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SOME SUFFICIENT CONDITIONS FOR STRONGLY STARLIKENESS

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Abstract. We consider strongly starlikeness of order α of functions $f(z) = z+a_{n+1}z^{n+1} + \cdots$ which are analytic in the unit disc and satisfy the condition of the form

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} - 1\right| < \lambda, \quad 0 < \mu < 1,\ 0 < \lambda < 1.$$

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1. Introduction and preliminaries. Let H denote the class of functions analytic in the unit disc $U = \{z : |z| < 1\}$ and let $A \subset H$ be the class of normalized analytic functions f in U such that $f(0) = f'(0) - 1 = 0$. For $n \geq 1$ we put

$$A_n = \{f : f(z) = z + a_{n+1}z^{n+1} + \cdots \text{ is analytic in } U\} \tag{1.1}$$

and $A_1 = A$.

A function $f \in A$ is said to be *strongly starlike of order* α, $0 < \alpha \leq 1$, if and only if

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}, \tag{1.2}$$

where $\prec$ denotes the usual *subordination*. We denote this class by $S(\alpha)$. If $\alpha = 1$, then $S(1) \equiv S^*$ is the well-known class of *starlike functions* in U (cf. [1]).

In this paper, we find a condition so that $f \in A_n$ satisfying

$$f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} \prec 1 + \lambda z, \quad 0 < \mu < 1,\ 0 < \lambda < 1, \tag{1.3}$$

is in $S(\alpha)$. Also, we consider an integral transformation.

We note that the author in [4] determined the values for λ in (1.3) which implies starlikeness in U. Recently, Ponnusamy and Singh [5] found the condition which implies the strongly starlikeness of order α, but for $\mu < 0$ in (1.3). By using the similar method as in [5] we consider strongly starlikeness in the case (1.3).

First, we cite the following lemma.

Lemma 1.1. *Let $Q \in H$ satisfy the subordination condition*

$$Q(z) \prec 1 + \lambda_1 z, \qquad Q(0) = 1, \tag{1.4}$$

where $0 < \lambda_1 \leq 1$. For $0 < \alpha \leq 1$, let $p \in H$, $p(0) = 1$ and p satisfy the condition

$$Q(z)p^{\alpha}(z) \prec 1 + \lambda z, \quad 0 < \lambda \leq 1. \tag{1.5}$$

Then for

$$\sin^{-1}\lambda + \sin^{-1}\lambda_1 \le \frac{\alpha\pi}{2} \tag{1.6}$$

we have $\operatorname{Re}\{p(z)\} > 0$ *in* U.

This is the special case of the more general lemma given in [5].

2. Results and consequences. For our results we also need the following two lemmas.

Lemma 2.1. *Let* $p \in H$, $p(z) = 1 + p_n z^n + \cdots$, $n \ge 1$, *satisfy the condition*

$$p(z) - \frac{1}{\mu} z p'(z) \prec 1 + \lambda z, \quad 0 < \mu < 1,\ 0 < \lambda \le 1. \tag{2.1}$$

Then

$$p(z) \prec 1 + \lambda_1 z, \tag{2.2}$$

where

$$\lambda_1 = \frac{\lambda\mu}{n-\mu}. \tag{2.3}$$

The proof of this lemma for $n = 1$ is given by [4]. For any $n \in N$ we also can apply Jack's lemma [3].

Lemma 2.2. *If* $0 < \mu < 1$, $0 < \lambda \le 1$ *and* $Q \in H$ *satisfying*

$$Q(z) \prec 1 + \frac{\lambda\mu}{n-\mu} z, \quad Q(0) = 1, \quad n \in N, \tag{2.4}$$

and if $p \in H$, $p(0) = 1$ *and satisfies*

$$Q(z) p^{\alpha}(z) \prec 1 + \lambda z, \tag{2.5}$$

where

$$0 < \lambda \le \frac{(n-\mu)\sin(\pi\alpha/2)}{|\mu + (n-\mu)e^{i\pi\alpha/2}|}, \tag{2.6}$$

then $\operatorname{Re}\{p(z)\} > 0$ *in* U.

Proof. If in Lemma 1.1 we put $\lambda_1 = \lambda\mu/(n-\mu)$, then the condition (1.6) is equivalent to

$$\sin^{-1}\lambda + \sin^{-1}\frac{\lambda\mu}{n-\mu} \le \frac{\alpha\pi}{2}. \tag{2.7}$$

This inequality is equivalent to

$$\sin^{-1}\left(\lambda\sqrt{1 - \frac{\lambda^2\mu^2}{(n-\mu)^2}} + \frac{\lambda\mu}{n-\mu}\sqrt{1-\lambda^2}\right) \le \frac{\alpha\pi}{2}, \tag{2.8}$$

or to the inequality

$$\lambda\left[\sqrt{(n-\mu)^2-\lambda^2\mu^2}+\mu\sqrt{1-\lambda^2}\right] \le (n-\mu)\sin\left(\frac{\alpha\pi}{2}\right). \tag{2.9}$$

From there, after some transformations, we get the following equivalent inequality

$$\begin{aligned}&\left\{\left[\mu^2+(n-\mu)^2\right]^2-4\mu^2(n-\mu)^2\cos^2\left(\frac{\alpha\pi}{2}\right)\right\}\lambda^4\\ &\quad -2(n-\mu)^2\left[\mu^2+(n-\mu)^2\right]\sin^2\left(\frac{\alpha\pi}{2}\right)\lambda^2+(1-\mu)^4\sin^4\left(\frac{\alpha\pi}{2}\right)\ge 0\end{aligned} \tag{2.10}$$

which is equivalent to the condition (2.6).

By Lemma 1.1 we have that $\operatorname{Re}\{p(z)\} > 0$ in U. □

Theorem 2.3. *Let $f \in A_n$, $0 < \mu < 1$ and f satisfy the subordination*

$$f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} \prec 1+\lambda z, \tag{2.11}$$

where

$$0 < \lambda \le \frac{n-\mu}{\sqrt{\mu^2+(n-\mu)^2}}. \tag{2.12}$$

Then $f \in S^*$.

Proof. If we put $Q(z) = (z/f(z))^\mu (= 1+q_nz^n+\cdots)$, then after some calculations, we get

$$Q(z)-\frac{1}{\mu}zQ'(z) = f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} \prec 1+\lambda z. \tag{2.13}$$

From Lemma 2.1 we have

$$Q(z) \prec 1+\lambda_1 z, \quad \lambda_1 = \frac{\lambda\mu}{n-\mu}. \tag{2.14}$$

The rest part of the proof is the same as in the case $n = 1$ (see [4, Theorem 1]) and we omit the details. □

Theorem 2.4. *Let $0 < \mu < 1$ and $0 < \alpha \le 1$. If $f \in A_n$ satisfies*

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}-1\right| < \frac{(n-\mu)\sin(\pi\alpha/2)}{\left|\mu+(n-\mu)e^{i\pi\alpha/2}\right|}, \quad z \in U, \tag{2.15}$$

then $f \in S(\alpha)$.

Proof. If we put $\lambda = (n-\mu)\sin(\pi\alpha/2)/|\mu+(n-\mu)e^{i\pi\alpha/2}|$, then, since $0 < \alpha \le 1$, we have $0 < \lambda \le (n-\mu)/\sqrt{\mu^2+(n-\mu)^2}$, and by Theorem 2.3, $f \in S^*$. Later, the function $Q(z) = (z/f(z))^\mu = 1+q_nz^n+\cdots$ is analytic in U and satisfies $Q(z) \prec 1+\lambda_1 z$, $\lambda_1 = \lambda\mu/(n-\mu)$. Now, if we define

$$p(z) = \left(\frac{zf'(z)}{f(z)}\right)^{1/\alpha}, \tag{2.16}$$

then p is analytic in U, $p(0) = 1$ and condition (2.15) is equivalent to

$$Q(z)p^{\alpha}(z) \prec 1+\lambda z. \tag{2.17}$$

Finally, from Lemma 2.2 we obtain

$$\left(\frac{zf'(z)}{f(z)}\right)^{1/\alpha} \prec \frac{1+z}{1-z} \left(\Longleftrightarrow \frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}\right), \tag{2.18}$$

that is, $f \in S(\alpha)$. □

We note that for $\alpha = 1$ we have the statement of Theorem 2.3.

For $n = 1$, $\mu = 1/2$, $\alpha = 2/3$ we get the following corollary.

Corollary 2.5. *Let $f \in A$ and let*

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^{3/2} - 1\right| < \frac{1}{2}, \quad z \in U. \tag{2.19}$$

Then

$$\left| \arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{3}, \quad z \in U, \tag{2.20}$$

that is, $f \in S(2/3)$.

Theorem 2.6. *Let $0 < \mu < 1$, $\operatorname{Re}\{c\} > -\mu$, and $0 < \alpha \le 1$. If $f \in A_n$ satisfies*

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} - 1\right| < \left|\frac{n+c-\mu}{c-\mu}\right| \frac{(n-\mu)\sin(\pi\alpha/2)}{|\mu+(n-\mu)e^{i\pi\alpha/2}|}, \quad z \in U, \tag{2.21}$$

then the function

$$F(z) = z\left[\frac{c-\mu}{z^{c-\mu}}\int_0^z \left(\frac{t}{f(t)}\right)^{\mu} t^{c-\mu-1}dt\right]^{-1/\mu} \tag{2.22}$$

belongs to $S(\alpha)$.

Proof. If we put that λ is equal to the right-hand side of the inequality (2.21) and

$$Q(z) = F'(z)\left(\frac{z}{F(z)}\right)^{1+\mu} (= 1+q_n z^n + \cdots), \tag{2.23}$$

then from (2.21) and (2.22) we obtain

$$Q(z) + \frac{1}{c-\mu} zQ'(z) = f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} \prec 1+\lambda z. \tag{2.24}$$

Hence, by using the result of Hallenbeck and Ruscheweyh [2, Theorem 1] we have that

$$Q(z) \prec 1+\lambda_1 z, \quad \lambda_1 = \frac{|(c-\mu)|\lambda}{|n+c-\mu|} = \frac{(n-\mu)\sin(\pi\alpha/2)}{|\mu+(n-\mu)e^{i\pi\alpha/2}|}, \tag{2.25}$$

and the desired result easily follows from Theorem 2.4. □

Remark 2.7. For $\alpha = 1$ and $n = 1$ we have the corresponding result given earlier in [4]. For $c = \mu + 1$, we have

Corollary 2.8. *Let* $0 < \mu < 1$ *and* $0 < \alpha \leq 1$. *If* $f \in A_n$ *satisfies the condition*

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} - 1 \right| < \frac{n(n-\mu)\sin(\pi\alpha/2)}{|\mu + (n-\mu)e^{i\pi\alpha/2}|}, \quad z \in U, \tag{2.26}$$

then the function

$$F(z) = z\left[\frac{1}{z}\int_0^z \left(\frac{t}{f(t)}\right)^{\mu} dt\right]^{-1/\mu} \tag{2.27}$$

belongs to $S(\alpha)$.

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