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STRUCTURE OF WEAKLY PERIODIC RINGS WITH POTENT EXTENDED COMMUTATORS

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Dedicated to the memory of Professor Hisao Tominaga

ABSTRACT. A well-known theorem of Jacobson (1964, page 217) asserts that a ring R with the property that, for each x in R, there exists an integer $n(x) > 1$ such that $x^{n(x)} = x$ is necessarily commutative. This theorem is generalized to the case of a weakly periodic ring R with a "sufficient" number of potent extended commutators. A ring R is called *weakly periodic* if every x in R can be written in the form $x = a + b$, where a is nilpotent and b is "potent" in the sense that $b^{n(b)} = b$ for some integer $n(b) > 1$. It is shown that a weakly periodic ring R in which certain extended commutators are potent must have a nil commutator ideal and, moreover, the set N of nilpotents forms an ideal which, in fact, coincides with the Jacobson radical of R.

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1. Introduction. A ring R is called *periodic* if for each x in R there exist distinct positive integers m and n such that $x^m = x^n$. An element x is called *potent* if for some integer $n = n(x) > 1$, $x^n = x$. R is called *weakly periodic* if every x in R can be written (not necessarily uniquely) as a sum of a nilpotent element and a potent element. It is well known that a periodic ring is necessarily weakly periodic (see [2]). Whether a weakly periodic ring is necessarily periodic is apparently not known. Moreover, by a theorem of Chacron (see [3]), R is periodic if and only if for each x in R, there exists a positive integer $k = k(x)$ and a polynomial $f(\lambda) = f_x(\lambda)$ with integer coefficients such that $x^k = x^{k+1}f(x)$. For x, y in R, $[x,y]_1 = [x,y] = xy - yx$ denotes the usual commutator, and for every positive integer $k > 1$, we define the *extended commutator* $[x,y]_k$ inductively by $[x,y]_k = [[x,y]_{k-1}, y]$.

2. Main results. We begin with some basic facts about weakly periodic rings.

LEMMA 2.1. *The homomorphic image of any weakly periodic ring is weakly periodic.*

This follows readily from the definition of a weakly periodic ring.

LEMMA 2.2. *A weakly periodic division ring is necessarily commutative.*

This follows from the "$x^{n(x)} = x$" theorem of Jacobson (see [5]).

LEMMA 2.3. *Let R be a weakly periodic ring, N the set of nilpotents, and J the Jacobson radical of R. Then $J \subseteq N$.*

Proof. Suppose $j \in J$. Then,

$$j = a + b, \quad a \in N, \qquad b^n = b \quad \text{for some } n > 1. \tag{2.1}$$

Suppose $a^q = 0$. Then,

$$(j - b)^{(n-1)q+1} = a^{(n-1)q+1} = 0, \tag{2.2}$$

which implies (since $j \in J$) $b^{(n-1)q+1} \in J$. But $b^{(n-1)q+1} = b$, since $b^n = b$, and hence $b \in J$. Since b^{n-1} is an *idempotent* element in J, $b^{n-1} = 0$. Therefore, $b = b^n = 0$, and hence by (2.1), $j = a \in N$. Thus, $J \subseteq N$. □

Theorem 2.4. *Let R be a weakly periodic ring and suppose that N is the set of nilpotents of R. Let $n > 1$ be a fixed integer. Suppose that for all $x_1, \dots, x_n$ in $R \backslash N$, σ is a permutation in S_n such that $\sigma(n) \neq n$. Suppose, further, that for all $x_1, \dots, x_n$ in $R \backslash N$, there exists a positive integer k such that the following extended commutator is potent, namely*

$$[x_1 \cdots x_n,\ x_{\sigma(1)} \cdots x_{\sigma(n)}]_k \text{ is potent}, \quad \forall x_i \in R \backslash N. \tag{2.3}$$

Then,

(i) *The commutator ideal of R is nil, $(C(R) \subseteq N)$.*
(ii) *N is an ideal of R.*
(iii) *$N = J$, the Jacobson radical of R.*
(iv) *R is periodic.*

Proof. The semisimple ring R/J is isomorphic to a subdirect sum of primitive rings R_i $(i \in \Gamma)$. By Lemma 2.1, each R_i is a weakly periodic ring. Now, by Jacobson's density theorem, we must have

(a) R_i is a division ring, or
(b) for some positive integer $m > 1$, there exists a subring T_i of R_i which maps homomorphically onto D_m, the complete matrix ring of $m \times m$ matrices over some division ring D.

In case (a), R_i is commutative, by Lemma 2.2. In case (b), D_m must satisfy (2.3), since (2.3) is inherited by all subrings and all homomorphic images of R, where $m > 1$. The net result is:

$(*)$ The ring D_m of all $m \times m$ matrices over the division ring D satisfies (2.3), where $m > 1$.

That statement $(*)$ is false can be seen by taking

$$x_1 = x_2 = \cdots = x_{n-1} = E_{11}, \qquad x_n = E_{11} + E_{12}. \tag{2.4}$$

In verifying this, note that

$$\begin{gathered} x_1 \cdots x_n = E_{11}(E_{11} + E_{12}) = E_{11} + E_{12}; \\ x_{\sigma(1)} \cdots x_{\sigma(n)} = (E_{11} + E_{12})E_{11} = E_{11}; \\ [x_1 \cdots x_n, x_{\sigma(1)} \cdots x_{\sigma(n)}] = -E_{12}; \\ [x_1 \cdots x_n, x_{\sigma(1)} \cdots x_{\sigma(n)}]_k = (-1)^k E_{12}, \end{gathered} \tag{2.5}$$

which is not potent. This contradiction shows that case (b) never occurs, and hence *each R_i is a commutative division ring* (case (a)). Thus, R/J is indeed commutative, and hence

$$C(R) \subseteq J, \quad (C(R) \text{ denotes the commutator ideal of } R). \tag{2.6}$$

Combining this with Lemma 2.3, we see that $C(R) \subseteq N$, which proves (i). Part (ii) follows at once from part (i) by considering the *commutative* ring $R/C(R)$. To prove (iii), observe that $N \subseteq J$ (since N is an ideal) while $J \subseteq N$, by Lemma 2.3. Thus, $N = J$.

To prove part (iv), let $x \in R$. Then, by definition of weakly periodic ring, there exist elements a and b such that

$$x = a+b, \quad a \in N; \qquad b^{\gamma} = b, \quad \text{for some } \gamma > 1. \tag{2.7}$$

Hence, since N is an *ideal* (part (ii)) and $a \in N$,

$$x-a = b = b^{\gamma} = (x-a)^{\gamma} = x^{\gamma} + a_0 \quad (a_0 \in N). \tag{2.8}$$

Therefore, $x - x^{\gamma} = a + a_0 \in N$, and thus $(x-x^{\gamma})^{\alpha} = 0$ for some positive integer α. Hence, $x^{\alpha} = x^{\alpha+1} f(x)$, $f(\lambda) \in \mathbb{Z}[\lambda]$, and thus by Chacron's theorem (see Section 1) R is periodic. This proves the theorem. □

In preparation for the proof of the next theorem, we first prove the following lemmas.

Lemma 2.5. *Let R be an arbitrary ring (not necessarily weakly periodic), and suppose N is the set of nilpotents of R. Let $n > 1$ be a fixed integer. Suppose that for all $x_1,\dots,x_n$ in $R\backslash N$, σ is a permutation in S_n such that $\sigma(1) \neq 1$ and $\sigma(n) \neq n$. Suppose, further, that for all $x_1,\dots,x_n$ in $R\backslash N$, there exists a positive integer k such that*

$$[x_1 \cdots x_n, x_{\sigma(1)} \cdots x_{\sigma(n)}]_k \text{ is potent}, \quad \forall x_i \in R\backslash N. \tag{2.9}$$

Then, the set E of idempotents of R is central.

Proof. Suppose $e \in E$, $x \in R$, $a = ex - exe$, $f = e+a$. We claim that $ef = fe$. Suppose $ef \neq fe$, then $e \neq 0$, $f \neq 0$, and (since $e^2 = e$, $f^2 = f$) hence $e \notin N$, $f \notin N$. Therefore, by (2.9) with $x_1 = \cdots = x_{n-1} = e$, $x_n = f$, we have $[ef, fe]_k$ is potent, and hence $[f,e]_k = (-1)^k a$ is potent. Since $a^2 = 0$, it follows that $a = 0$. Hence, $f = e+a = e$, which contradicts the hypothesis $ef \neq fe$. This contradiction shows that $ef = fe$, and hence $e(e+a) = (e+a)e$. Thus, $a = ea = ae = 0$. Hence, $ex = exe$. A similar argument, using $a' = xe - exe$, $f' = e + a'$ shows that $xe = exe$, and hence $ex = xe$. This proves the lemma. □

Lemma 2.6. *Suppose that R is a weakly periodic ring which satisfies the hypotheses of Theorem 2.4. Suppose $\delta : R \to R^*$ is a homomorphism of R onto R^*, and let N be the set of nilpotents of R. Then, the set N^* of nilpotents of R^* coincides with the set $\delta(N)$.*

Proof. By Theorem 2.4(iv), R is periodic. The lemma now follows from [1]. □

Lemma 2.7. *Let R be a subdirectly irreducible ring. Then, the only central idempotents of R are 0 and 1 (if $1 \in R$).*

This lemma is well known, and we omit the proof.

Lemma 2.8. *Let R be a ring, and let $x,y \in R$. Suppose that $[x,y]$ commutes with x. Then, for all positive integers k, we have*

$$[x^k, y] = kx^{k-1}[x,y]. \tag{2.10}$$

(Equivalently, $[y,x^k] = kx^{k-1}[y,x]$.)

This follows at once, by induction.

Lemma 2.9. *Let R be a periodic ring with the set N of nilpotents commutative. If for each $a \in N$ and $x \in R$ there exists a positive integer k such that $[a,x]_k = 0$, then R is commutative.*

This lemma was proved by Bell [4].

We are now in a position to prove our next theorem.

Theorem 2.10. *Let R be a weakly periodic ring, and let N denote the set of nilpotents of R. Let $n > 1$ be a fixed integer. Suppose that for all $x_1,\dots,x_n$ in $R\backslash N$, σ is a permutation in S_n such that $\sigma(1) = n$ and $\sigma(n) = 1$. Suppose that, for all $x_1,\dots,x_n$ in $R\backslash N$, there exists a positive integer k such that*

$$[x_1\cdots x_n, x_{\sigma(1)}\cdots x_{\sigma(n)}]_k \text{ is potent}, \quad \forall x_i \in R\backslash N. \tag{2.11}$$

Suppose, further, that

$$[a,b] \text{ is potent} \quad \forall a,b \in N. \tag{2.12}$$

Then, R is commutative.

Proof. In view of Lemma 2.6, all the hypotheses are inherited by homomorphic images of R; and since every ring is isomorphic to a subdirect sum of subdirectly irreducible rings, we may assume that R is subdirectly irreducible. Since N is an ideal, by Theorem 2.4(ii), we see that for all a,b in N, $[a,b]$ is both potent (see (2.12)) and nilpotent, and hence $[a,b] = 0$, which implies that N is commutative.

Now, R is periodic, by Theorem 2.4(iv), and hence some power of each element of R is idempotent. Therefore, by Lemmas 2.5 and 2.7, either R is nil or R has an identity 1. In the first case, $R = N$ is commutative and there is nothing further to prove. So we assume that $1 \in R$.

Let $a \in N$ and $x \in R\backslash N$. Since $1 + a \notin N$, there exists a positive integer k such that

$$\begin{aligned} &[(1+a)\cdot 1\cdot 1\cdots 1\cdot x, x\cdot 1\cdot 1\cdots 1\cdot(1+a)]_k \text{ is potent},\\ &\text{and thus } [x+ax, x+xa]_k \text{ is potent.} \end{aligned} \tag{2.13}$$

Next, we show, by induction, that

$$[x+ax, x+xa]_m = [a,x]_{m+1} \quad \text{for all positive integers } m. \tag{2.14}$$

To begin with, observe that

$$[x+ax, x+xa]_1 = [x,xa] + [ax,x] + [ax,xa]. \tag{2.15}$$

Since N is a commutative ideal of R and $a \in N$, therefore $[ax, xa] = 0$, and hence (2.15) is equivalent to

$$[x+ax, x+xa]_1 = -[xa, x] + [ax, x] = [a, x]_2. \tag{2.16}$$

Hence (2.14) is true for $m = 1$. Now, suppose (2.14) is true for $m = q$. That is, suppose that

$$[x+ax, x+xa]_q = [a, x]_{q+1}. \tag{2.17}$$

This induction hypothesis implies that

$$\begin{aligned}[x+ax, x+xa]_{q+1} &= [[a,x]_{q+1}, x+xa] \\ &= [[a,x]_{q+1}, x] + [[a,x]_{q+1}, xa] \\ &= [[a,x]_{q+1}, x] = [a,x]_{q+2},\end{aligned}$$

since $[[a,x]_{q+1}, xa] = 0$ (recall that $a \in N$ and N is a commutative ideal of R). Thus, (2.14) is true for $m = q+1$, completing this induction proof of (2.14). Now, combining (2.13) and (2.14), we see that

$$[a, x]_{k+1} \text{ is potent} \quad (a \in N,\ x \in R). \tag{2.18}$$

Since $[a, x]_{k+1}$ is also in the ideal N, therefore this extended commutator is both nilpotent and potent, and hence

$$[a, x]_{k+1} = 0. \tag{2.19}$$

Keeping in mind that R is periodic and N is commutative, and combining (2.19) with Lemma 2.9, it follows that R is commutative, and the theorem is proved. □

Corollary 2.11. *Suppose R is a periodic ring with commuting nilpotents and with the property that, for all x, y in R, there exists a positive integer k such that $[xy, yx]_k = 0$. Then, R is commutative.*

Proof. Since R is periodic, R is also weakly periodic (see Section 1). Therefore, all the hypotheses of Theorem 2.10 are satisfied (take $n = 2$ in (2.11)), and hence R is commutative. □

The following is another corollary which yields a result proved by Putcha and Yaqub [6].

Corollary 2.12. *A periodic ring with commuting nilpotents and central commutators is commutative.*

Proof. Let x, y be any elements of R. Since $[x, y]$ is in the center of R, therefore $[[x, y], y] = 0$; that is, $[x, y]_2 = 0$. Thus,

$$[x, y]_2 = 0 \quad \forall x, y \in R. \tag{2.20}$$

In particular, $[xy, yx]_2 = 0$, and the corollary now follows by taking $k = 2$ in Corollary 2.11. □

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