

α -MINIMAL SETS AND RELATED TOPICS IN TRANSFORMATION SEMIGROUPS (II)

MASOUD SABBAGHAN and FATEMAH AYATOLLAH ZADEH SHIRAZI

(Received 20 January 1999 and in revised form 12 September 1999)

ABSTRACT. We give some generalizations of proximal relation and distal structure relation of a transformation semigroup in terms of A -minimal sets and A -minimal sets instead of minimal right ideals and conclude similar results.

2000 Mathematics Subject Classification. Primary 54H15.

1. Preliminaries. By a transformation semigroup (X, S, ρ) (or simply (X, S)) we mean a compact Hausdorff topological space X , a discrete topological semigroup S with identity e , and a continuous map $\rho : X \times S \rightarrow X$ ($\rho(x, s) = xs \forall x \in X, \forall s \in S$), such that

- (1) $xe = x \forall x \in X$;
- (2) $x(st) = (xs)t \forall x \in X, \forall s, t \in S$.

In the transformation semigroup (X, S) , for each $s \in S$ define $\pi^s : X \rightarrow X$ by $\pi^s(x) = xs$ ($\forall x \in X$). We assume the semigroup S acts effectively on X , that is, for each $s, t \in S, s \neq t$ if and only if $\pi^s \neq \pi^t$. The closure of $\{\pi^s \mid s \in S\}$ in X^X (with pointwise convergence topology) is called the enveloping semigroup (or Ellis semigroup) of (X, S) and is denoted by $E(X, S)$ (or simply $E(X)$), $E(X)$ has a semigroup structure [1]. A nonempty subset I of $E(X)$ is called a right ideal of $E(X)$ if $IE(X) \subseteq I$, moreover, if the right ideal I of $E(X)$ does not have any proper subset which is a right ideal of $E(X)$, then I is called a minimal right ideal of $E(X)$, the set of all minimal right ideals of $E(X)$ is denoted by $\text{Min}(E(X))$. An element u of $E(X)$ is called idempotent if $u^2 = u$. For $p \in E(X)$ and $a \in X$ the maps $L_p : E(X) \rightarrow E(X)$ and $\theta_a : E(X) \rightarrow X$ defined by $L_p(q) = pq$ and $\theta_a(q) = aq$ ($q \in E(X)$), respectively, are continuous [2, Propositions 3.2 and 3.3]. Let I be a right ideal of $E(X)$, $B \subseteq E(X)$, $C \subseteq X$ ($B, C \neq \emptyset$) and $a \in X$. Standing notations:

$$\begin{aligned}
 S(I) &= \{p \in I \mid L_p : I \rightarrow I \text{ is surjective}\}, & F(a, B) &= \{p \in B \mid ap = a\}, \\
 I(I) &= \{p \in I \mid L_p : I \rightarrow I \text{ is injective}\}, & F(C, B) &= \bigcap_{c \in C} F(c, B), \\
 B(I) &= \{p \in I \mid L_p : I \rightarrow I \text{ is bijective}\}, & \bar{F}(C, B) &= \{p \in B \mid Cp = C\}, \\
 J(B) &= \{u \in B \mid u^2 = u\}.
 \end{aligned} \tag{1.1}$$

A nonempty subset Z of X is called invariant if $ZS \subseteq Z$, moreover, a closed invariant subset Z of X is called minimal if it does not have any proper closed invariant subset. Also $a \in X$ is called almost periodic if $\bar{aS} = aE(X)$ is a minimal subset of X [3, Theorems 1.15 and 1.17]. Let K be a closed right ideal of $E(X)$, then K is called

an A - $\overline{\text{minimal}}$ set if for each $b \in A$, $bK = bE(X)$ and K does not contain any closed right ideal L of $E(X)$ such that $K \neq L$ and for each $b \in A$, $bL = bE(X)$, also K is called an A - $\overline{\text{minimal}}$ set if $AK = AE(X)$ and K does not contain any closed right ideal L of $E(X)$ such that $K \neq L$ and $AL = AE(X)$; the collection of all A - $\overline{\text{minimal}}$ sets is denoted by $\overline{M}_{(X,S)}(A)$ or simply $\overline{M}(A)$ and the collection of all A - $\overline{\text{minimal}}$ sets is denoted by $\overline{\overline{M}}_{(X,S)}(A)$ or simply $\overline{\overline{M}}(A)$; we use $M_{(X,S)}(a)$ (or simply $M(a)$) instead of $\overline{M}_{(X,S)}(\{a\})$ and its elements are called a -minimal sets; in addition we introduce the following sets:

$$\begin{aligned} \overline{M}(X,S) &= \{D \subseteq X \mid D \neq \emptyset, \forall K \in \overline{M}(D) J(F(D,K)) \neq \emptyset\}, \\ \overline{\overline{M}}(X,S) &= \{D \subseteq X \mid D \neq \emptyset, \overline{\overline{M}}(D) \neq \emptyset, \forall K \in \overline{\overline{M}}(D) J(F(D,K)) \neq \emptyset\}, \end{aligned} \tag{1.2}$$

the transformation semigroup (X,S) is called A - $\overline{\text{distal}}$ (or simply A - $\overline{\text{distal}}$) if for each $b \in A$, $E(X) \in M(b)$, and it is called A - $\overline{\overline{\text{M}}}$ - $\overline{\text{distal}}$ (respectively, A - $\overline{\overline{\text{M}}}$ - $\overline{\text{distal}}$) if $E(X) \in \overline{M}(A)$ (respectively, $E(X) \in \overline{\overline{M}}(A)$).

Let (X,S) and (Y,S) be transformation semigroups, then the continuous map $\varphi : (X,S) \rightarrow (Y,S)$ is called a homomorphism if $\varphi(xs) = \varphi(x)s$ ($\forall x \in X, \forall s \in S$), if φ is onto, then there exists a unique induced homomorphism $\hat{\varphi} : (E(X),S) \rightarrow (E(Y),S)$ which is onto and for each $x \in X$, the following diagram commutes:

$$\begin{array}{ccc} (E(X),S) & \xrightarrow{\hat{\varphi}} & (E(Y),S) \\ \theta_x \downarrow & & \downarrow \theta_x(x) \\ (X,S) & \xrightarrow{\varphi} & (Y,S) \end{array} \tag{1.3}$$

moreover, $\hat{\varphi}$ is a semigroup homomorphism; if φ is onto and one-to-one, it is called an isomorphism, and $\hat{\varphi}$ is an isomorphism too [2, Proposition 3.8]. An equivalence relation \mathfrak{R} on X is called invariant if \mathfrak{R} is an invariant subset of the transformation semigroup $(X \times X,S)$. Let \mathfrak{R} be an equivalence relation on X , then $\pi_{\mathfrak{R}} : X \rightarrow X/\mathfrak{R}$ ($\pi_{\mathfrak{R}}(x) = [x]_{\mathfrak{R}}$ ($\forall x \in X$)) is the natural canonical map.

For the remainder of this paper (X,S) is a fixed transformation semigroup, with e as the identity element of S and $\Delta_A = \{(x,x) \mid x \in A\}$.

DEFINITION 1.1. Let A be a nonempty subset of X and let

$$\begin{aligned} \mathfrak{I} &= \{\mathfrak{R} \mid \mathfrak{R} \text{ is a closed invariant equivalence relation on } X \text{ such that} \\ &\quad (X/\mathfrak{R},S) \text{ is distal}\}, \\ \mathfrak{I}_0 &= \{\mathfrak{R} \mid \mathfrak{R} \text{ is a closed invariant equivalence relation on } X \text{ such that } (X/\mathfrak{R},S) \\ &\quad \text{is } [A]_{\mathfrak{R}}\text{-distal}\}, \\ \mathfrak{I}_1 &= \{\mathfrak{R} \mid \mathfrak{R} \text{ is a closed invariant equivalence relation on } X \text{ such that } (X/\mathfrak{R},S) \\ &\quad \text{is } [A]_{\mathfrak{R}}\overline{\text{M}}\text{-distal}\}, \\ \mathfrak{I}_2 &= \{\mathfrak{R} \mid \mathfrak{R} \text{ is a closed invariant equivalence relation on } X \text{ such that } (X/\mathfrak{R},S) \\ &\quad \text{is } [A]_{\mathfrak{R}}\overline{\overline{\text{M}}}\text{-distal}\}, \end{aligned} \tag{1.4}$$

then $\bigcap_{\mathfrak{R} \in \mathfrak{I}} \mathfrak{R}$, $\bigcap_{\mathfrak{R} \in \mathfrak{I}_0} \mathfrak{R}$, $\bigcap_{\mathfrak{R} \in \mathfrak{I}_1} \mathfrak{R}$, and $\bigcap_{\mathfrak{R} \in \mathfrak{I}_2} \mathfrak{R}$ are called, respectively, proximal structure relation, $A^{(-)}$ proximal structure relation (or simply A -proximal structure relation), $A^{(\overline{M})}$ proximal structure relation, and $A^{(\overline{M})}$ proximal structure relation (on X), for $a \in X$, instead of “ $\{a\}$ -proximal structure relation” we simply use “ a -proximal structure relation”; and the sets

$$\begin{aligned}
 P(X, S) &= \{(x, y) \in X \times X \mid \exists I \in \text{Min}(E(X)) \forall p \in I \ x p = y p\} \\
 &\quad \text{(or simply } P(X) \text{ or } P), \\
 P_A(X, S) &= \{(x, y) \in X \times X \mid \exists b \in A \exists I \in M(b) \forall p \in I \ x p = y p\} \\
 &\quad \text{(or simply } P_A(X) \text{ or } P_A), \\
 \overline{P}_A(X, S) &= \{(x, y) \in X \times X \mid \exists I \in \overline{M}(A) \forall p \in I \ x p = y p\} \\
 &\quad \text{(or simply } \overline{P}_A(X) \text{ or } \overline{P}_A), \\
 \overline{\overline{P}}_A(X, S) &= \{(x, y) \in X \times X \mid \exists I \in \overline{\overline{M}}(A) \forall p \in I \ x p = y p\} \\
 &\quad \text{(or simply } \overline{\overline{P}}_A(X) \text{ or } \overline{\overline{P}}_A),
 \end{aligned}
 \tag{1.5}$$

are called, respectively, proximal relation, $A^{(-)}$ proximal relation (or simply A -proximal relation), $A^{(\overline{M})}$ proximal relation, and $A^{(\overline{M})}$ proximal relation (on X), if $a \in X$, then instead of “ $\{a\}$ -proximal relation” (respectively, “ $P_{\{a\}}(X)$ ”) we simply use “ a -proximal relation” (respectively, “ $P_a(X)$ ”).

THEOREM 1.2. *Let A be a nonempty subset of X , then by Definition 1.1, we have*

- (a) (i) if $\{\mathfrak{R}_\alpha\}_{\alpha \in \Gamma}$ is a nonempty collection in \mathfrak{I} , then $\bigcap_{\alpha \in \Gamma} \mathfrak{R}_\alpha \in \mathfrak{I}$,
- (ii) if $\{\mathfrak{R}_\alpha\}_{\alpha \in \Gamma}$ is a nonempty collection in \mathfrak{I}_0 , then $\bigcap_{\alpha \in \Gamma} \mathfrak{R}_\alpha \in \mathfrak{I}_0$,
- (iii) if $\{\mathfrak{R}_\alpha\}_{\alpha \in \Gamma}$ is a nonempty collection in \mathfrak{I}_1 such that for $Z = \{([x_\alpha]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid x \in X\} \subseteq \prod_{\alpha \in \Gamma} X/\mathfrak{R}_\alpha$ we have $\{([a]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid a \in A\} \in \overline{M}(Z, S)$, then $\bigcap_{\alpha \in \Gamma} \mathfrak{R}_\alpha \in \mathfrak{I}_1$,
- (iv) if $\{\mathfrak{R}_\alpha\}_{\alpha \in \Gamma}$ is a nonempty collection in \mathfrak{I}_2 such that for $Z = \{([x_\alpha]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid x \in X\} \subseteq \prod_{\alpha \in \Gamma} X/\mathfrak{R}_\alpha$ we have $\{([a]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid a \in A\} \in \overline{\overline{M}}(Z, S)$, then $\bigcap_{\alpha \in \Gamma} \mathfrak{R}_\alpha \in \mathfrak{I}_2$,
- (b) (i) $X \times X \in \mathfrak{I} \cap \mathfrak{I}_0 \cap \mathfrak{I}_1 \cap \mathfrak{I}_2$,
- (ii) $\bigcap_{\mathfrak{R} \in \mathfrak{I}} \mathfrak{R} \in \mathfrak{I}$, $\bigcap_{\mathfrak{R} \in \mathfrak{I}_0} \mathfrak{R} \in \mathfrak{I}_0$,
- (c) (i) (X, S) is distal if and only if $\Delta_X \in \mathfrak{I}$,
- (ii) (X, S) is A -distal if and only if $\Delta_X \in \mathfrak{I}_0$,
- (iii) (X, S) is $A^{(\overline{M})}$ distal if and only if $\Delta_X \in \mathfrak{I}_1$,
- (iv) (X, S) is $A^{(\overline{\overline{M}})}$ distal if and only if $\Delta_X \in \mathfrak{I}_2$.

PROOF. (a) (ii) Let $\{\mathfrak{R}_\alpha\}_{\alpha \in \Gamma}$ be a nonempty collection in \mathfrak{I}_0 , then for each $\alpha \in \Gamma$, $(X/\mathfrak{R}_\alpha, S)$ is $[A]_{\mathfrak{R}_\alpha}$ -distal, thus $(\prod_{\alpha \in \Gamma} X/\mathfrak{R}_\alpha, S)$ is $\{([a]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid a \in A\}$ -distal (since $\{([a]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid a \in A\} \subseteq \prod_{\alpha \in \Gamma} [A]_{\mathfrak{R}_\alpha}$), but $\{([a]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid a \in A\} \subseteq \{([x]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid x \in X\}$ and $\{([x]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid x \in X\}$ is a closed invariant subset of $\prod_{\alpha \in \Gamma} X/\mathfrak{R}_\alpha$, therefore $(\{([x]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid x \in X\}, S)$ is $\{([a]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid a \in A\}$ -distal [4, Theorem 1.23(c)]. On the other hand, $\varphi : (X/\bigcap_{\alpha \in \Gamma} \mathfrak{R}_\alpha, S) \rightarrow (\{([x]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid x \in X\}, S)$ defined by $\varphi([x]_{\bigcap_{\alpha \in \Gamma} \mathfrak{R}_\alpha}) = ([x]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma}$ ($\forall x \in X$) is an isomorphism and $(\{([x]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid x \in X\}, S)$ is

$\varphi([A]_{\bigcap_{\alpha \in \Gamma} \mathfrak{R}_\alpha})$ -distal, therefore $(X/\bigcap_{\alpha \in \Gamma} \mathfrak{R}_\alpha, S)$ is $[A]_{\bigcap_{\alpha \in \Gamma} \mathfrak{R}_\alpha}$ -distal and $\bigcap_{\alpha \in \Gamma} \mathfrak{R}_\alpha \in \mathfrak{S}_0$.

(iii) Let $\{\mathfrak{R}_\alpha\}_{\alpha \in \Gamma}$ be a nonempty collection in \mathfrak{S}_1 , then for each $\alpha \in \Gamma$, $(X/\mathfrak{R}_\alpha, S)$ is $[A]_{\mathfrak{R}_\alpha}^{(\overline{M})}$ -distal, thus for each $\alpha \in \Gamma$, $J(F([A]_{\mathfrak{R}_\alpha}, E(X/\mathfrak{R}_\alpha))) = \{e\}$, therefore $J(F(\{([a]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid a \in A\}, E(\prod_{\alpha \in \Gamma} X/\mathfrak{R}_\alpha))) = \{e\}$, but $\{([x]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid x \in X\}$ is a closed invariant subset of $\prod_{\alpha \in \Gamma} X/\mathfrak{R}_\alpha$, and by the hypothesis $\{([a]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid a \in A\} \in \overline{M}(\{([x]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid x \in X\}, S)$, thus $(\{([x]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid x \in X\}, S)$ is $\{([a]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid a \in A\}^{(\overline{M})}$ -distal [4, Theorem 1.23(d)]. On the other hand, $\varphi : (X/\bigcap_{\alpha \in \Gamma} \mathfrak{R}_\alpha, S) \rightarrow (\{([x]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid x \in X\}, S)$ defined by $\varphi([x]_{\bigcap_{\alpha \in \Gamma} \mathfrak{R}_\alpha}) = ([x]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} (\forall x \in X)$ is an isomorphism, and $(\{([x]_{\mathfrak{R}_\alpha})_{\alpha \in \Gamma} \mid x \in X\}, S)$ is $\varphi([A]_{cap_{\alpha \in \Gamma} \mathfrak{R}_\alpha})^{(\overline{M})}$ -distal, therefore $(X/\bigcap_{\alpha \in \Gamma} \mathfrak{R}_\alpha, S)$ is $[A]_{\bigcap_{\alpha \in \Gamma} \mathfrak{R}_\alpha}^{(\overline{M})}$ -distal and $\bigcap_{\alpha \in \Gamma} \mathfrak{R}_\alpha \in \mathfrak{S}_1$.

(iv) The proof is similar to (iii).

(b) Let $\mathfrak{R} = X \times X$, then X/\mathfrak{R} is singleton, thus it is clear that $(X/\mathfrak{R}, S)$ is distal, $[A]_{\mathfrak{R}}$ -distal, $[A]_{\mathfrak{R}}^{(\overline{M})}$ -distal, and $[A]_{\mathfrak{R}}^{(\overline{M})}$ -distal, thus $X \times X = \mathfrak{R} \in \mathfrak{S} \cap \mathfrak{S}_0 \cap \mathfrak{S}_1 \cap \mathfrak{S}_2$. On the other hand, by (a) ((i) and (ii)) we have $\bigcap_{\mathfrak{R} \in \mathfrak{S}} \mathfrak{R} \in \mathfrak{S}$ and $\bigcap_{\mathfrak{R} \in \mathfrak{S}_0} \mathfrak{R} \in \mathfrak{S}_0$.

(c) (ii) Let $\mathfrak{R} = \Delta_X$, then the canonical map $\pi_{\mathfrak{R}} : (X, S) \rightarrow (X/\mathfrak{R}, S)$ is an isomorphism, thus (X, S) is A -distal if and only if $(X/\mathfrak{R}, S)$ is $[A]_{\mathfrak{R}}$ -distal if and only if $\Delta_X = \mathfrak{R} \in \mathfrak{S}_0$. □

NOTE 1.3. Let A be a nonempty subset of X , then

- (a) (i) $P(X)$ is a reflexive and symmetric relation on X ,
- (ii) $P_A(X)$ is a reflexive and symmetric relation on X ,
- (iii) $\overline{P}_A(X)$ is a reflexive and symmetric relation on X ,
- (iv) if $\overline{M}(A) \neq \emptyset$, then $\overline{\overline{P}}_A(X)$ is a reflexive and symmetric relation on X ,
- (b) if S is abelian, then $P(X), P_A(X), \overline{P}_A(X)$, and $\overline{\overline{P}}_A(X)$ (this latter case when $\overline{M}(A) \neq \emptyset$) are invariant relations on X ,
- (c) (i) for each nonempty subset B of A we have $\overline{P}_A(X) \subseteq \overline{P}_B(X) \subseteq P_B(X) \subseteq P_A(X) \subseteq P(X)$,
- (ii) $P_A(X) = \cup_{a \in A} P_a(X)$,
- (iii) $\overline{P}_A(X) \subseteq P(X)$,
- (d) (i) if all of the points of A are almost periodic, then $\overline{\overline{P}}_A(X) = \overline{P}_A(X) = P_A(X) = P(X)$,
- (ii) $P_A(X) = \Delta_X \Leftrightarrow (\forall a \in A P_a(X) = \Delta_X)$,
- (iii) $P_A(X) = \Delta_X \Rightarrow (\forall a \in A \forall K \in \overline{M}(a) J(F(a, K)) = J(S(K)) = \{e\})$,
- (iv) $\overline{P}_A(X) = \Delta_X \Rightarrow (\forall K \in \overline{M}(A) J(F(A, K)) \subseteq J(S(K)) \subseteq \{e\})$,
- (v) $\overline{\overline{P}}_A(X) = \Delta_X \Rightarrow (\forall K \in \overline{M}(A) J(F(A, K)) \subseteq J(S(K)) \subseteq \{e\})$.

PROOF. (a) and (b) are clear.

(c) (i) Let B be a nonempty subset of A , then for each $(x, y) \in X \times X$ we have

$$\begin{aligned} (x, y) \in \overline{P}_A(X) &\Rightarrow \exists K \in \overline{M}(A) \forall p \in K, xp = yp \\ &\Rightarrow \exists K \in \overline{M}(A) \exists L \in \overline{M}(B) \forall p \in K, (xp = yp \wedge L \subseteq K) \text{ (by [4, Corollary 1.3])} \\ &\Rightarrow \exists L \in \overline{M}(B) \forall p \in L, xp = yp \\ &\Rightarrow (x, y) \in \overline{P}_B(X) \end{aligned}$$

$$\begin{aligned}
 &(x, y) \in \overline{P}_B(X) \\
 &\Rightarrow \exists L \in \overline{M}(B) \forall p \in L, xp = yp \\
 &\Rightarrow \exists L \in \overline{M}(B) \forall b \in B \exists K \in M(b) \forall p \in L \\
 &\quad (xp = yp \wedge K \subseteq L) \text{ (by [4, Corollary 1.3])} \\
 &\Rightarrow \forall b \in B \exists K \in M(b) \forall p \in K, xp = yp \\
 &\Rightarrow \exists b \in B \exists K \in M(b) \forall p \in K, xp = yp \\
 &\Rightarrow (x, y) \in P_B(X) \\
 &(x, y) \in P_B(X) \\
 &\Rightarrow \exists b \in B \exists K \in M(b) \forall p \in K, xp = yp \\
 &\Rightarrow \exists a \in A \exists K \in M(a) \forall p \in K, xp = yp \\
 &\Rightarrow (x, y) \in P_A(X) \\
 &(x, y) \in P_A(X) \\
 &\Rightarrow \exists a \in A \exists K \in M(a) \forall p \in K, xp = yp \\
 &\Rightarrow \exists a \in A \exists K \in M(a) \exists L \in \text{Min}(E(X)) \forall p \in K, (xp = yp \wedge L \subseteq K) \\
 &\Rightarrow \exists L \in \text{Min}(E(X)) \forall p \in L, xp = yp \\
 &\Rightarrow (x, y) \in P(X). \tag{1.6}
 \end{aligned}$$

(ii) It is clear by Definition 1.1.

(iii) It is clear by Definition 1.1 and the fact that each closed right ideal of $E(X)$ contains at least one element of $\text{Min}(E(X))$.

(d) (i) If all of the points of A are almost periodic, then for each $a \in A$, $M(a) = \overline{M}(A) = \overline{M}(a) = \text{Min}(E(X))$ [4, Note 1.12], thus $\overline{P}_A(X) = \overline{P}_A(X) = P_A(X) = P(X)$.

(ii) Since for each $a \in A$, $\Delta_X \subseteq P_a(X) \subseteq P_A(X) = \cup_{b \in A} P_b(X)$ (use (c) (ii)), thus $P_A(X) = \Delta_X$ if and only if for each $a \in A$, $P_a(X) = \Delta_X$.

(iv) Let $\overline{P}_A(X) = \Delta_X$ and $K \in \overline{M}(A)$, then $J(F(A, K)) \subseteq J(S(K))$ [4, Corollary 1.5(Table 1.3)], if $u \in J(S(K))$, then $uK = K$ and for each $x \in X$, $(xu)u = xu$, thus for each $p \in K$, $(xu)up = xup$, that is, for each $q \in K$, $(xu)q = xq$ and $(xu, x) \in \overline{P}_A(X) = \Delta_X$, therefore for each $x \in X$, $xu = x$ and $u = e$.

Considering (ii), (iii) is a special case of (iv). The proof of (v) has a similar argument. \square

THEOREM 1.4. *Let A be a nonempty subset of X , then:*

- (a) (i) (X, S) is distal if and only if $P(X) = \Delta_X$,
- (ii) (X, S) is A -distal if and only if $P_A(X) = \Delta_X$,
- (iii) if $A \in \overline{M}(X, S)$, then (X, S) is $A^{(\overline{M})}$ -distal if and only if $\overline{P}_A(X) = \Delta_X$,
- (iv) if $A \in \overline{M}(X, S)$, then (X, S) is $A^{(\overline{M})}$ -distal if and only if $\overline{P}_A(X) = \Delta_X$,
- (b) if $(x, y) \in X \times X$, then:
 - (i) the following statements are equivalent:
 - (1) $(x, y) \in P(X)$,
 - (2) $\exists u \in J(E(X))$, $xu = yu$,
 - (3) $\exists p \in E(X)$, $xp = yp$,

- (ii) *the following statements are equivalent:*
 - (1) $(x, y) \in P_A(X)$,
 - (2) $\exists a \in A \exists u \in J(F(a, E(X))), xu = yu$,
 - (3) $\exists a \in A \exists p \in F(a, E(X)), xp = yp$,
- (iii) *if $A \in \overline{\mathcal{M}}(X, S)$, then the following statements are equivalent:*
 - (1) $(x, y) \in \overline{P}_A(X)$,
 - (2) $\exists u \in J(F(A, E(X))), xu = yu$,
 - (3) $\exists p \in F(A, E(X)), xp = yp$.

PROOF. In each case for the sake of brevity we prove (iii).

(a) (iii) If (X, S) is $A^{\overline{\mathcal{M}}}$ -distal, then $\overline{\mathcal{M}}(A) = \{E(X)\}$, and if $(x, y) \in \overline{P}_A(X)$, then for each $p \in E(X)$, $xp = yp$, thus $x = xe = ye = y$ and $\overline{P}_A(X) \subseteq \Delta_X$, therefore $\overline{P}_A(X) = \Delta_X$. On the other hand, let $A \in \overline{\mathcal{M}}(X, S)$ and $\overline{P}_A(X) = \Delta_X$, take $K \in \overline{\mathcal{M}}(A)$ and $u \in J(F(A, K)) (\neq \emptyset)$, then $uK = K$ and for each $x \in X$ and $p \in K$, $xp = x(Up) = (xu)p$, so $(x, xu) \in \overline{P}_A(X) = \Delta_X$, that is, for each $x \in X$, $xu = x$ and $u = e$ so $K = E(X)$, therefore (X, S) is $A^{\overline{\mathcal{M}}}$ -distal.

(b) (iii) We have

$$\begin{aligned}
 (1) &\Rightarrow \exists K \in \overline{\mathcal{M}}(A) \forall p \in K, xp = yp \\
 &\Rightarrow \exists K \in \overline{\mathcal{M}}(A) \exists u \in J(F(A, K)), xu = yu \text{ (since } A \in \overline{\mathcal{M}}(X, S)\text{)} \\
 &\Rightarrow (2), \\
 (3) &\Rightarrow \exists p \in F(A, E(X)) \forall q \in pE(X), xq = yq \\
 &\Rightarrow \exists p \in F(A, E(X)) \exists L \in \overline{\mathcal{M}}(A) \forall q \in pE(X), \\
 &\quad (xq = yq \wedge L \subseteq pE(X)) \text{ (by [4, Corollary 1.3])} \\
 &\Rightarrow \exists L \in \overline{\mathcal{M}}(A) \forall q \in L, xq = yq \\
 &\Rightarrow (1). \qquad \qquad \qquad \square
 \end{aligned}
 \tag{1.7}$$

THEOREM 1.5. *Let A be a nonempty subset of X , then*

- (a) (i) *the following statements are equivalent:*
 - (1) $\text{Min}(E(X))$ is singleton,
 - (2) $P(X)$ is a transitive relation on X ,
 - (3) $P(X)$ is an equivalence relation on X ,
- (ii) *if $A \in \overline{\mathcal{M}}(X, S)$, then the following statements are equivalent:*
 - (1) $\overline{\mathcal{M}}(A)$ is singleton,
 - (2) $\overline{P}_A(X)$ is a transitive relation on X ,
 - (3) $\overline{P}_A(X)$ is an equivalence relation on X ,
- (iii) *if $A \in \overline{\overline{\mathcal{M}}}(X, S)$, then the following statements are equivalent:*
 - (1) $\overline{\overline{\mathcal{M}}}(A)$ is singleton,
 - (2) $\overline{\overline{P}}_A(X)$ is a transitive relation on X ,
 - (3) $\overline{\overline{P}}_A(X)$ is an equivalence relation on X ,
- (b) *if S is an abelian semigroup, then:*
 - (i) *if $P(X)$ is a closed relation on X , then $P(X)$ is an equivalence relation on X ,*
 - (ii) *if $A \in \overline{\mathcal{M}}(X, S)$ and $\overline{P}_A(X)$ is a closed relation on X , then $\overline{P}_A(X)$ is an equivalence relation on X ,*
 - (iii) *if $A \in \overline{\overline{\mathcal{M}}}(X, S)$ and $\overline{\overline{P}}_A(X)$ is a closed relation on X , then $\overline{\overline{P}}_A(X)$ is an equivalence relation on X .*

PROOF. (a) (ii) By [Note 1.3\(a\)](#), it is enough to show that (3) implies (1). Let $\bar{P}_A(X)$ be an equivalence relation on X , $K, L \in \bar{M}(A)$ and $u \in J(F(A, K))$, there exists $v \in J(F(A, L))$ such that $uv = u$ and $vu = v$ [[4, Theorem 7.1\(a\)](#)], moreover, $uE(X) = uK = K$, $vE(X) = vL = L$ [[4, Corollary 1.5\(Table 3\)](#)], and for each $x \in X$, $p \in K$ and $q \in L$ we have: $(xu)p = x(up) = xp$ and $(xv)q = x(vq) = xq$. Therefore $(xu, x), (x, xv) \in \bar{P}_A(X)$ and by the transitivity of $\bar{P}_A(X)$, $(xu, xv) \in \bar{P}_A(X)$, thus there exists $N \in \bar{M}(A)$ such that for each $l \in N$, $xul = xvl$. We know there exists $w \in J(F(A, N))$, such that $uw = u$ and $vw = (vu)w = v(uw) = vu = v$ [[4, Theorem 1.7\(a\)](#)] thus $xu = xuw = xvw = xv$ (for each $x \in X$), so $u = v$ and $K = uE(X) = vE(X) = L$. Therefore $\bar{M}(A)$ is singleton.

(b) (ii) Let $A \in \bar{M}(X, S)$ and $\bar{P}_A(X)$ be a closed relation on X , then for each $(x, y), (y, z) \in \bar{P}_A(X)$, there exists $K \in \bar{M}(A)$ such that for each $p \in K$, $xp = yp$. Let $u \in J(F(A, K))$ ($\neq \emptyset$), then $xu = yu$. Now by [Note 1.3\(b\)](#), we have $(yu, zu) \in \bar{P}_A(X)$. Choose $L \in \bar{M}(A)$ such that for each $q \in L$, $yuq = zuq$. There exists $v \in J(F(A, L))$, such that $uv = u$ [[4, Theorem 1.7\(a\)](#)] thus $xu = yu = yuv = zuv = zu$, by [Theorem 1.4\(iii\)](#), $(x, z) \in \bar{P}_A(X)$ and $\bar{P}_A(X)$ is a transitive relation on X , thus by (a) (ii) $\bar{P}_A(X)$ is an equivalence relation on X . □

NOTE 1.6. Let A be a nonempty subset of X , let $\varphi : (X, S) \rightarrow (Y, S)$ be an onto homomorphism. Define $\varphi \times \varphi : X \times X \rightarrow Y \times Y$ by $\varphi \times \varphi(x, y) = (\varphi(x), \varphi(y))$ ($\forall (x, y) \in X \times X$), using [Definition 1.1](#), we have

- (a) if $K \in \bar{M}(A)$, then there exists $L \in \bar{M}(\varphi(A))$ such that $L \subseteq \hat{\varphi}(K)$,
- (b) (i) $\varphi \times \varphi(P(X)) \subseteq P(Y)$,
- (ii) $\varphi \times \varphi(P_A(X)) \subseteq P_{\varphi(A)}(Y)$,
- (iii) $\varphi \times \varphi(\bar{P}_A(X)) \subseteq \bar{P}_{\varphi(A)}(Y)$,
- (c) (i) $P(X) \subseteq \bigcap_{\mathfrak{R} \in \mathfrak{J}} \mathfrak{R}$,
- (ii) $P_A(X) \subseteq \bigcap_{\mathfrak{R} \in \mathfrak{J}_0} \mathfrak{R}$,
- (iii) $\bar{P}_A(X) \subseteq \bigcap_{\mathfrak{R} \in \mathfrak{J}_1} \mathfrak{R}$.

PROOF. (a) If $K \in \bar{M}(A)$, then $\hat{\varphi}(K)$ is a closed right ideal of $E(Y)$. On the other hand, for each $a \in A$, $aK = aE(X)$ thus $\varphi(a)\hat{\varphi}(K) = \varphi(aK) = \varphi(aE(X)) = \varphi(a)\hat{\varphi}(E(X)) = \varphi(a)E(Y)$, therefore there exists $L \in \bar{M}(\varphi(A))$ such that $L \subseteq \hat{\varphi}(K)$ [[4, Corollary 1.3\(b\)](#)].

(b) Let $(x, y) \in X \times X$.

(ii) If $(x, y) \in P_A(X)$, then there exists $a \in A$ and $K \in M(a)$ such that for each $p \in K$, $xp = yp$ and $\varphi(x)\hat{\varphi}(p) = \varphi(y)\hat{\varphi}(p)$, by (a) there exists $L \in M(\varphi(a))$ such that $L \subseteq \hat{\varphi}(K)$, so for each $q \in L$, $\varphi(x)q = \varphi(y)q$, therefore, $\varphi \times \varphi(x, y) = (\varphi(x), \varphi(y)) \in P_{\varphi(A)}(Y)$.

(iii) If $(x, y) \in \bar{P}_A(X)$, then there exists $K \in \bar{M}(A)$ such that for each $p \in K$, $xp = yp$ and $\varphi(x)\hat{\varphi}(p) = \varphi(y)\hat{\varphi}(p)$, by (a) there exists $L \in \bar{M}(\varphi(A))$ such that $L \subseteq \hat{\varphi}(K)$, so for each $q \in L$, $\varphi(x)q = \varphi(y)q$, therefore, $\varphi \times \varphi(x, y) = (\varphi(x), \varphi(y)) \in \bar{P}_{\varphi(A)}(Y)$.

(c) (ii) Let $\mathfrak{R} \in \mathfrak{J}_0$, then

$$\begin{aligned} \mathfrak{R} \in \mathfrak{J}_0 &\implies \left(\frac{X}{\mathfrak{R}}, S \right) \text{ is } [A]_{\mathfrak{R}}\text{-distal} \\ &\implies P_{[A]_{\mathfrak{R}}} \left(\frac{X}{\mathfrak{R}} \right) = \Delta_{X/\mathfrak{R}} \end{aligned}$$

$$\begin{aligned}
&\Rightarrow \pi_{\mathfrak{R}} \times \pi_{\mathfrak{R}}(P_A(X)) \subseteq P_{\pi_{\mathfrak{R}}(A)}\left(\frac{X}{\mathfrak{R}}\right) = P_{[A]_{\mathfrak{R}}}\left(\frac{X}{\mathfrak{R}}\right) = \Delta_{X/\mathfrak{R}} \\
&\Rightarrow \pi_{\mathfrak{R}} \times \pi_{\mathfrak{R}}(P_A(X)) = \Delta_{X/\mathfrak{R}} \\
&\Rightarrow \forall (x, y) \in P_A(X), [x]_{\mathfrak{R}} = [y]_{\mathfrak{R}} \\
&\Rightarrow \forall (x, y) \in P_A(X), (x, y) \in \mathfrak{R} \\
&\Rightarrow P_A(X) \subseteq \mathfrak{R}
\end{aligned} \tag{1.8}$$

so $P_A(X) \subseteq \bigcap_{\mathfrak{R} \in \mathfrak{S}_0} \mathfrak{R}$. □

DEFINITION 1.7. Let $\varphi : (X, S) \rightarrow (Y, S)$ be an onto homomorphism, $R(\varphi) = \{(x, y) \in X \times X \mid \varphi(x) = \varphi(y)\}$, and let A be a nonempty subset of X , and let B be a nonempty subset of Y , then

- (a) (Y, S) is a distal factor of (X, S) (under φ) if $R(\varphi) \cap P(X) = \Delta_X$,
- (b) (Y, S) is an $A^{(-)}$ distal (or simply A -distal) factor of (X, S) (under φ) if $R(\varphi) \cap P_A(X) = \Delta_X$,
- (c) (Y, S) is an $A^{\overline{(M)}}$ distal factor of (X, S) (under φ) if $R(\varphi) \cap \overline{P}_A(X) = \Delta_X$,
- (d) (Y, S) is an $A^{\overline{(M)}}$ distal factor of (X, S) (under φ) if $R(\varphi) \cap \overline{\overline{P}}_A(X) = \Delta_X$,
- (e) (X, S) is a distal extension of (Y, S) (under φ) if $R(\varphi) \cap P(X) = \Delta_X$,
- (f) (X, S) is a $B^{(-)}$ distal (or simply B -distal) extension of (Y, S) (under φ) if $R(\varphi) \cap P_{\varphi^{-1}(B)}(X) = \Delta_X$,
- (g) (X, S) is a $B^{\overline{(M)}}$ distal extension of (Y, S) (under φ) if $R(\varphi) \cap \overline{P}_{\varphi^{-1}(B)}(X) = \Delta_X$,
- (h) (X, S) is a $B^{\overline{(M)}}$ distal extension of (Y, S) (under φ) if $R(\varphi) \cap \overline{\overline{P}}_{\varphi^{-1}(B)}(X) = \Delta_X$,
- (a') (Y, S) is a proximal factor of (X, S) (under φ) if $R(\varphi) \subseteq P(X)$,
- (b') (Y, S) is an $A^{(-)}$ proximal (or simply A -proximal) factor of (X, S) (under φ) if $R(\varphi) \subseteq P_A(X)$,
- (c') (Y, S) is an $A^{\overline{(M)}}$ proximal factor of (X, S) (under φ) if $R(\varphi) \subseteq \overline{P}_A(X)$,
- (d') (Y, S) is an $A^{\overline{(M)}}$ proximal factor of (X, S) (under φ) if $R(\varphi) \subseteq \overline{\overline{P}}_A(X)$,
- (e') (X, S) is a proximal extension of (Y, S) (under φ) if $R(\varphi) \subseteq P(X)$,
- (f') (X, S) is a $B^{(-)}$ proximal (or simply B -proximal) extension of (Y, S) (under φ) if $R(\varphi) \subseteq P_{\varphi^{-1}(B)}(X)$,
- (g') (X, S) is a $B^{\overline{(M)}}$ proximal extension of (Y, S) (under φ) if $R(\varphi) \subseteq \overline{P}_{\varphi^{-1}(B)}(X)$,
- (h') (X, S) is a $B^{\overline{(M)}}$ proximal extension of (Y, S) (under φ) if $R(\varphi) \subseteq \overline{\overline{P}}_{\varphi^{-1}(B)}(X)$.

THEOREM 1.8. Let $\varphi : (X, S) \rightarrow (Y, S)$ be an onto homomorphism, let A be a nonempty subset of X , let B be a nonempty subset of Y , and consider the following statements:

- (π_1) (Y, S) is a distal factor of (X, S) under φ ,
- (π_2) (Y, S) is an A -distal factor of (X, S) under φ ,
- (π_3) (Y, S) is an $A^{\overline{(M)}}$ distal factor of (X, S) under φ ,
- (π_4) (Y, S) is an $A^{\overline{(M)}}$ distal factor of (X, S) under φ (by the assumption $\overline{\overline{M}}(A) \neq \emptyset$),
- (ρ_1) (X, S) is a distal extension of (Y, S) under φ ,
- (ρ_2) (X, S) is a B -distal extension of (Y, S) under φ ,
- (ρ_3) (X, S) is a $B^{\overline{(M)}}$ distal extension of (Y, S) under φ ,
- (ρ_4) (X, S) is a $B^{\overline{(M)}}$ distal extension of (Y, S) under φ (by the assumption $\overline{\overline{M}}(\varphi^{-1}(B)) \neq \emptyset$),

- (π'_1) (Y, S) is a proximal factor of (X, S) under φ ,
- (π'_2) (Y, S) is an A -proximal factor of (X, S) under φ ,
- (π'_3) (Y, S) is an $A^{\overline{(M)}}$ proximal factor of (X, S) under φ ,
- (π'_4) (Y, S) is an $A^{\overline{(M)}}$ proximal factor of (X, S) under φ (by the assumption $\overline{M}(A) \neq \emptyset$),
- (ρ'_1) (X, S) is a proximal extension of (Y, S) under φ ,
- (ρ'_2) (X, S) is a B -proximal extension of (Y, S) under φ ,
- (ρ'_3) (X, S) is a $B^{\overline{(M)}}$ proximal extension of (Y, S) under φ ,
- (ρ'_4) (X, S) is a $B^{\overline{(M)}}$ proximal extension of (Y, S) under φ (by the assumption $\overline{M}(\varphi^{-1}(B)) \neq \emptyset$),

then we have the following tables:

TABLE 1.1. The mark “ \surd ” indicates that for the corresponding case we have: “ $(\pi_i \Rightarrow \pi_j) \wedge (\rho_i \Rightarrow \rho_j)$ ”

$\frac{j}{i}$	1	2	3	4
1	\surd	\surd	\surd	\surd
2		\surd	\surd	
3			\surd	
4				\surd

TABLE 1.2. The mark “ \surd ” indicates that for the corresponding case we have: “ $(\pi'_i \Rightarrow \pi'_j) \wedge (\rho'_i \Rightarrow \rho'_j)$ ”

$\frac{j}{i}$	1	2	3	4
1	\surd			
2	\surd	\surd		
3	\surd	\surd	\surd	
4	\surd			\surd

PROOF. We have the following conditional statements:

$$\begin{aligned}
 (\pi_1) &\Rightarrow R(\varphi) \cap P(X) = \Delta_X \\
 &\Rightarrow (R(\varphi) \cap \overline{P}_A(X) \subseteq R(\varphi) \cap P_A(X) \subseteq R(\varphi) \cap P(X) = \Delta_X \\
 &\quad \wedge R(\varphi) \cap \overline{\overline{P}}_A(X) \subseteq R(\varphi) \cap P(X) = \Delta_X) \text{ (by Note 1.3(c))} \\
 &\Rightarrow (R(\varphi) \cap \overline{P}_A(X) = R(\varphi) \cap P_A(X) = \Delta_X \wedge R(\varphi) \cap \overline{\overline{P}}_A(X) \subseteq \Delta_X) \\
 &\Rightarrow (\pi_2 \wedge \pi_3 \wedge \pi_4), \\
 (\rho_1) &\Rightarrow R(\varphi) \cap P(X) = \Delta_X \\
 &\Rightarrow (R(\varphi) \cap \overline{P}_{\varphi^{-1}(B)}(X) \subseteq R(\varphi) \cap P_{\varphi^{-1}(B)}(X) \subseteq R(\varphi) \cap P(X) = \Delta_X \\
 &\quad \wedge R(\varphi) \cap \overline{\overline{P}}_{\varphi^{-1}(B)}(X) \subseteq R(\varphi) \cap P(X) = \Delta_X) \text{ (by Note 1.3(c))} \\
 &\Rightarrow (R(\varphi) \cap \overline{P}_{\varphi^{-1}(B)}(X) = R(\varphi) \cap P_{\varphi^{-1}(B)}(X) = \Delta_X \\
 &\quad \wedge R(\varphi) \cap \overline{\overline{P}}_{\varphi^{-1}(B)}(X) \subseteq \Delta_X)
 \end{aligned}$$

$$\begin{aligned}
&\Rightarrow (\rho_2 \wedge \rho_3 \wedge \rho_4), \\
(\pi_2) &\Rightarrow R(\varphi) \cap P_A(X) = \Delta_X \\
&\Rightarrow R(\varphi) \cap \bar{P}_A(X) \subseteq R(\varphi) \cap P_A(X) = \Delta_X \text{ (by Note 1.3(c))} \\
&\Rightarrow R(\varphi) \cap \bar{P}_A(X) = \Delta_X \\
&\Rightarrow (\pi_3), \\
(\rho_2) &\Rightarrow R(\varphi) \cap P_{\varphi^{-1}(B)}(X) = \Delta_X \\
&\Rightarrow R(\varphi) \cap \bar{P}_{\varphi^{-1}(B)}(X) \subseteq R(\varphi) \cap P_{\varphi^{-1}(B)}(X) = \Delta_X \text{ (by Note 1.3(c))} \\
&\Rightarrow R(\varphi) \cap \bar{P}_{\varphi^{-1}(B)}(X) = \Delta_X \\
&\Rightarrow (\rho_3) \tag{1.9}
\end{aligned}$$

these complete the proof of Table 1.1, also

$$\begin{aligned}
(\pi'_3) &\Rightarrow R(\varphi) \subseteq \bar{P}_A(X) \\
&\Rightarrow R(\varphi) \subseteq \bar{P}_A(X) \subseteq P_A(X) \text{ (by Note 1.3(c))} \\
&\Rightarrow (\pi'_2) \\
&\Rightarrow R(\varphi) \subseteq P_A(X) \subseteq P(X) \text{ (by Note 1.3(c))} \\
&\Rightarrow (\pi'_1), \\
(\rho'_3) &\Rightarrow R(\varphi) \subseteq \bar{P}_{\varphi^{-1}(B)}(X) \\
&\Rightarrow R(\varphi) \subseteq \bar{P}_{\varphi^{-1}(B)}(X) \subseteq P_{\varphi^{-1}(B)}(X) \text{ (by Note 1.3(c))} \\
&\Rightarrow (\rho'_2) \\
&\Rightarrow R(\varphi) \subseteq P_{\varphi^{-1}(B)}(X) \subseteq P(X) \text{ (by Note 1.3(c))} \\
&\Rightarrow (\rho'_1), \\
(\pi'_4) &\Rightarrow R(\varphi) \subseteq \bar{\bar{P}}_A(X) \\
&\Rightarrow R(\varphi) \subseteq \bar{\bar{P}}_A(X) \subseteq P(X) \text{ (by Note 1.3(c))} \\
&\Rightarrow (\pi'_1), \\
(\rho'_4) &\Rightarrow R(\varphi) \subseteq \bar{\bar{P}}_{\varphi^{-1}(B)}(X) \\
&\Rightarrow R(\varphi) \subseteq \bar{\bar{P}}_{\varphi^{-1}(B)}(X) \subseteq P(X) \text{ (by Note 1.3(c))} \\
&\Rightarrow (\rho'_1), \tag{1.10}
\end{aligned}$$

□

these complete the proof of Table 1.2.

THEOREM 1.9. *Let $\varphi : (X, S) \rightarrow (Y, S)$ be an onto homomorphism and $\emptyset \neq C \subseteq A \subseteq X$, and $\emptyset \neq D \subseteq B \subseteq Y$, then*

- (a) *if (Y, S) is an A -distal factor of (X, S) , then (Y, S) is a C -distal factor of (X, S) ,*
- (b) *if (Y, S) is a $C^{\overline{M}}$ -distal factor of (X, S) , then (Y, S) is an $A^{\overline{M}}$ -distal factor of (X, S) ,*
- (c) *if (X, S) is a B -distal extension of (Y, S) , then (X, S) is a D -distal extension of (Y, S) ,*
- (d) *if (X, S) is a $D^{\overline{M}}$ -distal extension of (Y, S) , then (X, S) is a $B^{\overline{M}}$ -distal extension of (Y, S) ,*

- (e) if (Y, S) is a C -proximal factor of (X, S) , then (Y, S) is an A -proximal factor of (X, S) ,
- (f) if (Y, S) is an $A^{\overline{M}}$ -proximal factor of (X, S) , then (Y, S) is a $C^{\overline{M}}$ -proximal factor of (X, S)
- (g) if (X, S) is a D -proximal extension of (Y, S) , then (X, S) is a B -proximal extension of (Y, S) ,
- (h) if (X, S) is a $B^{\overline{M}}$ -proximal extension of (Y, S) , then (X, S) is a $D^{\overline{M}}$ -proximal extension of (Y, S) ,

the factors and extensions are under φ .

PROOF. (a) (Y, S) is an A -distal factor of (X, S)

$$\begin{aligned} &\Rightarrow R(\varphi) \cap P_A(X) = \Delta_X \\ &\Rightarrow R(\varphi) \cap P_C(X) \subseteq R(\varphi) \cap P_A(X) = \Delta_X \text{ (by Note 1.3(c))} \\ &\Rightarrow R(\varphi) \cap P_C(X) = \Delta_X \\ &\Rightarrow (Y, S) \text{ is a } C\text{-distal factor of } (X, S), \end{aligned} \tag{1.11}$$

(b) (Y, S) is a $C^{\overline{M}}$ -distal factor of (X, S)

$$\begin{aligned} &\Rightarrow R(\varphi) \cap \overline{P}_C(X) = \Delta_X \\ &\Rightarrow R(\varphi) \cap \overline{P}_A(X) \subseteq R(\varphi) \cap \overline{P}_C(X) = \Delta_X \text{ (by Note 1.3(c))} \\ &\Rightarrow R(\varphi) \cap \overline{P}_A(X) = \Delta_X \\ &\Rightarrow (Y, S) \text{ is an } A^{\overline{M}}\text{-distal factor of } (X, S), \end{aligned} \tag{1.12}$$

(e) (Y, S) is a C -proximal factor of (X, S)

$$\begin{aligned} &\Rightarrow R(\varphi) \subseteq P_C(X) \\ &\Rightarrow R(\varphi) \subseteq P_C(X) \subseteq P_A(X) \text{ (by Note 1.3(c))} \\ &\Rightarrow (Y, S) \text{ is an } A\text{-proximal factor of } (X, S), \end{aligned} \tag{1.13}$$

(f) (Y, S) is an $A^{\overline{M}}$ -proximal factor of (X, S)

$$\begin{aligned} &\Rightarrow R(\varphi) \subseteq \overline{P}_A(X) \\ &\Rightarrow R(\varphi) \subseteq \overline{P}_A(X) \subseteq \overline{P}_C(X) \text{ (by Note 1.3(c))} \\ &\Rightarrow (Y, S) \text{ is a } C^{\overline{M}}\text{-proximal factor of } (X, S). \quad \square \end{aligned} \tag{1.14}$$

THEOREM 1.10 (associative and inheritance laws). *Let $\varphi : (X, S) \rightarrow (Y, S)$ and $\psi : (Y, S) \rightarrow (Z, S)$ be two onto homomorphisms, and let A be a nonempty subset of X , and B be a nonempty subset of Y , then we have*

(a) ASSOCIATIVE LAWS.

- (i) $((Z, S) \text{ is a distal factor of } (Y, S) \text{ (under } \psi)) \wedge ((Y, S) \text{ is a distal factor of } (X, S) \text{ (under } \varphi)) \Rightarrow ((Z, S) \text{ is a distal factor of } (X, S) \text{ (under } \psi \circ \varphi))$,
- (ii) $((Z, S) \text{ is a } \varphi(A)\text{-distal factor of } (Y, S) \text{ (under } \psi)) \wedge ((Y, S) \text{ is an } A\text{-distal factor of } (X, S) \text{ (under } \varphi)) \Rightarrow ((Z, S) \text{ is an } A\text{-distal factor of } (X, S) \text{ (under } \psi \circ \varphi))$,

- (iii) $((Z, S)$ is a $\varphi(A)^{\overline{M}}$ distal factor of (Y, S) (under ψ) \wedge $((Y, S)$ is an $A^{\overline{M}}$ distal factor of (X, S) (under φ)) \Rightarrow $((Z, S)$ is an $A^{\overline{M}}$ distal factor of (X, S) (under $\psi \circ \varphi$)),
- (i)' $((X, S)$ is a distal extension of (Y, S) (under φ) \wedge $((Y, S)$ is a distal extension of (Z, S) (under ψ)) \Rightarrow $((X, S)$ is a distal extension of (Z, S) (under $\psi \circ \varphi$)),
- (ii)' $((X, S)$ is a $\psi^{-1}(B)$ -distal extension of (Y, S) (under φ) \wedge $((Y, S)$ is a B -distal extension of (Z, S) (under ψ)) \Rightarrow $((X, S)$ is a B -distal extension of (Z, S) (under $\psi \circ \varphi$)),
- (iii)' $((X, S)$ is a $\psi^{-1}(B)^{\overline{M}}$ distal extension of (Y, S) (under φ) \wedge $((Y, S)$ is a $B^{\overline{M}}$ distal extension of (Z, S) (under ψ)) \Rightarrow $((X, S)$ is a $B^{\overline{M}}$ distal extension of (Z, S) (under $\psi \circ \varphi$)),

(b) INHERITANCE LAWS.

- (i) $((Z, S)$ is a distal factor of (X, S) (under $\psi \circ \varphi$) \Rightarrow $((Y, S)$ is a distal factor of (X, S) (under φ)),
- (ii) $((Z, S)$ is an A -distal factor of (X, S) (under $\psi \circ \varphi$) \Rightarrow $((Y, S)$ is an A -distal factor of (X, S) (under φ)),
- (iii) $((Z, S)$ is an $A^{\overline{M}}$ distal factor of (X, S) (under $\psi \circ \varphi$) \Rightarrow $((Y, S)$ is an $A^{\overline{M}}$ distal factor of (X, S) (under φ)),
- (iv) $((Z, S)$ is an $A^{\overline{M}}$ distal factor of (X, S) (under $\psi \circ \varphi$) \Rightarrow $((Y, S)$ is an $A^{\overline{M}}$ distal factor of (X, S) (under φ)),
- (v) $((Z, S)$ is a proximal factor of (X, S) (under $\psi \circ \varphi$) \Rightarrow $((Y, S)$ is a proximal factor of (X, S) (under φ)),
- (vi) $((Z, S)$ is an A -proximal factor of (X, S) (under $\psi \circ \varphi$) \Rightarrow $((Y, S)$ is an A -proximal factor of (X, S) (under φ)),
- (vii) $((Z, S)$ is an $A^{\overline{M}}$ proximal factor of (X, S) (under $\psi \circ \varphi$) \Rightarrow $((Y, S)$ is an $A^{\overline{M}}$ proximal factor of (X, S) (under φ)),
- (viii) $((Z, S)$ is an $A^{\overline{M}}$ proximal factor of (X, S) (under $\psi \circ \varphi$) \Rightarrow $((Y, S)$ is an $A^{\overline{M}}$ proximal factor of (X, S) (under φ)),
- (i)' $((X, S)$ is a distal extension of (Z, S) (under $\psi \circ \varphi$) \Rightarrow $((X, S)$ is a distal extension of (Y, S) (under φ)),
- (ii)' $((X, S)$ is a B -distal extension of (Z, S) (under $\psi \circ \varphi$) \Rightarrow $((X, S)$ is a $\psi^{-1}(B)$ -distal extension of (Y, S) (under φ)),
- (iii)' $((X, S)$ is a $B^{\overline{M}}$ distal extension of (Z, S) (under $\psi \circ \varphi$) \Rightarrow $((X, S)$ is a $\psi^{-1}(B)^{\overline{M}}$ distal extension of (Y, S) (under φ)),
- (iv)' $((X, S)$ is a $B^{\overline{M}}$ distal extension of (Z, S) (under $\psi \circ \varphi$) \Rightarrow $((X, S)$ is a $\psi^{-1}(B)^{\overline{M}}$ distal extension of (Y, S) (under φ)),
- (v)' $((X, S)$ is a proximal extension of (Z, S) (under $\psi \circ \varphi$) \Rightarrow $((X, S)$ is a proximal extension of (Y, S) (under φ)),
- (vi)' $((X, S)$ is a B -proximal extension of (Z, S) (under $\psi \circ \varphi$) \Rightarrow $((X, S)$ is a $\psi^{-1}(B)$ -proximal extension of (Y, S) (under φ)),
- (vii)' $((X, S)$ is a $B^{\overline{M}}$ proximal extension of (Z, S) (under $\psi \circ \varphi$) \Rightarrow $((X, S)$ is a $\psi^{-1}(B)^{\overline{M}}$ proximal extension of (Y, S) (under φ)),

(viii)' $((X,S)$ is a $B^{\overline{M}}$ proximal extension of (Z,S) (under $\psi \circ \varphi$) $\Rightarrow ((X,S)$ is a $\psi^{-1}(B)^{\overline{M}}$ proximal extension of (Y,S) (under φ)).

PROOF. (a) (ii) Let (Z,S) be a $\varphi(A)$ -distal factor of (Y,S) under ψ , and let (Y,S) be an A -distal factor of (X,S) under φ , then $R(\psi) \cap P_{\varphi(A)}(Y) = \Delta_Y$ and $R(\varphi) \cap P_A(X) = \Delta_X$. Moreover, using the symbols of Note 1.6, we have $\varphi \times \varphi(R(\psi \circ \varphi)) \subseteq R(\psi)$ so $\varphi \times \varphi(R(\psi \circ \varphi) \cap P_A(X)) \subseteq R(\psi) \cap P_{\varphi(A)}(Y) = \Delta_Y$, thus $\varphi \times \varphi(R(\psi \circ \varphi) \cap P_A(X)) = \Delta_Y$, that is, $R(\psi \circ \varphi) \cap P_A(X) \subseteq R(\varphi)$, thus $R(\psi \circ \varphi) \cap P_A(X) \subseteq R(\varphi) \cap P_A(X)$, therefore $R(\psi \circ \varphi) \cap P_A(X) = \Delta_X$ and (Z,S) is an A -distal factor of (X,S) (under $\psi \circ \varphi$).

(b) Use $R(\varphi) \subseteq R(\psi \circ \varphi)$. □

THEOREM 1.11. Let B be a nonempty subset of X , let $\Sigma = \{\varphi_\alpha \mid \alpha \in \Gamma\}$ be a nonempty collection of the extensions of (X,S) , $\alpha_0 \in \Gamma$, $\times_\Sigma X_\alpha = \{(x_\alpha)_{\alpha \in \Gamma} \in \prod_{\alpha \in \Gamma} X_\alpha \mid \forall \alpha \in \Gamma \varphi_\alpha(x_\alpha) = \varphi_{\alpha_0}(x_{\alpha_0})\}$, for each $y \in \Gamma$ let $\pi_y : \times_\Sigma X_\alpha \rightarrow X_y$ be the projection map on the y th coordinate, and $\varphi : \times_\Sigma X_\alpha \rightarrow X$ be such that $\varphi((x_\alpha)_{\alpha \in \Gamma}) = \varphi_{\alpha_0}(x_{\alpha_0})$, then

(a) for each $y \in \Gamma$, the following diagram commutes:

$$\begin{array}{ccc}
 (\times_\Sigma X_\alpha, S) & \xrightarrow{\pi_\delta} & (X_y, S) \\
 \varphi \downarrow & \swarrow \varphi_y & \\
 (X, S) & &
 \end{array}
 \tag{1.15}$$

- (b) (i) if for each $\alpha \in \Gamma$, (X_α, S) is a distal extension of (X,S) (under φ_α), then $(\times_\Sigma X_\alpha, S)$ is a distal extension of (X,S) (under φ),
- (ii) if for each $\alpha \in \Gamma$, (X_α, S) is a B -distal extension of (X,S) (under φ_α), then $(\times_\Sigma X_\alpha, S)$ is a B -distal extension of (X,S) (under φ),
- (iii) if for each $\alpha \in \Gamma$, (X_α, S) is a $B^{\overline{M}}$ -distal extension of (X,S) (under φ_α), then $(\times_\Sigma X_\alpha, S)$ is a $B^{\overline{M}}$ -distal extension of (X,S) (under φ).

PROOF. (b) (ii) By the definition of $\times_\Sigma X_\alpha$, we have

$$\begin{aligned}
 R(\varphi) &= \{((x_\alpha)_{\alpha \in \Gamma}, (y_\alpha)_{\alpha \in \Gamma}) \in (\times_\Sigma X_\alpha) \times (\times_\Sigma X_\alpha) \mid \forall \alpha \in \Gamma (x_\alpha, y_\alpha) \in R(\varphi_\alpha)\} \\
 &= \{((x_\alpha)_{\alpha \in \Gamma}, (y_\alpha)_{\alpha \in \Gamma}) \in (\times_\Sigma X_\alpha) \times (\times_\Sigma X_\alpha) \mid \exists \alpha \in \Gamma (x_\alpha, y_\alpha) \in R(\varphi_\alpha)\},
 \end{aligned}
 \tag{1.16}$$

moreover, for each $((x_\alpha)_{\alpha \in \Gamma}, (y_\alpha)_{\alpha \in \Gamma}) \in P_{\varphi^{-1}(B)}(\times_\Sigma X_\alpha)$ and $y \in \Gamma$, we have $(x_y, y_y) \in P_{\varphi_y^{-1}(B)}(X_y)$, so if $((x_\alpha)_{\alpha \in \Gamma}, (y_\alpha)_{\alpha \in \Gamma}) \in R(\varphi) \cap P_{\varphi^{-1}(B)}(\times_\Sigma X_\alpha)$, then $(x_y, y_y) \in R(\varphi_y) \cap P_{\varphi_y^{-1}(B)}(X_y)$, this will give the desired result, that is, if for each $\alpha \in \Gamma$, $R(\varphi_\alpha) \cap P_{\varphi_\alpha^{-1}(B)}(X_\alpha) = \Delta_{X_\alpha}$, then $R(\varphi) \cap P_{\varphi^{-1}(B)}(\times_\Sigma X_\alpha) = (\Delta \times_\Sigma X_\alpha)$. □

ACKNOWLEDGEMENT. The authors would like to express their appreciation to the referee for his comments and suggestions which improved the original version.

REFERENCES

[1] R. Ellis, *A semigroup associated with a transformation group*, Trans. Amer. Math. Soc. **94** (1960), 272-281. MR 23#A961. Zbl 094.17402.
 [2] ———, *Lectures on Topological Dynamics*, W. A. Benjamin, New York, 1969. MR 42#2463. Zbl 193.51502.

- [3] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, M. B. Porter Lectures, Princeton University Press, New Jersey, 1981. [MR 82j:28010](#). [Zbl 459.28023](#).
- [4] M. Sabbaghan and F. A. Z. Shirazi, *a-minimal sets and related topics in transformation semigroups (I)*, *Int. Math. Math. Sci* 25 (2001), no. 10, 637–654.

MASOUD SABBAGHAN: DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, THE UNIVERSITY OF TEHRAN, ENGHELAB AVE., TEHRAN, IRAN
E-mail address: sabbagh@khayam.ut.ac.ir

FATEMAH AYATOLLAH ZADEH SHIRAZI: DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, THE UNIVERSITY OF TEHRAN, ENGHELAB AVE., TEHRAN, IRAN
E-mail address: fatemah@khayam.ut.ac.ir