\Lambda))^\alpha$ and that

$$\sum_{k=0}^\infty |a_k x_k| \le \|a\|_{\mathscr{C}_\alpha^p(\Lambda)} \|x\|_{\tilde{c}_\infty^p(\Lambda)}. \tag{3.4}$$

Conversely, we assume $a \in c_0^p(\Lambda)$. We define the maps $f_a^{(m)} : c_0^p(\Lambda) \to \mathbb{R}$ by $f_a^{(m)}(x) = \sum_{k=0}^m |a_k x_k|$ $(x \in X)$. Then $(f_a^{(m)})_{m=0}^\infty$ is a sequence of seminorms on $c_0^p(\Lambda)$ which are continuous, since $c_0^p(\Lambda)$ is a BK space by [1, Theorem 1]. Further, $f_a^{(m)}(x) \le \sum_{k=0}^\infty |a_k x_k| = M(x) < \infty$ for all $m \in \mathbb{N}_0$ and for all $x \in X$. By the uniform boundedness principle, there is a constant M such that

$$\sum_{k=0}^\infty |a_k x_k| \le M \quad \forall x \in c_0^p(\Lambda) \text{ with } \|x\|_{\tilde{c}_\infty^p(\Lambda)} \le 1. \tag{3.5}$$

Since $a \in (c_0^p(\Lambda))^\alpha$ and $1/\Lambda = (1/\lambda_k)_{k=0}^\infty \in c_0^p(\Lambda)$, the numbers R_n are defined for all n. We put

$$S_\mu = \sum_{l=n(\mu)}^{n(\mu+1)-1} R_l^q \quad \forall \mu = 0,1,\dots. \tag{3.6}$$

Let $\nu(m) \in \mathbb{N}_0$ be given. We define the sequence $x^{\nu(m)}$ by

$$x_n^{\nu(m)} = \begin{cases} \dfrac{1}{\lambda_n}\left(\displaystyle\sum_{\mu=0}^{\nu-1}\lambda_{n(\mu+1)}S_\mu^{-1/p}\sum_{k=n(\mu)}^{n(\mu+1)-1}R_k^{q-1} + \lambda_{n(\nu+1)}S_\nu^{-1/p}\sum_{k=n(\nu)}^{n}R_k^{q-1}\right) \\ \qquad\qquad (n \in N^{\langle\nu\rangle};\ 0 \le \nu \le \nu(m)), \\ \dfrac{1}{\lambda_n}\displaystyle\sum_{\mu=0}^{\nu(m)}\lambda_{n(\mu+1)}S_\mu^{-1/p}\sum_{k=n(\mu)}^{n(\mu+1)-1}R_k^{q-1} \quad (n \ge n(\nu(m)+1)). \end{cases} \tag{3.7}$$

Then

$$\begin{aligned}(\Delta(\Lambda x^{\nu(m)}))_n &= \begin{cases}\lambda_{n(\nu+1)}S_\nu^{-1/p}R_n^{q-1} & (n \in N^{\langle\nu\rangle};\ \nu = 0,1,\dots,\nu(m)), \\ 0 & (n \in N^{\langle\nu\rangle};\ \nu \ge \nu(m)+1),\end{cases} \\ \sum_\nu |(\Delta(\Lambda x^{\nu(m)}))_n| &= \begin{cases}\lambda_{n(\nu+1)}^p S_\nu^{-1}\sum_\nu R_n^q = \lambda_{n(\nu+1)}^p & (0 \le \nu \le \nu(m)), \\ 0 & (\nu \ge \nu(m+1)).\end{cases}\end{aligned} \tag{3.8}$$

Thus $x^{\nu(m)} \in c_0^p(\Lambda)$ and $\|x^{\nu(m)}\|_{\tilde{c}_\infty^p(\Lambda)} = 1$. Now, by (3.5) and (3.8) and since $x_k^{\nu(m)} \geq 0$ for all $k = 0, 1, \dots,$

$$\begin{aligned}
\sum_{\nu=0}^{\nu(m)} \lambda_{n(\nu+1)} \left(\sum_\nu R_n^q\right)^{1/q} &= \sum_{\nu=0}^{\nu(m)} \lambda_{n(\nu+1)} \left(\sum_\nu R_n^q\right) S_\nu^{-1/p} = \sum_{\nu=0}^{\nu(m)} \sum_\nu \left(\lambda_{n(\nu+1)} S_\nu^{-1/p} R_n^{q-1}\right) R_n \\
&= \sum_{\nu=0}^{\nu(m)} \sum_\nu \left|\left(\Delta(\Lambda x^{\nu(m)})\right)_n\right| R_n \leq \sum_{n=0}^{\infty} \left|\left(\Delta(\Lambda x^{\nu(m)})\right)_n\right| \sum_{k=n}^{\infty} \frac{|a_k|}{\lambda_k} \\
&= \sum_{k=0}^{\infty} \frac{|a_k|}{\lambda_k} \left|\sum_{n=0}^{k} \left(\Delta(\Lambda x^{\nu(m)})\right)_n\right| = \sum_{k=0}^{\infty} \frac{|a_k|}{\lambda_k} \lambda_k \left|x_k^{\nu(m)}\right| \\
&= \sum_{k=0}^{\infty} |a_k| \left|x_k^{\nu(m)}\right| \leq M.
\end{aligned} \tag{3.9}$$

Since $\nu(m) \in \mathbb{N}_0$ was arbitrary, we have

$$\sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \left(\sum_\nu R_n^q\right)^{1/q} \leq \sum_{k=0}^{\infty} |a_k x_k| < \infty, \tag{3.10}$$

that is, $a \in \mathscr{C}_\alpha^p(\Lambda)$.

Therefore we have shown $(c_\infty^p(\Lambda))^\alpha = (c_0^p(\Lambda))^\alpha = \mathscr{C}_\alpha^p(\Lambda)$. Since $c_0^p(\Lambda) \subset c^p(\Lambda) \subset c_\infty^p(\Lambda)$ for $1 < p < \infty$ (cf. [1, Lemma 1(b)]), we also have $(c^p(\Lambda))^\alpha = \mathscr{C}_\alpha^p(\Lambda)$.

Finally, $\mathscr{C}_\alpha^p(\Lambda)$ is a BK space with $\|\cdot\|_{\mathscr{C}_\alpha^p(\Lambda)}$ by [4, Theorem 4.3.15, page 64], (3.4), and (3.10). □

4. The second duals of the sets $c_0^p(\Lambda)$, $c^p(\Lambda)$, and $c_\infty^p(\Lambda)$ for $0 < p \leq 1$. We put

$$\begin{aligned}
\mathscr{C}(\Lambda) &= \left\{a \in \omega : \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max_\nu \left|\sum_{k=n}^{\infty} \frac{a_k}{\lambda_k}\right| < \infty\right\}, \\
\|a\|_{\mathscr{C}(\Lambda)} &= \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max_\nu \left|\sum_{k=n}^{\infty} \frac{a_k}{\lambda_k}\right|.
\end{aligned} \tag{4.1}$$

In [1, Theorem 4], it was shown that if $X^p(\Lambda)$ is any of the sets $c_0^p(\Lambda)$ or $c_\infty^p(\Lambda)$ and $\dagger$ stands for any of the symbols β, γ, or f, then $(X^p(\Lambda))^\dagger = \mathscr{C}(\Lambda)$ and that this also holds when $X^p(\Lambda) = c(\Lambda)$ or $X^p(\Lambda) = c^p(\Lambda)$ for $0 < p < 1$ whenever

$$\sup_n \frac{1}{\mu_n^p} \sum_{k=0}^{n} \left|\left(\Delta(\mu x)\right)_k\right|^p < \infty; \tag{4.2}$$

otherwise $(c^p(\Lambda))^\beta = \mathscr{C}(\Lambda) \cap cs$ and $(c^p(\Lambda))^\gamma = \mathscr{C}(\Lambda) \cap bs$. Furthermore, it was shown that the continuous dual $(c_0^p(\Lambda))^*$ of $c_0^p(\Lambda)$ is norm isomorphic to $\mathscr{C}(\Lambda)$ when $c_0^p(\Lambda)$ has the p-norm $\|\cdot\|_{\tilde{c}_\infty^p(\Lambda)}$ and $\|a\|^*_{\tilde{c}_\infty^p(\Lambda)} = \|a\|_{\mathscr{C}(\Lambda)}$ on $c_\infty^p(\Lambda)$. Finally, that $f \in c^*(\Lambda)$ if and only if $f(x) = l\chi_f + \sum_{n=0}^\infty a_n x_n$ for all $x \in c(\Lambda)$ where $a \in \mathscr{C}(\Lambda)$, $l \in \mathbb{C}$ with $x - le \in c_0(\Lambda)$ and $\chi_f = f(e) - \sum_{n=0}^\infty a_n$, and that $\|f\|$ is equivalent to $|\chi_f| + \|a\|_{\mathscr{C}(\Lambda)}$; if condition (4.2) is satisfied, then this also holds for $c^p(\Lambda)$ $(0 < p < 1)$.

Therefore the following result gives the second duals.

Theorem 4.1. (a) *The space $\mathscr{C}(\Lambda)$ with $\|\cdot\|_{\mathscr{C}(\Lambda)}$ is a BK space with AK.*

(b) *The set $c_\infty(\Lambda)$ is β* perfect, *that is, $c_\infty^{\beta\beta}(\Lambda) = c_\infty(\Lambda)$. Further $\|a\|^*_{\mathscr{C}(\Lambda)} = \|a\|_{\tilde{c}_\infty(\Lambda)}$ for all $a \in \mathscr{C}^\beta(\Lambda)$.*

(c) *Finally, $\mathscr{C}^f(\Lambda) = \mathscr{C}^\gamma(\Lambda) = \mathscr{C}^\beta(\Lambda)$.*

Proof. We apply Abel's summation by parts. If $a \in cs$, then we write $R(a)$ for the sequence with $R_n(a) = \sum_{k=n}^\infty a_k$ $(n = 0,1,\dots)$. Then

$$\sum_{n=0}^{m-1} a_n y_n = \sum_{n=0}^{m} R_n(a)(\Delta y)_n - R_m(a) y_m \quad \forall m = 0,1,\dots. \tag{4.3}$$

(a) The space $\mathscr{W}(\Lambda)$ is a BK space with $\|\cdot\|_{\mathscr{W}(\Lambda)}$ (cf. [2, Theorem 2]). Further, the matrix A defined by $a_{nk} = 1/\lambda_k$ for $k \ge n$ and $a_{nk} = 0$ for $0 \le n-1$ $(n = 0,1,\dots)$ is one-to-one, and $x = A(y) \in \mathscr{C}(\Lambda)$ if and only if $y \in \mathscr{W}(\Lambda)$. So, by [4, Theorem 4.3.2, page 61], $\mathscr{C}(\Lambda)$ is a BK space with $\|x\|_{\mathscr{C}(\Lambda)} = \|A(y)\|_{\mathscr{W}(\Lambda)}$. Now, we show that $\mathscr{C}(\Lambda)$ has AK. First, we observe that $\phi \subset \mathscr{C}(\Lambda)$, since $\mathscr{C}(\Lambda)$ is the β-dual of a sequence space. Now, let $x \in \mathscr{C}(\Lambda)$ and let $\varepsilon > 0$. For each $m \in \mathbb{N}_0$, let ν_m denote the uniquely determined integer for which $m \in N^{\langle \nu_m \rangle}$. We choose $m_0 \in \mathbb{N}_0$ such that

$$\sum_{\nu=\nu_m}^{\infty} \lambda_{n(\nu+1)} \max_{\nu} |R_n(x/\Lambda)| < \varepsilon \quad \forall m \ge m_0. \tag{4.4}$$

Let $m \ge m_0$. Since the sequence $\Lambda = (\lambda_n)_{n=0}^\infty$ is exponentially bounded, there is $t \in (0,1)$ such that, by (1.5),

$$\begin{aligned}
\|x - x^{[m]}\|_{\mathscr{C}(\Lambda)} &= \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max_{\nu} |R_n((x - x^{[m]})/\Lambda)| \\
&\le \sum_{\nu=0}^{\nu_m-1} \lambda_{n(\nu+1)} |R_{m+1}(x/\Lambda)| + \sum_{\nu=\nu_m}^{\infty} \lambda_{n(\nu+1)} \max_{\nu} |R_n(x/\Lambda)| \\
&< \varepsilon + \sum_{\nu=0}^{\nu_m-1} \frac{\lambda_{n(\nu+1)}}{\lambda_{n(\nu_m+1)}} \lambda_{n(\nu_m+1)} \max_{\nu_m} |R_n(x/\Lambda)| \\
&< \varepsilon + \varepsilon \sum_{\nu=0}^{\nu_m-1} t^{\nu_m-\nu} < \varepsilon \frac{1}{1-t}.
\end{aligned} \tag{4.5}$$

This shows that $\mathscr{C}(\Lambda)$ has AK.

(b) First, we show that $\mathscr{C}^\beta(\Lambda) = c_\infty(\Lambda)$.

For any $X \subset \omega$, $X \subset X^{\beta\beta}$ by [4, Theorem 7.1.2, page 105]. So we have to show $c_\infty(\Lambda) \subset \mathscr{C}^\beta(\Lambda)$ by [1, Theorem 4].

Let $a \in \mathscr{C}^\beta(\Lambda)$. We define $f_a : \mathscr{C}(\Lambda) \to \mathbb{C}$ by $f_a(x) = \sum_{k=0}^\infty a_k x_k$ for all $x \in \mathscr{C}(\Lambda)$. Then $f_a \in \mathscr{C}^*(\Lambda)$ by [4, Theorem 7.2.9, page 107], and so

$$|f_a(x)| \le \|f_a\| \|x\|_{\mathscr{C}(\Lambda)} < \infty \quad \forall x \in \mathscr{C}(\Lambda). \tag{4.6}$$

Let $m \in \mathbb{N}_0$ be given and ν_m the uniquely determined integer such that $m \in N^{\langle \nu_m \rangle}$. Since Λ is exponentially bounded, there are $s, t \in (0,1)$ such that, by (1.5),

$$\begin{aligned} \|e^{(m)}\|_{\mathscr{C}(\Lambda)} &= \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max_{\nu} \left| R_n\left(\frac{e^{(m)}}{\Lambda_k}\right) \right| = \sum_{\nu=0}^{\nu_m} \frac{\lambda_{n(\nu+1)}}{\lambda_m} \\ &\leq \sum_{\nu=0}^{\nu_m} \frac{\lambda_{n(\nu+1)}}{\lambda_{n(\nu_m+1)}} \frac{\lambda_{n(\nu_m+1)}}{\lambda_{n(\nu_m)}} \leq \frac{1}{s} \sum_{\nu=0}^{\nu_m} t^{\nu_m-\nu} \leq \frac{1}{s(1-t)} < \infty. \end{aligned} \tag{4.7}$$

Now (4.6) implies

$$|a_m| = |f_a(e^{(m)})| \leq \|f_a\| \|e^{(m)}\|_{\mathscr{C}(\Lambda)} \leq \|f_a\| \frac{1}{s(1-t)} \quad \forall m \in \mathbb{N}_0, \tag{4.8}$$

and so $a \in l_\infty$. Further, $x \in \mathscr{C}(\Lambda)$ implies that $R_n(x/\Lambda) \in cs$ for all n, and $\Lambda R(x/\Lambda) \in c_0$. Therefore $a\Lambda R(x/\Lambda) \in c_0$. Now (4.3) yields

$$\sum_{n=0}^{\infty} a_n x_n = \sum_{n=0}^{\infty} R_n(x/\Lambda)\big(\Delta(\Lambda a)\big)_n \quad \forall x \in \mathscr{C}(\Lambda). \tag{4.9}$$

Thus $R(x/\Lambda)\Delta(\Lambda a) \in cs$ for all $x \in \mathscr{C}(\Lambda)$. Now $x \in \mathscr{C}(\Lambda)$ if and only if $R(x/\Lambda) \in \mathscr{W}(\Lambda)$ and, by [2, Theorem 4], $\Delta(\Lambda a) \in \mathscr{W}^\beta(\Lambda) = w_\infty(\Lambda)$. But this means that $a \in c_\infty(\Lambda)$. Thus we have shown that $\mathscr{C}^\beta(\Lambda) \subset c_\infty(\Lambda)$.

Now we show

$$\|a\|^*_{\mathscr{C}(\Lambda)} = \|a\|_{\tilde{c}_\infty(\Lambda)} \quad \forall a \in \mathscr{C}^\beta(\Lambda). \tag{4.10}$$

Let $a \in \mathscr{C}^\beta(\Lambda) = c_\infty(\Lambda)$, by what we have just shown. Then by (4.9), for all $x \in \mathscr{C}(\Lambda)$,

$$\begin{aligned} \left| \sum_{n=0}^{\infty} a_n x_n \right| &\leq \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max_{\nu} |R_n(x/\Lambda)| \frac{1}{\lambda_{n(\nu+1)}} \sum_{\nu} \big|\big(\Delta(\Lambda a)\big)_n\big| \\ &\leq \|a\|_{\tilde{c}_\infty(\Lambda)} \|x\|_{\mathscr{C}(\Lambda)}, \end{aligned} \tag{4.11}$$

and so

$$\|a\|^*_{\mathscr{C}(\Lambda)} \leq \|a\|_{\tilde{c}_\infty(\Lambda)}. \tag{4.12}$$

Let $\nu_m \in \mathbb{N}_0$. By $\nu_{0,m}$, we denote the smallest integer with $0 \leq \nu_{0,m} \leq \nu_m$ for which

$$\frac{1}{\lambda_{n(\nu_{0,m}+1)}} \sum_{\nu_{0,m}} \big|\big(\Delta(\Lambda a)\big)_n\big| = \max_{0 \leq \nu \leq \nu_m} \left(\frac{1}{\lambda_{n(\nu+1)}} \sum_{\nu} \big|\big(\Delta(\Lambda a)\big)_n\big| \right). \tag{4.13}$$

We define the sequences $R^{(m)}$ and $x^{(m)}$ by

$$R_n^{(m)} = \begin{cases} \dfrac{1}{\lambda_{n(\nu_{0,m}+1)}} \operatorname{sgn}\big(\big(\Delta(\Lambda a)\big)_n\big) & \text{for } n \in N^{\langle \nu_{0,m} \rangle}, \\ 0 & \text{for } n \notin N^{\langle \nu_{0,m} \rangle}, \end{cases} \tag{4.14}$$

and $x_n^{(m)} = R_n^{(m)} - R_{n+1}^{(m)}$ $(n = 0, 1, \dots)$. Then we have

$$\begin{gathered} \|x^{(m)}\|_{\mathscr{C}(\Lambda)} = \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max_{\nu} \left| R_n^{(m)} \right| = \lambda_{n(\nu_{0,m}+1)} \max_{\nu_{0,m}} \left| R_n^{(m)} \right| \le 1, \\ \left| \sum_{n=0}^{\infty} a_n x_n^{(m)} \right| = \max_{0 \le \nu \le \nu_m} \frac{1}{\lambda_{n(\nu+1)}} \sum_{\nu} \left| \left(\Delta(\Lambda a)\right)_n \right| \le \|a\|_{\mathscr{C}(\Lambda)}^{*} \|x\|_{\mathscr{C}(\Lambda)} \le \|a\|_{\mathscr{C}(\Lambda)}^{*}. \end{gathered} \tag{4.15}$$

Since ν_m was arbitrary, we obtain $\|a\|_{\tilde{c}_\infty(\Lambda)} \le \|a\|_{\mathscr{C}(\Lambda)}^{*}$. Together with (4.12), this yields (4.10).

(c) Since $\mathscr{C}(\Lambda)$ has AK by part (b) and so AD, part (c) follows from [4, Theorem 7.2.7(ii) and (iii), page 106]. □

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