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APPROXIMATING FIXED POINTS OF NONEXPANSIVE TYPE MAPPINGS

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ABSTRACT. In a uniformly convex Banach space, the convergence of Ishikawa iterates to a unique fixed point is proved for nonexpansive type mappings under certain conditions.

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1. Introduction. Let D be a nonempty, closed, and convex subset of a uniformly convex Banach space B, and $T : D \to D$ with fixed point set $F(T)$. Recently, Ghosh and Debnath [1] introduced the generalized versions of the conditions of Senter and Dotson [6] as: the mapping T with $F(t) \neq \emptyset$ is said to satisfy the following conditions.

Condition 1.1. If there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$||(1-TT_\mu)x|| \geq f(d(x,F)) \quad \forall x \in D, \tag{1.1}$$

where $T_\mu x = (1-\mu)x + \mu Tx$ with $0 \leq \mu \leq \beta < 1$ and $d(x,F) = \inf_{z\in F} \|x-z\|$.

Condition 1.2. If there exists a positive real number k such that

$$||(1-TT_\mu)x|| \geq k\, d(x,F(T)) \quad \forall x \in D. \tag{1.2}$$

When $\mu = 0$, both conditions reduce to those of Senter and Dotson [6]. It may be noted that the mapping which satisfies Condition 1.2 also satisfies Condition 1.1.

In this paper, we wish to use Conditions 1.1 and 1.2 to prove the convergence of Ishikawa iterates [3] of certain nonexpansive type mappings.

2. Ishikawa's iterative process. Let D be a convex subset of a Banach space B and $T : D \to D$. For $x_1 \in D$, Ishikawa [3] defined a sequence $\{x_n\}$ such that

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T[(1-\beta_n)x_n + \beta_n T x_n], \tag{2.1}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences of nonnegative numbers with $0 \leq \alpha_n \leq \beta_n \leq 1$, $\lim_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$.

Using notation for $T_\mu x$ of Section 1, (2.1) may be written as

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T T_{\beta_n} x_n. \tag{2.2}$$

In this paper, we assume that α_n and β_n satisfy

(i) $0 < a \le \alpha_n < b < 1$,

(ii) $0 \le \beta_n \le \beta < 1$.

We denote the sequence (2.1) by $M(x_1, \alpha_n, \beta_n, T)$, where α_n and β_n satisfy (i) and (ii). We also assume that $\alpha_n = \lambda$ and $\beta_n = \mu$ for all n in the Ishikawa iterates defined above, that is,

$$x_{n+1} = T^n_{\lambda,\mu} x_1, T_{\lambda,\mu} = (1-\lambda)I + \lambda T[(1-\mu)I + \mu T]. \tag{2.3}$$

3. Nonexpansive type mappings and convergence theorems. Before we state and prove our main results we need to recall several definitions.

Definition 3.1. A mapping $T : D \to D$ is called nonexpansive if for all $x, y \in D$,

$$||Tx - Ty|| \le ||x - y||. \tag{3.1}$$

Definition 3.2. A mapping $T : D \to D$ is called generalized nonexpansive if it satisfies the condition, for all $X, Y \in D$,

$$||Tx - Ty|| \le a||x - y|| + b\{||x - Tx|| + ||y - Ty||\} + c\{||x - Ty|| + ||y - Tx||\}, \tag{3.2}$$

where $a, c \ge 0$, $b > 0$, and $a + 2b + 2c \le 1$. This type of mapping was introduced by Hardy and Rogers [2] in metric spaces.

Definition 3.3. A mapping $T : D \to D$ is said to satisfy Condition 1.1 if for all $x, y \in D$,

$$||Tx - Ty|| \le \max\left\{\beta||x - y||, \frac{[||x - Tx|| + ||y - Ty||]}{2}, \frac{[||x - Ty|| + ||y - Tx||]}{2}\right\}, \tag{3.3}$$

and T is said to satisfy Condition 1.2 if for all $x, y \in D$,

$$||Tx - Ty|| \le \max\left\{\beta||x - y||, \frac{[||x - Tx|| + ||y - Ty||]}{2}, ||x - Ty||, \beta||y - Tx||\right\}, \tag{3.4}$$

where $0 \le \mu \le \beta < 1$.

Remark 3.4. It is to be noted that

(i) a nonexpansive mapping is generalized nonexpansive,

(ii) generalized nonexpansive mappings and mappings satisfying Condition 1.1 also satisfy Condition 1.2, but the converse is not true as can be seen from the following example.

Example 3.5. Let $B = R$ with the usual norm and let $D = D_1 \bigcup D_2$ where

$$\begin{aligned} D_1 &= \frac{m}{n}, \quad m = 0, 1, 3, 9, \ldots; \ n = 1, 4, \ldots, 3k+1, \\ D_2 &= \frac{m}{n}, \quad m = 1, 3, 9, 27, \ldots; \ n = 2, 5, \ldots, 3k+2. \end{aligned} \tag{3.5}$$

Define $T : D \to D$ by

$$Tx = \begin{cases} \frac{3x}{4}, & x \in D_1, \\ \frac{x}{2}, & x \in D_2. \end{cases} \tag{3.6}$$

Then T satisfies Condition 1.2, but it does not satisfy Condition 1.1 and coincidentally that T is not a generalized nonexpansive mapping; for instance, take $x = 1$, $y = 3/5$. Then

$$\begin{aligned} ||Tx - Ty|| &= \frac{9}{20} \geq \max\left\{\frac{2}{5}\beta, \frac{11}{40}, \frac{17}{40}\right\} \\ &= \max\left\{\frac{2}{5}\beta, \frac{1}{2}\left[\frac{1}{4} + \frac{3}{10}\right], \frac{1}{2}\left[\frac{7}{10} + \frac{3}{20}\right]\right\} \\ &= \max\left\{\beta||x - y||, \frac{[||x - Tx|| + ||y - Ty||]}{2}, \frac{[||x - Ty|| + ||y - Tx||]}{2}\right\}. \end{aligned} \tag{3.7}$$

We now show that a mapping T satisfying Condition 1.2 is a quasi-nonexpansive mapping. Suppose p is a fixed point of T. Then putting $y = p$ in (3.4) and for $Tx \neq p$, we obtain

$$\begin{aligned} 0 < ||Tx - p|| &= ||Tx - Tp|| \\ &\leq \max\left\{\beta||Tx - p||, \ \frac{1}{2}\|x - Tx\|, ||x - p||, \ \beta||p - Tx||\right\} \\ &\leq \max\left\{\beta||Tx - p||, \ \frac{1}{2}[||x - p|| + ||p - Tx||], ||x - p||, \beta||p - Tx||\right\}. \end{aligned} \tag{3.8}$$

Since $\|Tx - p\| \leq \beta\|p - Tx\|$ is not possible, we have

$$||Tx - p|| \leq \max\left\{\frac{1}{2}[||x - p|| + ||p - Tx||], ||x - p||\right\} \tag{3.9}$$

which implies that

$$||Tx - p|| \leq ||x - p||. \tag{3.10}$$

Therefore, T is quasi-nonexpansive. Next we show that

$$F(T) = F(T_{\lambda,\mu}) = F(TT_\mu). \tag{3.11}$$

Obviously $F(T) \subset F(T_{\lambda,\mu})$.

Let $p \in F(T_{\lambda,\mu})$. Then $T_{\lambda,\mu}p = p$ implies that $T_{\lambda,\mu}p = (1-\lambda)Ip + \lambda T[(1-\mu)Ip + \mu Tp] = (1-\lambda)p + \lambda TT_\mu p$ and so $TT_\mu p = p$.

It follows from (3.4) that

$$\begin{aligned} ||Tp - p|| &= ||Tp - TT_\mu p|| \\ &\leq \max\left\{\beta||p - T_\mu p||, \frac{1}{2}[||p - Tp|| + ||T_\mu p - p||], 0, \beta||T_\mu p - Tp||\right\} \\ &= \max\left\{\beta\mu||p - Tp||, \frac{1}{2}(1 + \mu)||p - Tp||, 0, \beta(1 - \mu)||p - Tp||\right\}, \end{aligned} \tag{3.12}$$

whence we obtain $Tp = p$, since $\max\{\beta\mu, (1/2)(1+\mu), \beta(1-\mu)\} < 1$. Thus, $F(T_{\lambda,\mu}) \subset F(T)$ leading to the result (3.11).

Now, we show that the mapping T satisfies Condition 1.2. We have from (3.4)

$$\begin{aligned}
||TT_\mu x - p|| &= ||TT_\mu x - Tp|| \\
&\le \max\left\{\beta||T_\mu X - P||, \frac{1}{2}||T_\mu x - TT_\mu x||, ||T_\mu x - p||, \beta||p - TT_\mu x||\right\} \\
&= \max\left\{||T_\mu x - p||, \frac{1}{2}||T_\mu x - TT_\mu x||, \beta||p - TT_\mu x||\right\} \\
&\le \max\left\{||T_\mu x - p||, \frac{1}{2}||T_\mu x - TT_\mu x||, \beta||p - T_\mu x||\right\} \\
&= \max\left\{||T_\mu x - p||, \frac{1}{2}||T_\mu x - TT_\mu x||\right\} \\
&\le \max\left\{[(1-\mu)||x-p|| + \mu||Tx - P||], \frac{1}{2}[||x - T_\mu x|| + ||x - TT_\mu x||]\right\} \\
&\le \max\left\{[(1-\mu)||x-p|| + \mu||x - p||], \frac{1}{2}[\mu||x - Tx|| + ||x - TT_\mu x||]\right\} \\
&\le \max\left\{||x-p||, \frac{1}{2}[\mu(||x-p|| + ||p - Tx||) + ||x - TT_\mu x||]\right\} \\
&\le \max\left\{||x-p||, \frac{1}{2}[2\mu||x-p|| + ||x - TT_\mu x||]\right\}.
\end{aligned} \tag{3.13}$$

Also,we know that

$$||TT_\mu x - p|| \ge ||x - p|| - ||x - TT_\mu x||. \tag{3.14}$$

From (3.13) and (3.14), we deduce that

$$\max\left\{||x-p||, \frac{1}{2}[2\mu||x-p|| + ||x - TT_\mu x||]\right\} \ge ||x-p|| - ||x - TT_\mu x|| \tag{3.15}$$

which implies $\|x - TT_\mu x\| \ge (2(1-\mu)/3)\|x - p\|$. Then we may write

$$||x - TT_\mu x|| \ge k||x - p||, \tag{3.16}$$

where

$$0 < k = \frac{2(1-\mu)}{3} < 1. \tag{3.17}$$

Thus T satisfies Condition 1.2 with $0 < k < 1$. Consequently, by Maiti and Ghosh [4, Theorem 1, page 114], we have the following.

Theorem 3.6. *Let D be a closed convex Banach space B, and let $T : D \to D$ be a mapping which satisfies (3.4) and has a fixed point in D. Then T satisfies Condition 1.2 and, for any $x_1 \in D$, $M(x_1, \alpha_n, \beta_n, T)$ converges to the fixed point of D.*

We next consider a mapping T which satisfies Condition 1.1, and a variant of Theorem 3.6 is stated below.

Theorem 3.7. *Let D be a closed convex bounded subset of a uniformly convex Banach space B, and let $T : D \to D$ be a mapping satisfying (3.3). Then T satisfies Condition 1.1, and for any $x_1 \in D$, $M(x_1, \alpha_n, \beta_n, T)$ converges to the unique fixed point of T.*

Proof. The mapping T satisfying (3.3) also satisfies Naimpally and Singh [5, Condition II(D)], and so T has a unique fixed point. We now show that T is quasinonexpansive. Let $p \in F(T)$. Then, for any $x \in D$, we have from (3.3),

$$\begin{aligned} \|Tx-p\| = \|Tx-Tp\| &\le \max\left\{\beta\|x-p\|, \frac{1}{2}\|x-Tx\|, \frac{1}{2}[\|x-p\|+\|p-Tx\|]\right\} \\ &\le \max\left\{\beta\|x-p\|, \frac{1}{2}[\|x-p\|+\|p-Tx\|]\right\} \end{aligned} \tag{3.18}$$

implying

$$\|Tx-p\| \le \|x-p\|. \tag{3.19}$$

Next, we show that T satisfies Condition 1.1.

Let $p \in F(T)$. Then we have from (3.1),

$$\begin{aligned} \|TT_\mu x-p\| &= \|TT_\mu x-T\| \\ &\le \max\left\{\beta\|T_\mu x-p\|, \frac{1}{2}\|T_\mu x-TT_\mu x\|, \frac{1}{2}[\|T_\mu x-p\|+\|p-TT_\mu x\|]\right\} \\ &\le \max\left\{\beta\|T_\mu x-p\|, \frac{1}{2}[\|T_\mu x-p\|+\|p-TT_\mu x\|]\right\} \\ &\le \max\left\{\beta\|T_\mu x-p\|, \|T_\mu x-p\|\right\} \\ &= \|T_\mu x-p\| = \|x-p\|. \end{aligned} \tag{3.20}$$

From (3.14) and (3.20), we derive

$$\|x-p\| \ge \|x-p\| - \|x-TT_\mu x\| \tag{3.21}$$

which implies that

$$\|x-TT_\mu x\| \ge 0 = f(0). \tag{3.22}$$

Thus T satisfies all conditions which ensure the convergence of $M(x_1, \alpha_n, \beta_n, T)$. □

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