IJMMS 25:8 (2001) 559–563
PII. S0161171201003945
http://ijmms.hindawi.com

A NOTE ON θ-GENERALIZED CLOSED SETS

C. W. BAKER

(Received 22 September 1999)

Abstract. The purpose of this note is to strengthen several results in the literature concerning the preservation of θ-generalized closed sets. Also conditions are established under which images and inverse images of arbitrary sets are θ-generalized closed. In this process several new weak forms of continuous functions and closed functions are developed.

2000 Mathematics Subject Classification. Primary 54C10.

1. Introduction. Recently Dontchev and Maki [5] have introduced the concept of a θ-generalized closed set. This class of sets has been investigated also by Arockiarani et al. [1]. The purpose of this note is to strengthen slightly some of the results in [5] concerning the preservation of θ-generalized closed sets. This is done by using the notion of a θ-c-closed set developed by Baker [2]. These sets turn out to be a very natural tool to use in investigating the preservation of θ-generalized closed sets. In this process we introduce a new weak form of a continuous function and a new weak form of a closed function, called θ-g-c-continuous and θ-g-c-closed, respectively. It is shown that θ-g-c-continuity is strictly weaker than strong θ-continuity and that θ-g-c-closed is strictly weaker than θ-g-closed.

2. Preliminaries. The symbols X and Y denote topological spaces with no separation axioms assumed unless explicitly stated. If A is a subset of a space X, then the closure and interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. The θ-closure of A [8], denoted by $\mathrm{Cl}_\theta(A)$, is the set of all $x \in X$ for which every closed neighborhood of x intersects A nontrivially. A set A is called θ-closed if $A = \mathrm{Cl}_\theta(A)$. The θ-interior of A [8], denoted by $\mathrm{Int}_\theta(A)$, is the set of all $x \in X$ for which A contains a closed neighborhood of x. A set A is said to be θ-open provided that $A = \mathrm{Int}_\theta(A)$. Furthermore, the complement of a θ-open set is θ-closed and the complement of a θ-closed set is θ-open.

Definition 2.1 (Dontchev and Maki [5]). A set A is said to be θ-generalized closed (or briefly θ-g-closed) provided that $\mathrm{Cl}_\theta(A) \subseteq U$ whenever $A \subseteq U$ and U is open. A set is called θ-generalized open (or briefly θ-g-open) if its complement is θ-generalized closed.

The following theorem from [5] gives a useful characterization of θ-g-openness.

Theorem 2.2 (Dontchev and Maki [5]). *A set A is θ-g-open if and only if $F \subseteq \text{Int}_\theta(A)$ whenever $F \subseteq A$ and F is closed.*

Definition 2.3 (Dontchev and Maki [5]). A function $f : X \to Y$ is said to be θ-g-closed provided that $f(A)$ is θ-g-closed in Y for every closed subset F of X.

Definition 2.4 (Dontchev and Maki [5]). A function $f : X \to Y$ is said to be θ-g-irresolute (θ-g-continuous), if for every θ-g-closed (closed) subset A of Y, $f^{-1}(A)$ is θ-g-closed in X.

Definition 2.5 (Noiri [7]). A function $f : X \to Y$ is said to be strongly θ-continuous provided that, for every $x \in X$ and every open neighborhood V of $f(x)$, there exists an open neighborhood U of x for which $f(\text{Cl}(U)) \subseteq V$.

3. Sufficient conditions for images of θ-g-closed sets to be θ-g-closed.

Dontchev and Maki [5] proved that the θ-g-closed, continuous image of a θ-g-closed set is θ-g-closed. In this section, we strengthen this result by replacing both the θ-g-closed and continuous requirements with weaker conditions. Our replacement for the θ-g-closed condition uses the concept of a θ-c-open set from [2].

Definition 3.1 (Baker [2]). A set A is said to be θ-c-closed provided there is a set B for which $A = \text{Cl}_\theta(B)$.

We define a function $f : X \to Y$ to be θ-g-c-closed if $f(A)$ is θ-g-closed in Y for every θ-c-closed set A in X. Since θ-c-closed sets are obviously closed, θ-g-closed implies θ-g-c-closed. The following example shows that the converse implication does not hold.

Example 3.2. Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$ and let $f : X \to X$ be the identity mapping. Since the θ-closure of every nonempty set is X, f is obviously θ-g-c-closed. However, since $f(\{c\})$ fails to be θ-g-closed, f is not θ-g-closed.

Theorem 3.3. *If $f : X \to Y$ is continuous and θ-g-c-closed, then $f(A)$ is θ-g-closed in Y for every θ-g-closed set A in X.*

Proof. Assume A is a θ-g-closed subset of X and that $f(A) \subseteq V$, where V is an open subset of Y. Then $A \subseteq f^{-1}(V)$, which is open. Since A is θ-g-closed, $\text{Cl}_\theta(A) \subseteq f^{-1}(V)$ and hence $f(\text{Cl}_\theta(A)) \subseteq V$. Because $\text{Cl}_\theta(A)$ is θ-c-closed and f is θ-g-c-closed, $f(\text{Cl}_\theta(A))$ is θ-g-closed. Therefore $\text{Cl}_\theta(f(\text{Cl}_\theta(A))) \subseteq V$ and hence $\text{Cl}_\theta(f(A)) \subseteq \text{Cl}_\theta(f(\text{Cl}_\theta(A))) \subseteq V$, which proves that $f(A)$ is θ-g-closed. □

Corollary 3.4 (Dontchev and Maki [5]). *If $f : X \to Y$ is continuous and θ-g-closed, then $f(A)$ is θ-g-closed in Y for every θ-g-closed subset A of X.*

Theorem 3.3 can be strengthened further by replacing continuity with a weaker condition. Instead of requiring inverse images of open sets to be open, we require that the inverse images of open sets interact with θ-g-closed sets in the same way as open sets.

Definition 3.5. A function $f: X \to Y$ is said to be approximately θ-continuous (or briefly a-θ-continuous) provided that $\mathrm{Cl}_\theta(A) \subseteq f^{-1}(V)$ whenever $A \subseteq f^{-1}(V)$, A is θ-g-closed, and V is open.

The proof of Theorem 3.3 is easily modified to obtain the following result.

Theorem 3.6. *If $f: X \to Y$ is a-θ-continuous and θ-g-c-closed, then $f(A)$ is θ-g-closed in Y for every θ-g-closed set A in X.*

Obviously continuity implies a-θ-continuity and the following example shows that a-θ-continuity is strictly weaker than continuity.

Example 3.7. Let (X,τ) be the space in Example 3.2 and let $\sigma = \{X, \emptyset, \{b\}\}$. Then the identity mapping $f: (X,\tau) \to (X,\sigma)$ is not continuous but is a-θ-continuous.

In [4] Dontchev defined a function to be contra-continuous provided that inverse images of open sets are closed. We modify this concept slightly to obtain a θ-contra-continuous function.

Definition 3.8. A function $f: X \to Y$ is said to be θ-contra-continuous if for every open subset V of Y, $f^{-1}(V)$ is θ-closed.

If the continuity requirement in Theorem 3.3 is replaced with θ-contra-continuity, then a much stronger result is obtained. The step in the proof of Theorem 3.3 where we obtain $\mathrm{Cl}_\theta(A) \subseteq f^{-1}(V)$ now holds for every set A, because $f^{-1}(V)$ is θ-closed. Therefore we have the following theorem.

Theorem 3.9. *If $f: X \to Y$ is θ-contra-continuous and θ-g-c-closed, then $f(A)$ is θ-g-closed in Y for every subset A of X.*

4. Sufficient conditions for θ-g-irresoluteness.

Dontchev and Maki [5] proved that a strongly θ-continuous, closed function is θ-g-irresolute. We strengthen this result slightly by replacing strong θ-continuity and closure with weaker conditions.

We define a function $f: X \to Y$ to be θ-g-c-continuous provided that, for every θ-c-closed subset A of Y, $f^{-1}(A)$ is θ-g-closed. Since strong θ-continuity is equivalent to the requirement that inverse images of closed sets be θ-closed [6], strong θ-continuity obviously implies θ-g-c-continuity. The function in Example 3.2 is θ-g-c-continuous but not strongly θ-continuous.

Theorem 4.1. *If $f: X \to Y$ is θ-g-c-continuous and closed, then f is θ-g-irresolute.*

Proof. Assume $A \subseteq Y$ is θ-g-closed and that $f^{-1}(A) \subseteq U$, where U is open. Then $X - U \subseteq X - f^{-1}(A)$ and we see that $f(X-U) \subseteq Y - A$. Since A is θ-g-closed, $Y - A$ is θ-g-open. Also, since f is closed, $f(X-U)$ is closed. Thus $f(X-U) \subseteq \mathrm{Int}_\theta(Y - A) = Y - \mathrm{Cl}_\theta(A)$ or $X - U \subseteq f^{-1}(Y - \mathrm{Cl}_\theta(A)) = X - f^{-1}(\mathrm{Cl}_\theta(A))$ and we have that $f^{-1}(\mathrm{Cl}_\theta(A)) \subseteq U$. Since f is θ-g-c-continuous, $f^{-1}(\mathrm{Cl}_\theta(A))$ is θ-g-closed. Therefore $\mathrm{Cl}_\theta(f^{-1}(A)) \subseteq \mathrm{Cl}_\theta(f^{-1}(\mathrm{Cl}_\theta(A))) \subseteq U$, which proves that $f^{-1}(A)$ is θ-g-closed. Thus f is θ-g-irresolute. □

Corollary 4.2 (Dontchev and Maki [5]). *If $f : X \to Y$ is strongly θ-continuous and closed, then f is θ-g-irresolute.*

Obviously θ-g-continuity implies θ-g-c-continuity. Therefore we have the following result.

Corollary 4.3. *If $f : X \to Y$ is θ-g-continuous and closed, then f is θ-g-irresolute.*

The function in Example 3.2 is θ-g-c-continuous but not θ-g-continuous.

Theorem 4.1 can be strengthened in much the same way as Theorem 3.3 was strengthened by replacing the closure requirement with a weaker condition.

Definition 4.4. A function $f : X \to Y$ is said to be approximately θ-closed (or briefly a-θ-closed) provided that $f(F) \subseteq \mathrm{Int}_\theta(A)$ whenever $f(F) \subseteq A$, F is closed, and A is θ-g-open.

Note that, under an a-θ-closed function, images of closed sets interact with θ-g-open sets in the same manner as closed sets. Obviously closed functions are a-θ-closed. The inverse of the function in Example 3.7 is a-θ-closed but not closed. The proof of the following theorem is analogous to that of Theorem 4.1.

Theorem 4.5. *If $f : X \to Y$ is θ-g-c-continuous and a-θ-closed, then f is θ-g-irresolute.*

Finally, Theorem 4.1 can be modified by replacing the requirement that the function be closed with a variation of a contra-closed function. Contra-closed functions, introduced by Baker [3], are characterized by having open images of closed sets.

Definition 4.6. A function $f : X \to Y$ is said to be θ-contra-closed provided that $f(F)$ is θ-open for every closed subset F of X.

Theorem 4.7. *If $f : X \to Y$ is θ-g-c-continuous and θ-contra-closed, then for every subset A of Y $f^{-1}(A)$ is θ-g-closed (and hence also θ-g-open).*

The proof of Theorem 4.7 follows that of Theorem 4.1, except that the step $f(X - U) \subseteq \mathrm{Int}_\theta(Y - A)$ holds for every subset A of Y because $f(X - U)$ is θ-open.

References

[1] I. Arockiarani, K. Balachandran, and M. Ganster, *Regular generalized locally closed sets and RGL-continuous functions*, Indian J. Pure Appl. Math. **28** (1997), no. 5, 661–669. CMP 1 459 608. Zbl 884.54006.

[2] C. W. Baker, *On θ-C open sets*, Int. J. Math. Math. Sci. **15** (1992), no. 2, 255–259. CMP 1 155 517. Zbl 774.54018.

[3] ______, *Contra-open functions and contra-closed functions*, Math. Today (Ahmedabad) **15** (1997), 19–24. MR 98f:54016. Zbl 903.54008.

[4] J. Dontchev, *Contra-continuous functions and strongly S-closed spaces*, Int. J. Math. Math. Sci. **19** (1996), no. 2, 303–310. MR 96m:54021. Zbl 927.54037.

[5] J. Dontchev and H. Maki, *On θ-generalized closed sets*, Int. J. Math. Math. Sci. **22** (1999), no. 2, 239–249. MR 2000b:54045. Zbl 927.54037.

[6] P. E. Long and L. L. Herrington, *Strongly θ-continuous functions*, J. Korean Math. Soc. **18** (1981/82), no. 1, 21–28. MR 83b:54011. Zbl 478.54006.

[7] T. Noiri, *On δ-continuous functions*, J. Korean Math. Soc. **16** (1979/80), no. 2, 161–166. MR 82b:54020. Zbl 435.54010.

[8] N. V. Veličko, *H-closed topological spaces*, Mat. Sb. (N.S.) **70(112)** (1966), 98–112. MR 33#6576.

C. W. Baker: Department of Mathematics, Indiana University Southeast, New Albany, IN 47150, USA

E-mail address: cbaker@ius.edu