Internat. J. Math. & Math. Sci.
Vol. 24, No. 3 (2000) 145–148
S0161171200003264

T_Ω-SEQUENCES IN ABELIAN GROUPS

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(Received 19 April 1999)

ABSTRACT. A sequence in an abelian group is called a T-sequence if there exists a Hausdorff group topology in which the sequence converges to zero. This paper describes the fundamental system for the finest group topology in which this sequence converges to zero. A sequence is a T_Ω-sequence if there exist uncountably many different Hausdorff group topologies in which the sequence converges to zero. The paper develops a condition which insures that a sequence is a T_Ω-sequence and examples of T_Ω-sequences are given.

Keywords and phrases. Abelian topological group, Hausdorff, sequential convergence.

2000 Mathematics Subject Classification. Primary 22A05.

1. Introduction. Let G be an abelian group and let $\langle a_n\rangle_{n=1}^{\infty}$ be a nontrivial sequence in G. If 0 is the identity element in G, we can ask what is the finest group topology on G such that $\langle a_n\rangle_{n=1}^{\infty}$ converges to zero? In the terminology of [2], we are placing the topology of a nonconstant sequence on the subspace $\{a_n\}_{n=1}^{\infty}\cup\{0\}\subseteq G$ and finding the associated Graev topology. When this topology is Hausdorff, Zelenyuk, and Protasov [4] say that $\langle a_n\rangle_{n=1}^{\infty}$ is a T-sequence. The purpose of this paper will be to extend some of the results of Zelenyuk and Protasov concerning T-sequences in specific abelian groups. We will develop a fundamental system approach to defining group topologies and use this approach to consider the cardinality of the set of Hausdorff group topologies in which a specific sequence converges to zero. This extends results found in [1].

We assume as additional hypothesis throughout this paper that G is an abelian group and that each sequence under consideration is a one-to-one function from the natural numbers $\mathbb{N}$ into G. Also the notations $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and S^1 will denote the integers, rationals, real, and the circle group, respectively. The subgroup of S^1 which is the set of solutions of the form k/p^n, where $k\in\mathbb{Z}$, p is prime and $n\in\mathbb{N}$, we will denote it as $\mathbb{Z}(p^\infty)$.

2. Fundamental systems generated by sequences. Since G is abelian it is possible to define various fundamental systems in a subgroup and use them as a fundamental system for the entire group. We shall use the terms of the sequence $\langle a_n\rangle_{n=1}^{\infty}$ to define such a fundamental system for the subgroup generated by $\{a_n\}_{n=1}^{\infty}$. Let $T(n)=\{0\}\cup\{a_k\}_{k=n}^{\infty}\cup\{-a_k\}_{k=n}^{\infty}$, where $-a_k$ denotes the inverse of a_k in G, and let ζ denote the collection of all increasing sequences in $\mathbb{N}$. Then for $C,D\in\zeta$ we define $U(C,D)=\{g_1+g_2+\cdots+g_k \mid g_i\in c_iT(d_i)$ for $i\in\{1,2,\dots,k\};\ k\in\mathbb{N}\}$.

PROPOSITION 2.1. *$\mathcal{F}=\{U(C,D)\mid C,D\in\zeta\}$ is a fundamental system for G.*

Proof. Suppose that $U(C,D)$ and $U(C',D')$ are elements of $\mathcal{F}$. For each $i \in \mathbb{N}$ let $c'' = \min\{c_i, c_i'\}$ and $d'' = \max\{d_i, d_i'\}$. Define $C'' = \langle c_i'' \rangle_{i=1}^{\infty}$ and $D'' = \langle d_i'' \rangle_{i=1}^{\infty}$. Clearly, both $C'', D'' \in \zeta$. Since $c_1 T(n) \subseteq c_2 T(n)$ whenever $c_1 \le c_2$ and $T(n) \subseteq T(m)$ whenever $m \le n$, we have that $c_i'' T(d_i'') \subseteq c_i T(d_i) \cap c_i' T(d_i')$. Therefore we have $U(C'', D'') \subseteq U(C,D) \cap U(C',D')$.

Now suppose $x \in U(C,D)$. Then $x = g_1 + g_2 + \cdots + g_k$ for some $k \in \mathbb{N}$ and each $g_i \in c_i T(d_i)$ for $i \in \{1,2,\ldots,k\}$. If $C' = \langle c_{k+1}, c_{k+2}, \ldots \rangle$ and $D' = \langle d_{k+1}, d_{k+2}, \ldots \rangle$ then $x + U(C',D') \subseteq U(C,D)$.

Let $U(C,D) \in \mathcal{F}$. For each $i \in \mathbb{N}$ we define

$$c_i' = \begin{cases} \dfrac{c_{2i}}{2} & \text{if } c_{2i} \text{ is even,} \\ \dfrac{c_{2i}-1}{2} & \text{if } c_{2i} \text{ is odd.} \end{cases} \tag{2.1}$$

If $C' = \langle c_i' \rangle$ then $C' \in \zeta$ since $C \in \zeta$. Also we have that $2c' \le c_{2i}$ for all $i \in \mathbb{N}$. Define $D' = \langle d_{2i} \rangle$. Then for each $i \in \mathbb{N}$ we have that $2c_i' T(d_{2i}) \subseteq c_{2i} T(d_{2i})$ and hence $2U(C',D') \subseteq U(C,D)$.

Finally, we note that since $U(C,D)^{-1} = U(C,D)$, $\mathcal{F}$ is a fundamental system. $\square$

Proposition 2.2. *The group topology generated by $\mathcal{F}$ is the finest group topology on G for which $\langle a_n \rangle_{n=1}^{\infty}$ converges to zero.*

Proof. Let τ be any group topology on G for which the sequence $\langle a_n \rangle_{n=1}^{\infty}$ converges to zero and let $0 \in W \in \tau$. We inductively define a sequence of open sets in τ, say $V_1, V_2, \ldots$, with $0 \in V_i$ for all i, $2V_1 \subseteq W$, and in general $(n+1)V_n \subseteq V_{n-1}$ for $n \ge 2$. We also may assume that each V_i is symmetric.

For any $k \in \mathbb{N}$ we have that $V_1 + 2V_2 + \cdots + kV_k \subseteq W$. Since $\langle a_n \rangle_{n=1}^{\infty}$ converges to zero in τ, we can find a tail of the sequence in V_i. We choose $d_i \in \mathbb{N}$ so that $T(d_i) \subseteq V_i$ and $d_i > \max\{d_1, \ldots, d_{i-1}\}$. Then we have that $kT(d_i) \subseteq kV_k$ and for $D = \langle d_i \rangle$, we have that $U(\mathbb{N}, D) \subseteq W$.

The technique used in Proposition 2.1 can be used to show that various subcollections of $\mathcal{F}$ are also fundamental systems for G. For example if $D = \langle d_i \rangle \in \zeta$ and for $k \in \mathbb{N}$, $D_k = \langle d_{ki} \rangle$, then $\mathcal{F}' = \{U(C, D_k) \mid C \in \zeta,\ k \in \mathbb{N}\}$ will also form a fundamental system. $\square$

T_Ω-sequences. Shelah [3] constructs an example of a nonabelian group that admits only the discrete and indiscrete topologies as group topologies. Certainly, the constant identity sequence in Shelah's group will be a T-sequence which converges in a unique Hausdorff group topology. On the other hand, the sparse sequences in $\mathbb{Q}$ described in [1] are shown to converge to the identity in uncountably many different Hausdorff group topologies. We will call any such sequence a T_Ω-sequence. As we shall see in this section, many sequences in abelian groups are actually T_Ω-sequences.

Our search for T_Ω-sequences will require that we focus our attention on various subcollections of the fundamental system described in Proposition 2.1. To refine our notation we define for $D = \langle d_n \rangle \in \zeta$, $U(\langle d_n \rangle) = \{ \sum_{i=1}^{n} g_i \mid g_i \in T(d_i)$ for $i \in \{1,2,\ldots,n\}$ and $n \in \mathbb{N}\}$ and $\mathcal{F}_D = \{U(\langle d_{kn} \rangle) \mid k \in \mathbb{N}\}$. Using techniques similar to those used in Proposition 2.1, it can be shown that $\mathcal{F}_D$ is a fundamental system for G.

We will also focus on a subcollection of ζ. For each $c \in \mathbb{R}$ with $c > 2$ we define $c_n = [n^c]$, the greatest integer in n^c. Clearly $C = \langle c_n \rangle_{n=1}^{\infty} \in \zeta$.

Lemma 2.3. *If c, d are real numbers with $2 < c < d$ and if $k \in \mathbb{N}$ then we can find $N_k \in \mathbb{N}$ such that for $m \geq N_k$, $c_{km} + m < d_m$.*

Proof. We can find $N_k \in \mathbb{N}$ such that for $m \geq N_k$ we have that $k^c + 2 < m^{d-c}$. Hence $[(km)^c] + m < [m^d]$ and thus $c_{km} + m < d_m$ for all $m > N_k$. □

Definition 2.4. Let $S \subseteq G$. For $n \in \mathbb{N}$ and $g \in G$ we say that g has an n-factorization in S if and only if there exits $\{s_1, \ldots, s_n\} \subseteq S - \{0\}$ with $g = s_1 + s_2 + \cdots + s_n$. The factorization is favorable if and only if $-s_i \notin \{s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n\}$.

Proposition 2.5. *Let $\langle a_n \rangle_{n=1}^{\infty}$ be a sequence in G and $S = \{\sum_{i=1}^{n} g_i \mid g_i \in T(i)$ for $1 \leq i \leq n$ and $n \in \mathbb{N}\}$. If*

(1) *every element of S has only finitely many favorable factorizations in S;*
(2) *if $a = \sum_{i=n}^{m} a_i$ for some $n, m \in \mathbb{N}$, then a has no other favorable factorizations in S;*

then the sequence $\langle a_n \rangle_{n=1}^{\infty}$ is a T_Ω-sequence.

Proof. For any sequence $D = \langle d_n \rangle_{n=1}^{\infty} \in \zeta$ we have that $U(\langle d_n \rangle_{n=1}^{\infty}) \subseteq S$. So by (1) we have that for every $g \in S$ there exists a $k \in \mathbb{N}$ such that no favorable factorization of g in S has a factor in $T(k)$. Hence $g \notin U(\langle d_{kn} \rangle_{n=1}^{\infty})$ and thus $\mathcal{F}_D$ generates a Hausdorff group topology.

Now choose $C = \langle c_n \rangle_{n=1}^{\infty}$ and $D = \langle d_n \rangle_{n=1}^{\infty}$ in ζ with the property that for each $k \in \mathbb{N}$ there exists $N_k \in \mathbb{N}$ such that $c_{km} + m < d_m$ for all $m \geq N_k$. Suppose that $U(\langle d_n \rangle_{n=1}^{\infty})$ is open in the topology generated by $\mathcal{F}_c$. Then there exists a k such that $U(\langle c_{kn} \rangle_{n=1}^{\infty}) \subseteq U(\langle d_n \rangle_{n=1}^{\infty})$. We have that $a = \sum_{i=1}^{N_k} a_{c_{kN_k}+i} \in U(\langle c_{kn} \rangle_{n=1}^{\infty})$. But by (2) and the fact that $c_{kN_k} + N_k < b_{N_k}$, we must conclude that $a \notin U(\langle d_n \rangle_{n=1}^{\infty})$. Hence the group topology generated by $\mathcal{F}_c$ is different from the group topology $\mathcal{F}_D$. By Lemma 2.3 we can find uncountably many different Hausdorff group topologies on G with the property that $\langle a_n \rangle_{n=1}^{\infty}$ converges to zero. □

Example 2.6. Let $\langle p^n \rangle_{n=1}^{\infty}$ be the sequence of powers of the prime p in $\mathbb{Z}$. $\langle p^n \rangle_{n=1}^{\infty}$ is a T_Ω- sequence.

Example 2.7. Let $k \in \mathbb{N}$ and let $\langle a_n \rangle_{n=1}^{\infty}$ be an increasing sequence in $\mathbb{Z}$ satisfying the inequality $a_{n+1}/a_n > n/k$ for all n. For $n > 2k$ we have that $\sum_{i=1}^{m} a_{n+i} < a_{n+m+1}$ for each $m \in \mathbb{N}$. Hence $\langle a_n \rangle_{n=1}^{\infty}$ is a T_Ω-sequence.

Example 2.8. Let $Z \in \mathbb{Z}(p^\infty)$. The order of Z is p^n if Z is a p^n-root of unity but not a p^{n-1}-root of unity. We denote the order of Z by $O(Z)$. Now if $O(Z) = p^m$ and $O(w) = p^n$ and $m < n$ we have that $O(Zw) = p^n$. Let $\langle Z_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{Z}(p^\infty)$ satisfying

$$O(Z_{p^n+i}) \geq p^{n+1} O(Z_{p^n+i-1}) \quad \text{for all } n \in \mathbb{N} \text{ and for } 0 \leq i < p. \tag{2.2}$$

By Proposition 2.5, $\langle Z_n \rangle_{n=1}^{\infty}$ is a T_Ω-sequence.

Example 2.9. Consider $\mathbb{R}$ as the direct sum of uncountably many copies of $\mathbb{Q}$. If $\langle r_n \rangle_{n=1}^{\infty}$ is any sequence of linearly independent real numbers, then $\langle r_n \rangle_{n=1}^{\infty}$ is a T_Ω-sequence.

We end this paper with a question. Does there exist a nontrivial sequence in a group G which is a T-sequence, but not a T_Ω-sequence?

References

[1] B. Clark and S. Cates, *Algebraic obstructions to sequential convergence in Hausdorff abelian groups*, Int. J. Math. Math. Sci. **21** (1998), no. 1, 93–96. MR 98k:22004. Zbl 895.22002.

[2] B. Clark and V. Schneider, *On constructing the associated Graev topology*, Arch. Math. (Basel) **55** (1990), no. 3, 296–297. MR 91m:22003. Zbl 716.22001.

[3] S. Shelah, *On a problem of Kurosh, Jónsson groups, and applications*, Word Problems, II (Conf. on Decision Problems in Algebra, Oxford, 1976) (Amsterdam), North-Holland, 1980, pp. 373–394. MR 81j:20047. Zbl 438.20025.

[4] E. G. Zelenyuk and I. V. Protasov, *Topologies on abelian groups*, Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), no. 5, 1090–1107, Translation: Math. USSR. Izv. 37 (1991), no. 2, 445–460. MR 93b:22001. Zbl 728.22003.

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