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GENERALIZATION OF SOME ABSOLUTE SUMMABILITY METHODS

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ABSTRACT. Mazhar (1971) gave the characterization for the series $\sum a_n\epsilon_n$ to be summable $|N,p_n|$ whenever $\sum a_n$ is summable $|C,\alpha|_k$, $\alpha\ge 0$, $k\ge 1$. Here we prove two theorems, the first concerns the sufficient conditions and the second the necessary conditions satisfied by $\{\epsilon_n\}$ in order to have $\sum a_n\epsilon_n$ summable $|\bar{N},p_n|_k$ whenever $\sum a_n$ is summable $|C,\alpha|_k$, $k\ge 1$.

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Let $\sum a_n$ be a given infinite series with partial sums s_n. Let σ_n^δ and r_n^δ denote the nth Cesàro mean of order δ $(\delta > -1)$ of the sequences $\{s_n\}$ and $\{na_n\}$, respectively. The series $\sum a_n$ is said to be summable $|C,\delta|_k$, $k\ge 1$ if

$$\sum_{n=1}^{\infty} n^{k-1}\left|\sigma_n^\delta-\sigma_{n-1}^\delta\right|^k < \infty, \tag{1}$$

or equivalently

$$\sum_{n=1}^{\infty} n^{-1}\left|r_n^\delta\right|^k < \infty. \tag{2}$$

Let $\{P_n\}$ be a sequence of positive real constants such that

$$P_n = p_0+p_1+\cdots+p_n, \quad P_n \longrightarrow \infty \text{ as } n\longrightarrow\infty,\ (P_{-1}=p_{-1}=0). \tag{3}$$

A series $\sum a_n$ is said to be summable $|\bar{N},p_n|_k$, $k\ge 1$, (see [1]), if

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|t_n-t_{n-1}\right|^k < \infty, \tag{4}$$

where

$$t_n = \left(\frac{1}{P_n}\right)\sum_{\nu=0}^{\infty} p_\nu s_\nu. \tag{5}$$

For $p_n = 1/(n+1)$, the summability $|\bar{N}, p_n|_k$ reduces to the well-known summability $|R, \log n, 1|_k$. In general, the two summabilities $|\bar{N}, p_n|_k$ and $|C, \alpha|_k$ are independent of each other, but for $p_n = 1$ and $\alpha = 1$ they are equivalent. For any real α and integer $n \geq 0$, we define

$$\Delta^\alpha \epsilon_n = \sum_{\nu=n}^{\infty} A_{\nu-n}^{-\alpha-1} \epsilon_\nu, \tag{6}$$

whenever the series is convergent.

Theorem A (see [2]). *The necessary and sufficient conditions for the series $\sum a_n \epsilon_n$ to be summable $|\bar{N}, p_n|$ whenever $\sum a_n$ is summable $|C, \alpha|_k$, $\alpha \geq 0$, $k \geq 1$ are*

(i) $n^{\alpha+1-(1/k')} \Delta^\alpha (\epsilon_n / n) \in \ell^{k'}$, $1/k + 1/k' = 1$;

(ii) $n^{-1/k'} \epsilon_n \in \ell^{k'}$, $0 \leq \alpha \leq 1$;

(iii) $n^{\alpha-(1/k')} (p_n / P_n) \epsilon_n \in \ell^{k'}$, $\alpha > 1$;

where (a) $p_n = O(p_{n+1})$, (b) $(n+1) p_n = O(P_n)$, *and* (c) $P_n = O(n^\alpha p_n)$ $(\alpha > 1)$. *Theorem A includes as special cases $k = 1$ and $p_n = 1$, the result of Mohapatra* [4] *and Mehdi* [3], *respectively.*

We need the following lemmas.

Lemma 1 (see [5]). *If $\sigma > \delta > 0$, then*

$$\sum_{n=\nu+1}^{\infty} \frac{(n-\nu)^{\delta-1}}{n^\sigma} = O(\nu^{\delta-\sigma}). \tag{7}$$

Lemma 2. *If $\sigma > \delta \geq 0$, then*

$$\sum_{n=\nu+1}^{\infty} \frac{A_{n-\nu}^{\delta-1}}{A_n^\sigma} = O(\nu^{\delta-\sigma}). \tag{8}$$

Proof. If $\delta > 0$, then by Lemma 1,

$$\sum_{n=\nu+1}^{\infty} \frac{A_{n-\nu}^{\delta-1}}{A_n^\sigma} = O(1) \sum_{n=\nu+1}^{\infty} \frac{(n-\nu)^{\delta-1}}{n^\sigma} = O(\nu^{\delta-\sigma}). \tag{9}$$

If $\delta = 0$, then by Lemma 1,

$$\sum_{n=\nu+1}^{\infty} \frac{A_{n-\nu}^{-1}}{A_n^\sigma} = O(1) \frac{1}{A_n^\sigma} \sum_{n=\nu+1}^{\infty} |A_{n-\nu}^{-1}| = O(n^{-\sigma}). \tag{10}$$

□

Lemma 3 (see [6]). *Let $k \geqslant 1$, $P_n = p_0 + p_1 + \cdots + p_n \to \infty$ as $n \to \infty$, $p_n > 0$. Then for all $\nu \geq 1$,*

$$\frac{1}{kP_{\nu-1}^k} \leqslant \sum_{n=\nu}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \leqslant \frac{1}{P_{\nu-1}^k}. \tag{11}$$

We prove the following theorems.

Theorem 4. *Let $\{\epsilon_n\}$ be monotonic nonincreasing sequence of constants. Suppose that $\{P_n\epsilon_n/n\}$ is nonincreasing, and*

$$np_n = O(P_n). \tag{12}$$

Then the following are sufficient conditions for $\sum a_n\epsilon_n$ to be summable $|\bar{N}, p_n|_k$ whenever $\sum a_n$ is summable $|C,\alpha|_k$, $0 < \alpha \leq \alpha k \leq 1$:

(i) $\epsilon_n = O(np_n/P_n)$,

(ii) $\Delta\epsilon_n = O(p_n/P_n)$.

Proof of Theorem 4. Define

$$T_n = \frac{(p_n/P_n)^{1/k}}{P_{n-1}} \sum_{\nu=1}^{n} P_{\nu-1} a_\nu \epsilon_\nu, \qquad t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} \nu \alpha_\nu,$$

$$\begin{aligned}
T_n &= \frac{(p_n/P_n)^{1/k}}{P_{n-1}} \sum_{\nu=1}^{n} \frac{P_{\nu-1}\epsilon_\nu}{\nu} \sum_{r=1}^{n} A_{\nu-r}^{-\alpha-1} A_r^\alpha t_r^\alpha \\
&= \frac{(p_n/P_n)^{1/k}}{P_{n-1}} \sum_{r=1}^{n} A_r^\alpha t_r^\alpha \sum_{\nu=r}^{n} \frac{P_{\nu-1}\epsilon_\nu}{\nu} A_{\nu-r}^{-\alpha-1} \\
&= \frac{(p_n/P_n)^{1/k}}{P_{n-1}} \sum_{r=1}^{n} A_r^\alpha t_r^\alpha \sum_{\nu=0}^{n-r} \frac{P_{\nu-r-1}\epsilon_{\nu+r}}{\nu+r} A_\nu^{-\alpha-1} \\
&= \frac{(p_n/P_n)^{1/k}}{P_{n-1}} \sum_{r=1}^{n} A_r^\alpha t_r^\alpha \left[\sum_{\nu=0}^{n-r-1} \left(\sum_{\mu=0}^{\nu} A_\mu^{-\alpha-1} \right) \Delta_\nu \left(\frac{P_{\nu+r-1}\epsilon_{\nu+r}}{\nu+r} \right) \right. \\
&\qquad\qquad \left. + \left(\sum_{\mu=0}^{n-r} A_\mu^{-\alpha-1} \right) \frac{P_{n-1}\epsilon_n}{n} \right] \\
&= \frac{(p_n/P_n)^{1/k}}{P_{n-1}} \sum_{r=1}^{n} A_r^\alpha t_r^\alpha \left[\sum_{\nu=0}^{n-r-1} A_\nu^{-\alpha} \left\{ \frac{P_{\nu+r-1}\epsilon_{\nu+r}}{(\nu+r)(\nu+r+1)} - \frac{p_{\nu+r}\epsilon_{\nu+r}}{\nu+r+1} \right. \right. \\
&\qquad\qquad \left. \left. + \frac{P_{\nu+r}\Delta_\nu\epsilon_{\nu+r}}{\nu+r+1} \right\} + A_{n-r}^{-\alpha} \frac{P_{n-1}}{n} \epsilon_n \right], \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}.
\end{aligned} \tag{13}$$

In order to prove sufficiency, by Minkowski's inequality, it is sufficient to show that

$$\sum_{r=1}^{m} |T_{n,r}|^k < \infty, \quad r = 1,2,3,4. \tag{14}$$

$$\begin{aligned}
\sum_{n=1}^{m}|T_{n,1}|^{k} &= \sum_{n=1}^{m}\frac{p_n}{P_nP_{n-1}^{k}}\left|\sum_{r=1}^{n}A_r^{\alpha}t_r^{\alpha}\sum_{\nu=0}^{n-r-1}A_\nu^{-\alpha}\frac{P_{\nu+r-1}\epsilon_{\nu+r}}{(\nu+r)(\nu+r+1)}\right|^{k}\\
&\le \sum_{n=1}^{m}\frac{p_n}{P_nP_{n-1}^{k}}\left|\sum_{r=1}^{n}A_r^{\alpha}t_r^{\alpha}\frac{P_{r-1}\epsilon_r}{r}\sum_{\nu=0}^{n-r-1}\frac{A_\nu^{-\alpha}}{\nu+r+1}\right|^{k}\\
&\qquad\qquad\text{as } \frac{P_n\epsilon_n}{n} \text{ being nonincreasing}\\
&= \sum_{n=1}^{m}\frac{p_n}{P_nP_{n-1}^{k}}\left|\sum_{r=1}^{n}A_r^{\alpha}t_r^{\alpha}\frac{P_{r-1}\epsilon_r}{r}\sum_{\nu=r}^{n}\frac{(\nu-r)^{-\alpha}}{\nu}\right|^{k}\\
&= O(1)\sum_{n=1}^{m}\frac{p_n}{P_nP_{n-1}^{k}}\left|\sum_{r=1}^{n}A_r^{\alpha}t_r^{\alpha}\frac{P_{r-1}\epsilon_r}{r}r^{-\alpha}\right|^{k}\\
&\le O(1)\sum_{n=1}^{m}\frac{p_n}{P_nP_{n-1}^{k}}\left|\sum_{r=1}^{n}p_rt_r^{\alpha}\frac{P_r}{rp_r}\epsilon_r\right|^{k}\\
&\le O(1)\sum_{n=1}^{m}\frac{p_n}{P_nP_{n-1}}\sum_{r=1}^{n}p_r|t_r^{\alpha}|^{k}\left(\frac{P_r}{rp_r}\right)^{k}|\epsilon_r|^{k}\left\{\sum_{r=1}^{n}\frac{p_r}{P_{n-1}}\right\}^{k-1}\\
&= O(1)\sum_{r=1}^{m}p_r|t_r^{\alpha}|^{k}\left(\frac{P_r}{rp_r}\right)^{k}|\epsilon_r|^{k}\sum_{n=r}^{m}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{r=1}^{m}\frac{p_r}{P_r}|t_r^{\alpha}|^{k}\left(\frac{P_r}{rp_r}\right)^{k}|\epsilon_r|^{k}\\
&= O(1)\sum_{r=1}^{m}\frac{1}{r}|t_r^{\alpha}|^{k}.
\end{aligned}\tag{15}$$

$$\begin{aligned}
\sum_{n=1}^{m}|T_{n,2}|^{k} &= \sum_{n=1}^{m}\frac{p_n}{P_nP_{n-1}^{k}}\left|\sum_{r=1}^{n}A_r^{\alpha}t_r^{\alpha}\sum_{\nu=0}^{n-r-1}A_\nu^{-\alpha}\frac{-p_{\nu+r}\epsilon_{\nu+r}}{\nu+r+1}\right|^{k}\\
&\le \sum_{n=1}^{m}\frac{p_n}{P_nP_{n-1}^{k}}\left|\sum_{r=1}^{n}A_r^{\alpha}t_r^{\alpha}p_r\epsilon_r\sum_{\nu=0}^{n-r-1}\frac{A_\nu^{-\alpha}}{\nu+r+1}\right|^{k}\\
&\le \sum_{n=1}^{m}\frac{p_n}{P_nP_{n-1}^{k}}\left|\sum_{r=1}^{n}t_r^{\alpha}p_r\epsilon_r\right|^{k}\\
&\le O(1)\sum_{n=1}^{m}\frac{p_n}{P_nP_{n-1}}\sum_{r=1}^{n}|t_r^{\alpha}|^{k}p_r|\epsilon_r|^{k}\left\{\sum_{r=1}^{n}\frac{p_r}{P_{n-1}}\right\}^{k-1}\\
&= O(1)\sum_{r=1}^{n}|t_r^{\alpha}|^{k}p_r|\epsilon_r|^{k}\sum_{n=r}^{m}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{r=1}^{m}\frac{p_r}{P_r}|t_r^{\alpha}|^{k}|\epsilon_r|^{k}\\
&= O(1)\sum_{r=1}^{m}\frac{1}{r}|t_r^{\alpha}|^{k}|\epsilon_r|^{k}\\
&= O(1)\sum_{r=1}^{m}\frac{1}{r}|t_r^{\alpha}|^{k}\quad\text{as } \epsilon_r=O(1) \text{ being nonincreasing.}
\end{aligned}\tag{16}$$

$$\begin{aligned}
\sum_{n=1}^{m} |T_{n,3}|^k &= \sum_{n=1}^{m} \frac{p_n}{P_n P_{n-1}^k} \left| \sum_{r=1}^{n} A_r^{\alpha} t_r^{\alpha} \sum_{\nu=0}^{n-r-1} A_\nu^{-\alpha} \frac{P_{\nu+r} \Delta_\nu \epsilon_{\nu+r}}{\nu+r+1} \right|^k \\
&= \sum_{n=1}^{m} \frac{p_n}{P_n P_{n-1}^k} \left| \sum_{r=1}^{n} A_r^{\alpha} t_r^{\alpha} P_r \Delta\epsilon_r \sum_{\nu=0}^{n-r-1} \frac{A_\nu^{-\alpha}}{\nu+r+1} \right|^k \\
&\le O(1) \sum_{n=1}^{m} \frac{p_n}{P_n P_{n-1}^k} \left| \sum_{r=1}^{n} t_r^{\alpha}\, p_r \Delta\epsilon_r \right|^k \\
&= O(1) \sum_{n=1}^{m} \frac{p_n}{P_n P_{n-1}} \sum_{r=1}^{n} p_r |t_r^{\alpha}|^k \left(\frac{P_r}{p_r}\right)^k |\Delta\epsilon_r|^k \left\{ \sum_{r=1}^{n} \frac{p_r}{P_{n-1}} \right\}^{k-1} \\
&= O(1) \sum_{r=1}^{m} p_r |t_r^{\alpha}|^k \left(\frac{P_r}{p_r}\right)^k |\Delta\epsilon_r|^k \sum_{n=r}^{m} \frac{p_n}{P_n P_{n-1}} \\
&= O(1) \sum_{r=1}^{m} \frac{p_r}{P_r} |t_r^{\alpha}|^k \left(\frac{P_r}{p_r}\right)^k |\Delta\epsilon_r|^k \\
&= O(1) \sum_{r=1}^{m} \frac{1}{r} |t_r^{\alpha}|^k .
\end{aligned} \tag{17}$$

$$\begin{aligned}
\sum_{n=1}^{m} |T_{n,4}|^k &= \sum_{n=1}^{m} \frac{p_n}{P_n P_{n-1}^k} \left| \sum_{r=1}^{n} A_r^{\alpha} t_r^{\alpha} A_{n-r}^{-\alpha} \frac{P_{n-1}}{n} \epsilon_n \right|^k \\
&= \sum_{n=1}^{m} \frac{p_n |\epsilon_n|^k}{P_n n^k} \sum_{r=1}^{n} (A_r^{\alpha})^k |t_r^{\alpha}|^k (A_{n-r}^{-\alpha})^k \left\{ \sum_{r=1}^{n} 1 \right\}^{k-1} \\
&= \sum_{n=1}^{m} \frac{p_n |\epsilon_n|^k}{P_n n} \sum_{r=1}^{n} (A_r^{\alpha})^k |t_r^{\alpha}|^k (A_{n-r}^{-\alpha})^k \\
&= \sum_{r=1}^{m} (A_r^{\alpha})^k |t_r^{\alpha}|^k \sum_{n=r}^{m} \frac{p_n |\epsilon_n|^k}{n P_n} (A_{n-r}^{-\alpha})^k \\
&\le O(1) \sum_{r=1}^{m} (A_r^{\alpha})^k |t_r^{\alpha}|^k |\epsilon_r|^k \sum_{n=r}^{m} \frac{A_{n-r}^{-\alpha k}}{n^2} \\
&= O(1) \sum_{r=1}^{m} (A_r^{\alpha})^k |t_r^{\alpha}|^k |\epsilon_r|^k r^{-1-\alpha k} \\
&= O(1) \sum_{r=1}^{m} \frac{1}{r} |t_r^{\alpha}|^k .
\end{aligned} \tag{18}$$

□

Theorem 5. *The necessary conditions for $\sum a_n \epsilon_n$ to be summable $|\bar{N}, p_n|_k$ whenever $\sum a_n$ is summable $|C,\alpha|_k$, $k > 1$, are*

(i) $\epsilon_n = O\{(P_n/p_n)^{1/k} \nu^{1-\alpha-(1/k)}\}$,

(ii) $\Delta\epsilon_n = O(\nu^{1-\alpha-(1/k)})$.

Proof of Theorem 5. For $k \geqslant 1$, we define

$$\begin{aligned}
A &= \Big\{ \{a_i\} : \textstyle\sum a_i \text{ is summable } |C,\alpha|_k \Big\}, \\
B &= \Big\{ \{a_i\} : \textstyle\sum a_i \epsilon_i \text{ is summable } |\bar{N}, p_n|_k \Big\}.
\end{aligned} \tag{19}$$

These are BK-spaces, if normed by

$$\|a\|_1 = \left\{ \sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k \right\}^{1/k}, \qquad \|a\|_2 = \left\{ \sum_{n=1}^{\infty} |T_n|^k \right\}^{1/k}, \tag{20}$$

respectively. Since $\sum (1/n)|t_n|^k < \infty \Rightarrow \sum |T_n|^k < \infty$ by the hypothesis of Theorem 5, then

$$\|a\|_1 < \infty \Longrightarrow \|a_2\| < \infty. \tag{21}$$

Consider the inclusion map $i : A \to B$ defined by $i(x) = x$, i is continuous which follows as A and B are BK-spaces. Therefore, there exists a constant M such that

$$\|a\|_2 \le M \|a\|_1. \tag{22}$$

Applying the values of t_n and T_n, stated in Theorem 5, to $a = e_\nu - e_{\nu+1}$ (where e_ν is the νth coordinate vector), we have

$$t_n = \begin{cases} 0, & \text{if } n < \nu, \\ \dfrac{\nu}{A_\nu^\alpha}, & \text{if } n = \nu, \\ \dfrac{\Delta_\nu (A_{n-\nu}^{\alpha-1})}{A_n^\alpha}, & \text{if } n > \nu, \end{cases} \tag{23}$$

$$T_n = \begin{cases} 0, & \text{if } n < \nu, \\ \left(\dfrac{p_\nu}{P_\nu}\right)^{1/k}, & \text{if } n = \nu, \\ \dfrac{(p_n/P_n)^{1/k}}{P_{n-1}} \Delta(P_{\nu-1}\epsilon_\nu), & \text{if } n > \nu. \end{cases} \tag{24}$$

Equalities (20) give

$$\begin{aligned} \|a\|_1 &= \left\{ \frac{1}{\nu}\left(\frac{\nu}{A_\nu^\alpha}\right)^k + \sum_{n=\nu+1}^{\infty} \frac{1}{\nu} \left| \frac{\Delta_\nu (A_{n-\nu}^{\alpha-1}\nu)}{A_n^\alpha} \right|^k \right\}^{1/k}, \\ \|a\|_2 &= \left\{ \frac{p_\nu}{P_\nu} |\epsilon_\nu|^k + \sum_{n=\nu+1}^{\infty} \frac{p_n}{P_n P_{n-1}^k} |\Delta(p_{\nu-1}\epsilon_\nu)|^k \right\}^{1/k}. \end{aligned} \tag{25}$$

By inequality (22),

$$\frac{p_\nu}{P_\nu} |\epsilon_\nu|^k + \sum_{n=\nu+1}^{\infty} \frac{p_n}{P_n P_{n-1}^k} |\Delta(P_{\nu-1}\epsilon_\nu)|^k \le M^k \left(\frac{1}{\nu}\left(\frac{\nu}{A_\nu^\alpha}\right)^k + \sum_{n=\nu+1}^{\infty} \frac{1}{\nu} \left| \frac{\Delta_\nu (\nu A_{n-\nu}^{\alpha-1})}{A_n^\alpha} \right|^k \right). \tag{26}$$

Since $\Delta(\nu A_{n-\nu}^{\alpha-1}) = -A_{n-\nu}^{\alpha-1} + \nu \Delta_\nu A_{n-\nu}^{\alpha-1}$, then by Minkowski's inequality

$$\begin{aligned} M^k \left(\frac{1}{\nu}\left(\frac{\nu}{A_\nu^\alpha}\right)^k + \sum_{n=\nu+1}^{\infty} \frac{1}{\nu} \left| \frac{\Delta_\nu (\nu A_{n-\nu}^{\alpha-1})}{A_n^\alpha} \right|^k \right) &= O(\nu^{k-\alpha k-1}) + O\left\{ \frac{1}{\nu} \sum_{n=\nu+1}^{\infty} \frac{(n-\nu)^{\alpha k-k}}{A_n^{\alpha k}} \right\} \\ &\quad + O\left\{ \frac{\nu^k}{\nu} \sum_{n=\nu+1}^{\infty} \frac{(n-\nu)^{\alpha k-2k}}{n^{\alpha k}} \right\} \\ &= O(\nu^{k-\alpha k-1}) + O(\nu^{-k}) + O(\nu^{-k}), \text{ by Lemma 1} \\ &= O(\nu^{k-\alpha k-1}). \end{aligned} \tag{27}$$

As each part of the left-hand side of (26) is $O(\nu^{k-\alpha k-1})$, we have

$$\begin{gathered}\epsilon_\nu = O\left(\frac{P_\nu}{p_\nu}\right)^{1/k}\nu^{1-\alpha-(1/k)},\\ |\Delta(P_{\nu-1}\epsilon_\nu)|^k \sum_{n=\nu+1}^{\infty}\frac{p_n}{P_nP_{n-1}^k} = O(\nu^{k-\alpha k-1}),\end{gathered} \tag{28}$$

which, by Lemma 3, implies

$$|\Delta(P_{\nu-1}\epsilon_\nu)|^k = O(P_{\nu-1}^k\nu^{k-\alpha k-1}). \tag{29}$$

Now, as $\Delta(P_{\nu-1}\epsilon_\nu) = -p_\nu\epsilon_\nu + P_\nu\Delta\epsilon_\nu$,

$$\begin{aligned}\Delta\epsilon_\nu &= \frac{p_\nu}{P_\nu}\epsilon_\nu + \frac{1}{P_\nu}\Delta(P_{\nu-1}\epsilon_\nu)\\ &= \frac{p_\nu}{P_\nu}O\left\{\left(\frac{P_\nu}{p_\nu}\right)^{1/k}\nu^{1-\alpha-(1/k)}\right\} + \frac{1}{P_\nu}O\left(P_{\nu-1}\nu^{1-\alpha-(1/k)}\right)\\ &= O\left\{\left(\frac{p_\nu}{P_\nu}\right)^{1-(1/k)}\nu^{1-\alpha-(1/k)}\right\} + \frac{1}{P_\nu}O\left(P_{\nu-1}\nu^{1-\alpha-(1/k)}\right)\\ &= O(\nu^{1-\alpha-(1/k)}).\end{aligned} \tag{30}$$

This completes the proof of the theorem. □

References

[1] H. Bor, *On two summability methods*, Math. Proc. Cambridge Philos. Soc. **97** (1985), no. 1, 147–149. MR 86d:40004. Zbl 554.40008.

[2] S. M. Mazhar, *On the absolute summability factors of infinite series*, Tôhoku Math. J. (2) **23** (1971), 433–451. MR 46#7758. Zbl 227.40005.

[3] M. R. Mehdi, *Summability factors for generalized absolute summability. I*, Proc. London Math. Soc. (3) **10** (1960), 180–200. MR 22#9760. Zbl 093.26203.

[4] R. N. Mohapatra, *On absolute Riesz summability factors*, J. Indian Math. Soc. (N.S.) **32** (1968), 113–129. MR 43#7811. Zbl 197.34404.

[5] W. T. Sulaiman, *On absolute Cesàro summability of infinite series*, Pure Appl. Math. Sci. **31** (1990), no. 1-2, 25–30. MR 91j:40005. Zbl 713.40002.

[6] ———, *A study on absolute weighted mean summability method*, Math. Sci. Res. Hot-Line **1** (1997), no. 11, 25–29. MR 98m:40004.

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