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A NEW COMBINATORIAL IDENTITY

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ABSTRACT. We prove a combinatorial identity which arose from considering the relation $r_p(x,y,z) = (x+y-z)^p - (x^p+y^p-z^p)$ in connection with Fermat's last theorem.

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The following combinatorial identity:

$$\sum_{l'\le l}\sum_{j'\le j}\frac{1}{(m-l')}\binom{m+l'-j'}{2l'-j'+1}\binom{m-l'+j'-1}{j'}\binom{m-l'}{2(l-l')-(j-j')}\binom{m-l'}{j-j'} = \frac{1}{2(m-l)}\binom{2m}{2l+1}\binom{2l+1}{j} = \frac{1}{(2l+1)}\binom{2m}{2l}\binom{2l+1}{j} \tag{1}$$

for all $m > l \ge 0$, where m, l, and j are nonnegative integers and $0 \le j \le 2l+1$, arose from considering

$$r_p(x,y,z) = (x+y-z)^p - (x^p+y^p-z^p) \tag{2}$$

in connection with Fermat's last theorem (FLT), which was proved in 1994 by Wiles and Taylor. Recall that FLT states that $x^p + y^p - z^p \ne 0$, where x, y, z, p are any nonzero integers and $p > 2$. We take, without loss of generality, that x, y, and z are relatively prime and p is prime. In general, $r_p(x,y,z)$ can be factored as $p(z-x)(z-y)(x+y)f_p(x,y,z)$ which are powers of p if $x^p+y^p-z^p = 0$. These factors result in the elementary Abel-Barlow relations known since the 1820's (see [2]).

However, the last factor $f_p(x,y,z)$ is

$$\sum_{l=0}^{m-1}\sum_{i=0}^{2l}\sum_{j=0}^{i}\frac{(-1)^{i-j}}{(m-l)}\binom{m+l-j}{2l-j+1}\binom{m-l+j-1}{j}x^{2l-i}y^i(z-x)^{m-l-1}(z-y)^{m-l-1} = \begin{cases} p^{kp-1}d^p, & p \nmid xyz, \\ d^p, & p \mid xyz, \end{cases} \tag{3}$$

where $p = 2m+1 \ge 5$ and $k > 0$. This formulation of $f_p(x,y,z)$, which is believed to be novel, establishes the new identity. However, it appears to offer no new insights into a possible elementary proof of FLT.

To discover the identity, note that

$$r_p(x,y,z) = p\sum_{l=0}^{2m}\sum_{j=0}^{2m}\frac{(-1)^l}{p}\binom{p}{l}\binom{p-l}{j}x^j y^{p-j-l}z^l, \tag{4}$$

where $j+l \neq 0$.

Alternatively, we have

$$r_p(x,y,z) = p\sum_{l'=0}^{m}(z-x)^{m-l'}(z-y)^{m-l'}\sum_{j'=0}^{2l'+1}a_{j',m-l'}x^{2l'-j'+1}y^{j'}. \tag{5}$$

Equating (4) and (5) for a given j and l, we get the recurrence

$$a_{j,m-l} = \frac{1}{2(m-l)}\binom{2m}{2l+1}\binom{2l+1}{j} - \sum_{l'<l}\sum_{j'\leq j}a_{j',m-l'}\binom{m-l'}{2(l-l')-(j-j')}\binom{m-l'}{j-j'}. \tag{6}$$

Now,

$$a_{j,m-l} = \frac{1}{(m-l)}\binom{m+l-j}{2l-j+1}\binom{m-l+j-1}{j} \tag{7}$$

satisfies the recurrence (6). Substituting the expression for $a_{j,m-l}$ and rearranging, we obtain the new identity.

The authors have reviewed the literature, notably Gould [1] and Riordan [3] as well as the relevant journals since 1980. Based on this review, (1) is believed to be novel.

Proof of the identity. We consider two special cases.

Case 1 ($j = 0$)**.** Equation (1) reduces to:

$$\sum_{0\leq l'\leq l}\frac{1}{m-l'}\binom{m+l'}{2l'+1}\binom{m-l'}{2l-2l'} = \frac{1}{2l+1}\binom{2m}{2l}. \tag{8}$$

Divide both sides of (8) by the right-hand side and denote the resulting left-hand side by $S(m,l)$. Then $S(m,l)$ satisfies the recurrence equation $S(m+1,l)-S(m,l) = 0$—obtained by using Zeilberger's [5] Ekhad, a computer algebra package which is available from http://www.math.temple.edu/˜zeilberg/—and hence the identity follows from the fact that $S(1,0) = 1$.

Case 2 ($j \neq 0$)**.** Equation (1) reduces to

$$\sum_{l'}\sum_{j'}\binom{m+l'-j'}{2l'-j'+1}\binom{m-l'+j'-1}{j-1}\binom{m-l'}{2(l-l')-(j-j')}\binom{j}{j'} = \binom{2m}{2l}\binom{2l}{j-1}, \tag{9}$$

which by multiplying both sides by $(2l-j+1)/j$ is also expressible as

$$\sum_{l'}\sum_{j'}\binom{m-1-l'+j'}{j'}\binom{m-l'}{j-j'}\binom{m+l'-j'}{2l-j}\binom{2l-j+1}{2l'-j'+1} = \binom{2m}{2l}\binom{2l}{j} = \binom{2m}{j,2l-j}, \tag{10}$$

where

$$\binom{a}{b,c} := \frac{a!}{b!c!(a-b-c)!}. \tag{11}$$

Equation (10) follows from the identity

$$\sum_{l'}\sum_{j'}\binom{p-1-l'+j'}{j'}\binom{p-l'}{j-j'}\binom{m+l'-j'}{k}\binom{k+1}{2l'-j'+m-p+1}=\binom{m+p}{j,k} \tag{12}$$

with $p = m$ and $k = 2l-$j.

Denote the left-hand side of (12) by $S(m,p,j,k)$. $S(m,p,j,k)$ satisfies $S(m+1,p,j,k) = S(m,p,j,k)$ and hence $S(m,p,j,k) = S(m+p,0,j,k)$. Hence to prove (12) it suffices to prove

$$S(n,0,j,k)=\binom{n}{j,k}\quad \forall n,j,k\in\mathbb{Z}_{\geq 0}. \tag{13}$$

Clearly (13) is true for $n = 0$. Now, let $n > 0$ and set $S(n,j,k) := S(n,0,j,k)$. Then $S(n,j,k)$ satisfies the recurrence equation

$$\begin{aligned}(-1+j-n)&S(n-1,j-1,k)-(1+k)S(n-1,j,k-1)\\&+(j-k-n-1)S(n-1,j,k)+(j+k-n-1)S(n,j-1,k)\\&+(k+1)S(n,j-1,k+1)+(j+2k-n+1)S(n,j,k)+2(1+k)S(n,j,k+1)=0\end{aligned} \tag{14}$$

that is obtained by using Wegschaider's [4] MultiSum, a computer algebra package which is available from http://www.risc.uni-linz.ac.at/research/combinat/risc/software/. Note that the right-hand side of (13) also satisfies (14). Hence by induction it follows that

$$S(n,j,k)=\binom{n}{j,k}\quad \forall n,j,k\in\mathbb{Z}_{\geq 0}. \tag{15}$$

□

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