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A GENERALIZATION THEOREM COVERING MANY ABSOLUTE SUMMABILITY METHODS

W. T. SULAIMAN

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ABSTRACT. A general theorem concerning many absolute summability methods is proved.

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1. Introduction. Let $\sum a_n$ be given infinite series with the sequence of partial sums (s_n). By σ_n^δ and μ_n^δ we denote nth Cesaro mean of order δ $(\delta > -1)$ of the sequences (s_n) and (na_n), respectively. The series $\sum a_n$ is said to be summable $|C,\delta|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} \left|\sigma_n^\delta - \sigma_{n-1}^\delta\right|^k < \infty, \tag{1.1}$$

or equivalently

$$\sum_{n=1}^{\infty} n^{-1} \left|\mu_n^\delta\right|^k < \infty. \tag{1.2}$$

Let (p_n) be a sequence of real or complex numbers such that

$$P_n = \sum_{\nu=0}^{n} p_\nu \longrightarrow \infty \quad \text{as } n \longrightarrow \infty \ (P_{-i} = p_{-i} = 0,\ i \geq 1). \tag{1.3}$$

The sequence-to-sequence transformation:

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu, \tag{1.4}$$

defines the sequence (t_n) of the Riesz means of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [4]). The series $\sum a_n$ is said to be summable $|R,p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left|t_n - t_{n-1}\right|^k < \infty. \tag{1.5}$$

The series $\sum a_n$ is said to be summable $|\bar{N},p_n|_k$, $k \geq 1$ (see [1]), if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|t_n - t_{n-1}\right|^k < \infty. \tag{1.6}$$

In the special case when $p_n = 1$ for all values of n (respectively, $k = 1$), then both of $|R,p_n|_k$ and $|\bar{N},p_n|_k$ is the same as $|C,1|_k$ (respectively, $|R,p_n|, |\bar{N},p_n|$) summability.

For $p_n = 1/(n+1)$, the summability $|\bar{N}, p_n|_k$ is equivalent to $|R, \log n, 1|_k$. The series $\sum a_n$ is said to be summable $|N, p_n|$, if

$$\sum_{n=1}^{\infty} |T_n - T_{n-1}| < \infty, \tag{1.7}$$

where

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v \quad (T_{-1} = 0). \tag{1.8}$$

We write $p = \{p_n\}$ and

$$M = \left\{ p : p_n > 0 \text{ and } \frac{p_{n+1}}{p_n} \le \frac{p_{n+2}}{p_{n+1}} \le 1,\ n = 0, 1, \dots \right\}. \tag{1.9}$$

It is known that for $p \in M$, equation (1.7) holds if and only if (see [3])

$$\sum_{n=1}^{\infty} \frac{1}{nP_n} \left| \sum_{v=1}^{n} p_{n-v} v a_v \right| < \infty. \tag{1.10}$$

For $p \in M$, the series $\sum a_n$ is said to be summable $|N, p_n|_k$, $k \ge 1$ (see [5]), if

$$\sum_{n=1}^{\infty} \frac{1}{nP_n^k} \left| \sum_{v=1}^{n} p_{n-v} v a_v \right|^k < \infty. \tag{1.11}$$

In the special case in which $p_n = A_n^{r-1}$, $r > -1$, where A_n^r is the coefficient of x^n in the power series of $(1-x)^{-r-1}$, for $|x| < 1$, $|N, p_n|_k$ summability reduces to $|C, r|_k$ summability, and for $p_n = 1/(n+1)$, $|N, p_n|_k$ is equivalent to $|N, 1/(n+1)|_k$ summability. We assume that $(f_n), (g_n), (G_n)$, and (H_n) are positive sequences.

2. Main result.

We prove the following theorem.

Theorem 2.1. *Let*

$$Y_n = \frac{1}{F_{n-1} H_n} \sum_{v=1}^{n} v f_{n-v} x_v \epsilon_v \tag{2.1}$$

such that

$$F_n = \sum_{v=1}^{n} f_v \longrightarrow \infty, \quad f \in M, \qquad X_n = \frac{1}{G_n} \sum_{v=1}^{n} g_v x_v. \tag{2.2}$$

Suppose that $\{nG_n\epsilon_n / F_n g_n H_n\}$ *is monotonic and*

$$g_{n+1} = O(g_n), \tag{2.3}$$

$$\Delta g_n = O\left(\frac{g_{n+1}}{n}\right), \tag{2.4}$$

$$\triangle\left(\frac{g_n H_n}{nG_n}\right) = O\left(\frac{g_n H_n}{nG_n F_{n+1}}\right), \tag{2.5}$$

$$\sum_{n=v+1}^{m} \frac{f_{n-v}}{F_{n-1} H_n^k} = O\left(\frac{1}{H_v^k}\right). \tag{2.6}$$

Then necessary and sufficient conditions that $\sum |Y_n|^k < \infty$ *whenever* $\sum |X_n|^k < \infty$ *are*

(i) $\epsilon_n = O(F_n g_n H_n / n G_n)$,

(ii) $\Delta\epsilon_n = O(g_{n+1} H_n / n G_n)$.

3. Lemmas

Lemma 3.1 (see [2]). *Let* $k \geq 1$, *and let* $A = (a_{n\nu})$ *be an infinite matrix. In order that* $A \in (\ell^k; \ell^k)$ *it is necessary that*

$$a_{n\nu} = O(1) \quad (\text{all } n, \nu). \tag{3.1}$$

Lemma 3.2 (see [6]). *Let* $p \in M$. *Then for* $0 < r \leq 1$,

$$\sum_{n=\nu+1}^{\infty} \frac{p_{n-\nu-1}}{n^r P_{n-1}} = O(\nu^{-r}). \tag{3.2}$$

Lemma 3.3. *Suppose that* $\epsilon_n = O(f_n g_n)$, $f_n, g_n \geq 0$, $\{\epsilon_n / f_n g_n\}$ *monotonic*, $\triangle g_n = O(1)$, *and* $\triangle f_n = O(f_n / g_{n+1})$. *Then* $\triangle_{\epsilon_n} = O(f_n)$.

Proof. Let $k_n = \epsilon_n / f_n g_n = O(1)$. If (k_n) is nondecreasing, then

$$\begin{aligned}
\triangle\epsilon_n &= k_n f_n g_n - k_{n+1} f_{n+1} g_{n+1} \leq k_n f_n g_n - k_n f_{n+1} g_{n+1} \\
&= k_n (\triangle f_n g_n) = k_n (f_n \triangle g_n + g_{n+1} \triangle f_n), \\
|\triangle\epsilon_n| &= O(f_n |\triangle g_n|) + O(g_{n+1} |\triangle f_n|) = O(f_n) + O(f_n) = O(f_n).
\end{aligned} \tag{3.3}$$

If (k_n) is nonincreasing, write $\nabla f_n = f_{n+1} - f_n$,

$$\begin{aligned}
\nabla\epsilon_n &= k_{n+1} f_{n+1} g_{n+1} - k_n f_n g_n \leq k_n \nabla (f_n g_n) \\
&= k_n (f_n \nabla g_n + g_{n+1} \nabla f_n), \\
|\triangle\epsilon_n| = |\nabla\epsilon_n| &= O(f_n |\nabla g_n|) + O(g_{n+1} |\nabla f_n|) \\
&= O(f_n |\nabla g_n|) + O(g_{n+1} |\triangle f_n|) \\
&= O(f_n) + O(f_n) = O(f_n).
\end{aligned} \tag{3.4}$$

□

4. Proof of Theorem 2.1

Sufficiency. We have via Abel's transformation:

$$\begin{aligned}
Y_n &= \frac{1}{F_{n-1} H_n} \sum_{\nu=1}^{n} g_\nu x_\nu \left(\nu \frac{f_{n-\nu}}{g_\nu} \epsilon_\nu \right) \\
&= \frac{1}{F_{n-1} H_n} \left[\sum_{\nu=1}^{n-1} \left(\sum_{r=1}^{\nu} g_r x_r \right) \triangle_\nu \left(\frac{f_{n-\nu}}{g_\nu} \epsilon_\nu \right) + \left(\sum_{r=1}^{n} g_r x_r \right) n \frac{f_0}{g_n} \epsilon_n \right] \\
&= \frac{1}{F_{n-1} H_n} \sum_{\nu=1}^{n-1} G_\nu X_\nu \bigg\{ - \frac{f_{n-\nu}}{g_\nu} \epsilon_\nu + (\nu+1) \triangle g_\nu^{-1} f_{n-\nu} \epsilon_\nu + (\nu+1) g_{\nu+1}^{-1} \triangle_\nu f_{n-\nu} \epsilon_\nu \\
&\qquad + (\nu+1) g_{\nu+1}^{-1} f_{n-\nu-1} \triangle\epsilon_\nu \bigg\} + \frac{n G_n X_n f_0}{F_{n-1} H_n g_n} \epsilon_n \\
&= Y_{n,1} + Y_{n,2} + Y_{n,3} + Y_{n,4} + Y_{n,5}.
\end{aligned} \tag{4.1}$$

By Minkowski's inequality,

$$\sum_{n=1}^{m}|Y_n|^k = O(1)\sum_{n=1}^{m}\sum_{r=1}^{5}|Y_{n,r}|^k. \tag{4.2}$$

Applying Hölder's inequality,

$$\begin{aligned}
\sum_{n=2}^{m+1}|Y_{n,1}|^k &= \sum_{n=2}^{m+1}\frac{1}{F_{n-1}^k H_n^k}\left|\sum_{\nu=1}^{n-1} f_{n-\nu}\frac{G_\nu}{g_\nu}X_\nu\epsilon_\nu\right|^k \\
&\le \sum_{n=2}^{m+1}\frac{1}{F_{n-1}H_n^k}\sum_{\nu=1}^{n-1} f_{n-\nu}\left(\frac{G_\nu}{g_\nu}\right)^k|X_\nu|^k|\epsilon_\nu|^k\left\{\sum_{\nu=1}^{n-1}\frac{f_{n-\nu}}{F_{n-1}}\right\}^{k-1} \\
&\le O(1)\sum_{\nu=1}^{m}\left(\frac{G_\nu}{g_\nu}\right)^k|X_\nu|^k|\epsilon_\nu|^k\sum_{n=\nu+1}^{m+1}\frac{f_{n-\nu}}{F_{n-1}H_n^k} \\
&\le O(1)\sum_{\nu=1}^{m}\frac{1}{H_\nu^k}\left(\frac{\nu}{F_{\nu-1}}\right)^k\left(\frac{G_\nu}{g_\nu}\right)^k|X_\nu|^k|\epsilon_\nu|^k, \\
\sum_{n=2}^{m+1}|Y_{n,2}|^k &= \sum_{n=2}^{m+1}\frac{1}{F_{n-1}^k H_n^k}\left|\sum_{\nu=1}^{n-1}(\nu+1)G_\nu\triangle g_\nu^{-1} f_{n-\nu}X_\nu\epsilon_\nu\right|^k \\
&\le O(1)\sum_{n=2}^{m+1}\frac{1}{F_{n-1}H_n^k}\sum_{\nu=1}^{n-1}\nu^k G_\nu^k|\triangle g_\nu^{-1}|^k f_{n-\nu}|X_\nu|^k|\epsilon_\nu|^k\left\{\sum_{\nu=1}^{n-1}\frac{f_{n-\nu}}{F_{n-1}}\right\}^{k-1} \\
&\le O(1)\sum_{\nu=1}^{m}\nu^k G_\nu^k|\triangle g_\nu^{-1}|^k|X_\nu|^k|\epsilon_\nu|\sum_{n=\nu+1}^{m+1}\frac{f_{n-\nu}}{F_{n-1}H_n^k} \\
&= O(1)\sum_{\nu=1}^{m}\left(\frac{\nu}{H_\nu}\right)^k G_\nu^k\frac{|\triangle g_\nu|^k}{g_\nu^k g_{\nu+1}^k}|X_\nu|^k|\epsilon_\nu|^k \\
&\le O(1)\sum_{\nu=1}^{m}\frac{1}{H_\nu^k}\left(\frac{\nu}{F_{\nu-1}}\right)^k\left(\frac{G_\nu}{g_\nu}\right)^k|X_\nu|^k|\epsilon_\nu|^k, \\
\sum_{n=2}^{m+1}|Y_{n,3}|^k &= \sum_{n=2}^{m+1}\frac{1}{F_{n-1}^k H_n^k}\left|\sum_{\nu=1}^{n-1}\nu g_{\nu+1}^{-1}G_\nu\triangle_\nu f_{n-\nu}X_\nu\epsilon_\nu\right|^k \\
&\le \sum_{n=2}^{m+1}\frac{1}{F_{n-1}^k H_n^k}\sum_{\nu=1}^{n-1}\nu^k\left(\frac{G_\nu}{g_{\nu+1}}\right)^k|\triangle_\nu f_{n-\nu}||X_\nu|^k|\epsilon_\nu|^k\left\{\sum_{\nu=1}^{n-1}|\triangle f_{n-\nu}|\right\}^{k-1} \\
&\le O(1)\sum_{\nu-1}^{m}\nu^k\left(\frac{G_\nu}{g_\nu}\right)^k|X_\nu|^k|\epsilon_\nu|^k\sum_{n=\nu+1}^{m+1}\frac{\triangle_\nu f_{n-\nu}}{F_{n-1}^k H_n^k} \\
&\le O(1)\sum_{\nu=1}^{m}\frac{1}{H_\nu^k}\left(\frac{\nu}{F_{\nu-1}}\right)^k\left(\frac{G_\nu}{g_\nu}\right)^k|X_\nu|^k|\epsilon_\nu|^k,
\end{aligned}$$

$$\begin{aligned}
\sum_{n=2}^{m+1}|Y_{n,4}|^k &= \sum_{n=2}^{m+1}\frac{1}{F_{n-1}^kH_n^k}\left|\sum_{\nu=1}^{n-1}\nu g_{\nu+1}^{-1}f_{n-\nu-1}G_\nu X_\nu\triangle\epsilon_\nu\right|^k\\
&\le \sum_{n=2}^{m+1}\frac{1}{F_{n-1}H_n^k}\sum_{\nu=1}^{n-1}\nu^k\left(\frac{G_\nu}{g_{\nu+1}}\right)^k f_{n-\nu-1}|X_\nu|^k|\triangle\epsilon_\nu|^k\left\{\sum_{\nu=1}^{n-1}\frac{f_{n-\nu-1}}{F_{n-1}}\right\}^{k-1}\\
&\le O(1)\sum_{\nu=1}^{m}\nu^k\left(\frac{G_\nu}{g_{\nu+1}}\right)^k|X_\nu|^k|\triangle\epsilon_\nu|^k\sum_{n=\nu+1}^{m+1}\frac{f_{n-\nu-1}}{F_{n-1}H_n^k}\\
&\le O(1)\sum_{\nu=1}^{m}\left(\frac{\nu}{H_\nu}\right)^k\left(\frac{G_\nu}{g_{\nu+1}}\right)^k|X_\nu|^k|\triangle\epsilon_\nu|^k,\\
\sum_{n=1}^{m}|Y_{n,5}|^k &= \sum_{n=1}^{m}\left|\frac{nG_nX_nf_0\epsilon_n}{F_{n-1}H_ng_n}\right|^k\\
&\le O(1)\sum_{n=1}^{m}\left(\frac{n}{F_{n-1}}\right)^k\left(\frac{G_n}{g_n}\right)^k\frac{1}{H_n^k}|X_n|^k|\epsilon_n|^k.
\end{aligned} \tag{4.3}$$

Sufficiency of (i) and (ii) follows. Necessity of (i): using the result of Bor in [2], the transformation from (X_n) into (Y_n) maps ℓ^k into ℓ^k and hence the diagonal elements of this transformation are bounded (by Lemma 3.1) and so (i) is necessary. Necessity of (ii): this follows from Lemma 3.3 and necessity of (i) by taking

$$f_n \equiv \frac{g_nH_n}{nG_n}, \quad g_n \equiv F_n \text{ using (2.3)}. \tag{4.4}$$

This completes the proof of the theorem.

Remarks. It may be mentioned that on putting
(1) $f_n = p_n$ and $H_n = n^{1/k}$, we obtain $|N,p_n|_k$ summability of $\sum a_n\epsilon_n$.
(2) $g_n = Q_{n-1}$ and $G_n = Q_{n-1}(Q_n/q_n)^{1/k}$, we obtain $|\bar{N},q_n|_k$ summability of $\sum a_n$.
(3) $g_n = Q_{n-1}$ and $G_n = n^{1/k-1}(Q_nQ_{n-1}/q_n)$, we obtain $|R,q_n|_k$ summability of $\sum a_n$.

5. Applications

Theorem 5.1. *Let $p\in M$, $\{(n\epsilon_n/P_n)(Q_n/nq_n)^{1/k}\}$ is monotonic,*

$$\triangle\left(\frac{1}{n}\left(\frac{nq_n}{Q_n}\right)^{1/k}\right) = O\left(\frac{1}{n}\left(\frac{nq_n}{Q_n}\right)^{1/k}\frac{1}{P_{n+1}}\right), \quad nq_n = O(Q_n). \tag{5.1}$$

Then necessary and sufficient conditions that $\sum a_n\epsilon_n$ be summable $|N,p_n|_k$ whenever $\sum a_n$ is summable $|\bar{N},q_n|_k$, $k\ge 1$, are

$$\epsilon_n = O\left\{\frac{P_n}{n}\left(\frac{nq_n}{Q_n}\right)^{1/k}\right\}, \qquad \triangle\epsilon_n = O\left\{\frac{1}{n}\left(\frac{nq_n}{Q_n}\right)^{1/k}\right\}. \tag{5.2}$$

Theorem 5.2. *Let $p\in M$, $\{Q_n\epsilon_n/P_nq_n\}$ is monotonic,*

$$\triangle\left(\frac{q_n}{Q_n}\right) = O\left(\frac{q_n}{Q_nP_{n+1}}\right), \quad nq_n = O(Q_n). \tag{5.3}$$

Then necessary and sufficient conditions that $\sum a_n\epsilon_n$ be summable $|N,p_n|_k$ whenever $\sum a_n$ is summable $|R,q_n|_k$, $k\ge 1$ are

$$\epsilon_n = O\left(\frac{P_n q_n}{Q_n}\right), \qquad \triangle\epsilon_n = O\left(\frac{q_n}{Q_n}\right). \tag{5.4}$$

Corollary 5.3. *Necessary and sufficient conditions that $\sum a_n\epsilon_n$ be summable $|C,\alpha|_K$, $0<\alpha\le 1$ whenever $\sum a_n$ is summable $|C,1|_k$, $k\ge 1$, are*

$$\epsilon_n = O(n^{\alpha-1}), \qquad \triangle\epsilon_n = O(n^{-1}), \tag{5.5}$$

provided $\{n^{1-\alpha}\epsilon_n\}$ is monotonic.

Corollary 5.4. *Necessary and sufficient conditions that $\sum a_n\epsilon_n$ be summable $|N,1/(n+1)|_k$ whenever $\sum a_n$ is summable $|C,1|_k$, $k\ge 1$, are*

$$\epsilon_n = O\left(\frac{\log n}{n}\right), \qquad \triangle\epsilon_n = O(n^{-1}), \tag{5.6}$$

provided $\{n\epsilon_n/\log n\}$ is monotonic.

Corollary 5.5. *Necessary and sufficient conditions that $\sum a_n\epsilon_n$ be summable $|N,1/(n+1)|_k$ whenever $\sum a_n$ is summable $|R,\log n,1|_k$, $k\ge 1$, are*

$$\epsilon_n = O\left\{\frac{(\log n)^{1-1/k}}{n}\right\}, \qquad \triangle\epsilon_n = O\left\{\frac{1}{n(\log n)^{1/k}}\right\}, \tag{5.7}$$

provided $\{n(\log n)^{1/k-1}\epsilon_n\}$ is monotonic.

Corollary 5.6. *Necessary and sufficient conditions that $\sum a_n\epsilon_n$ be summable $|C,\alpha|_k$, $0<\alpha\le 1$ whenever $\sum a_n$ is summable $|R,\log n,1|_k$, $k\ge 1$, are:*

$$\epsilon_n = O\left\{\frac{n^{\alpha-1}}{(\log n)^{1/k}}\right\}, \qquad \triangle\epsilon_n = O\left\{\frac{1}{n(\log n)^{1/k}}\right\}, \tag{5.8}$$

provided $\{n^{1-\alpha}(\log n)^{1/k}\epsilon_n\}$ is monotonic.

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W. T. Sulaiman: Department of Mathematics, College of Science, University of Qatar, Qatar