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CHARACTERIZATION OF AN H^*-ALGEBRAS IN TERMS OF A TRACE

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Abstract. Arbitrary proper H^*-algebra is characterized in terms of the trace defined on a certain subset of the algebra.

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This paper deals with characterizations of an arbitrary proper (not necessarily commutative) H^*-algebra. More specifically, we show that any Banach *-algebra with a partially defined trace, and some additional properties, has a Hilbert space structure with respect to which it is an H^*-algebra. In the past, the author worked in characterizations of commutative H^*-algebras (e.g., [3, 4]). As in [3], we do not assume both the existence of an inner product and the commutativity.

Now, we state our first result.

Theorem 1. *Let A be a Banach algebra with an involution $x \to x^*$, $x \in A$, such that $\|x^*\| = \|x\|$. Assume that the set $A^2 = \{xy : x, y \in A\}$ has a complex valued trace, that is, there is a complex valued function tr on A^2 with the following properties:*

(i) *if x, y, and $x+y$ belong to A^2, then $\mathrm{tr}(x+y) = \mathrm{tr}\,x + \mathrm{tr}\,y$,*

(ii) *$\mathrm{tr}(\lambda x) = \lambda\,\mathrm{tr}\,x$ for all $x \in A^2$ and any complex number λ,*

(iii) *$\mathrm{tr}(x^*x) \geq 0$ and $\mathrm{tr}\,x^*x = 0$ if and only if $x = 0$, $x \in A$,*

(iv) *$\mathrm{tr}\,x^* = \overline{\mathrm{tr}}x$ for all $x \in A^2$.*

Suppose also that

(v) *$\mathrm{tr}(xy) = \mathrm{tr}(yx)$ for all $x, y \in A$,*

(vi) *$|\,\mathrm{tr}(xy)| \leq \|x\| \cdot \|y\|$ for all $x, y \in A$,*

(vii) *for each bounded linear functional $f\,(f \in A^*)$, there exists $a \in A$ such that $f(x) = \mathrm{tr}(xa^*)$ for all $x \in A$.*

Then A is a proper H^*-algebra with respect to some scalar product whose corresponding norm $\|\;\|$ is equivalent to the original norm. This means that there exists a scalar product $(\,,)$ on A such that A is an H^*-algebra with respect to this scalar product (and the original involution) and such that $k_1\|x\|^2 \leq (x,x) \leq k_2\|x\|^2$ for some $k_1, k_2 \geq 0$ and all $x \in A$ (and $(xy,z) = (y,x^*z) = (x,zy^*)$ for all $x,y,z \in A$).

Remark 2. Note that each proper H^*-algebra A has all the properties stated in Theorem 1 [5].

Proof of Theorem 1. For any $x, y \in A$, let $(x,y) = \mathrm{tr}(xy^*) = \mathrm{tr}(y^*x)$. Then $(\,,)$ is an inner product on A [6] (in the terminology of Loomis [1], $(\,,)$ is a scalar product). Let $\|\;\|_2$ be the corresponding norm, $(x,x) = \|x\|_2^2$, $x \in A$.

We show that A is complete with respect to this new norm $\| \|_2$. Let $\{a_n\}$ be a Cauchy sequence, $\lim_{m,n} \|a_n - a_m\|_2 = 0$. Then there exists $M > 0$ such that $\|a_n\|_2 \le M$ for all n (every Cauchy sequence is bounded). For any fixed $x \in A$, the sequence $\{\mathrm{tr}(xa_n^*)\}$ of complex numbers is also Cauchy

$$(\,|\,\mathrm{tr}\,(xa_n^*) - \mathrm{tr}\,(xa_n^*)\,| \le \|x\|_2 ||a_n - a_m||_2). \tag{1}$$

So there is a complex number λ_x such that $\mathrm{tr}(xa_n^*) \to \lambda_x$, $n \to \infty$. Define the complex valued function f on A by setting $f(x) = \lambda_x$. It follows from

$$f(x) = \lim \mathrm{tr}\,(xa_n^*), \qquad ||a_n||_2 \le M, \tag{2}$$

and the linearity of tr that f is a bounded linear functional on $A(f \in A^*)$. Assumption (vii) in Theorem 1 implies that there exists $a \in A$ such that

$$(x,a) = f(x) = \lim_n \mathrm{tr}\, xa_n^*. \tag{3}$$

We show that $\lim_{n\to\infty} \|a_n - a\|_2 = 0$. Let $\epsilon > 0$ be arbitrary, and let n_0 be such that $\|a_n - a_m\|_2 < \epsilon/2$ for all $n, m > n_0$. Let $n > n_0$ be fixed and arbitrary. The following relation:

$$\begin{aligned} ||a - a_n||_2^2 &= |\,(a - a_n, a - a_m) + (a - a_n, a_m - a_n)\,| \\ &= |\,(a - a_n, a) - (a - a_n, a_m) + (a - a_n, a_m - a_n)\,| \\ &\le |\,f(a - a_n) - (a - a_n, a_m)\,| + ||a - a_n||_2 \cdot ||a_n - a_m||_2 \\ &\le |\,f(a - a_n) - (a - a_n, a_m)\,| + \frac{\epsilon}{2} ||a - a_n||_2 \end{aligned} \tag{4}$$

shows that $\|a - a_n\|_2^2 \le \epsilon \|a - a_n\|_2$, since we can always find $m > n_0$ so that $|f(a - a_n) - (a - a_n, a_m)| \le \epsilon/2 \|a - a_n\|_2$. Hence, $\|a - a_n\|_2 \le \epsilon$ for any $n > n_0$. This proves that A is complete in this new norm $\| \|_2$.

It follows from (vi) that $\|x\|_2 \le \|x\|$ $((x,x) \le \|x\|\,\|x^*\| = \|x\|^2$ for all $x \in A)$. Closed graph theorem [1] tells us that $\| \|_2$ is equivalent to the original norm.

Now, it is an easy exercise to show that A is an H^*-algebra with respect to the inner product $(\,,\,)$. □

Now we state a slightly different characterization. It may not look like much of improvement over Theorem 1, but it allows for a larger class of examples. In fact, if we take any proper H^*-algebra A and replace its norm by any other norm equivalent to the original one, we get a canonical example of a Banach algebra which both satisfies the conditions of the following theorem and is characterized by it.

Theorem 3. *Let A be a Banach algebra with continuous involution $x \to x^*$, $x \in A$. Assume that the set $A^2 = \{xy : x, y \in A\}$ has a trace* tr *with the following properties:*

(i) *if $x, y, x + y \in A^2$, then $\mathrm{tr}(x + y) = \mathrm{tr}\, x + \mathrm{tr}\, y$,*
(ii) *$\mathrm{tr}(\lambda x) = \lambda\, \mathrm{tr}\, x$ for all $x \in A^2$ and each complex number λ,*
(iii) *$\mathrm{tr}(x^* x) \ge 0$ and $\mathrm{tr}(x^* x) = 0$ if and only if $x = 0$ $(x \in A)$,*
(iv) *$\mathrm{tr}\, x^* = \overline{\mathrm{tr}} x$, $x \in A^2$,*
(v) *$\mathrm{tr}(xy) = \mathrm{tr}(yx)$ for all $x, y \in A$.*

Assume further that

(vi)′ *for each* $a \in A$ *the map* $T_a : x \to \operatorname{tr}(xa^*)(= T_a(x))$ *is continuous* ($T_a \in A^*$ *for each* $a \in A$),

(vii) *for each bounded linear functional* $f(f \in A^*)$ *there exists* $a \in A$ *such that* $f(x) = \operatorname{tr}(xa^*)$ *for all* $x \in A$.

Then A *has a structure of a proper* H^**-algebra with respect to some scalar product* $(\,,\,)$ *such that* $k\|x\|^2 \le (x,x) \le K\|x\|^2$ *for all* $x \in A$ *and some* k, $K > 0$.

Remark 4. Note that (vi)′ is equivalent to the following condition:
(vi)″ there exists $M > 0$ such that $|\operatorname{tr}(xy)| \le M\|x\| \cdot \|y\|$ for all $x, y \in A$.

It is a consequence of uniform boundness theorem [6, page 239]. Proof of this fact is similar to the proof of Lemma 1 in [2]. Note also that continuity of involution implies that there exists $B > 0$ such that $\|x^*\| \le B\|x\|$, $x \in A$.

Proof of Theorem 3. We leave it to the reader to modify the proof of Theorem 1 in order to verify validity of Theorem 3. □

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