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QUATERNION CR-SUBMANIFOLDS OF A QUATERNION KAEHLER MANIFOLD

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Abstract. We study the quaternion CR-submanifolds of a quaternion Kaehler manifold. More specifically we study the properties of the canonical structures and the geometry of the canonical foliations by using the Bott connection and the index of a quaternion CR-submanifold.

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1. Introduction. The notion of a CR-submanifold of a Kaehler manifold was introduced by Bejancu [3]. Subsequently a number of authors studied these submanifolds (see [4] for details). In [1], Barros et al. studied quaternion CR-submanifolds of a quaternion Kaehler manifold and obtained many interesting results. The aim of this paper is to continue the study of quaternion CR-submanifolds of a quaternion Kaehler manifold. The paper is organized as follows: in Section 2 we collect some basic formulas and results for later use and in Section 3 we study some properties of canonical structures, particularly its parallelism and QR-product. In Section 4 we study the geometry of the canonical foliations using the Bott connection and the index of a quaternion CR-submanifold. Finally, as an extension of the work of Chen [5] for the Kaehler manifolds we give a complete classification of the totally umbilical quaternion CR-submanifolds of a quaternion Kaehler manifold.

2. Preliminaries. Let $\bar{M}$ be a quaternion Kaehler manifold with metric tensor g and quaternion structure V [7]. We will denote by $\psi_1 = I$, $\psi_2 = J$, and $\psi_3 = K$ a local basis of almost Hermitian structures for V.

Let X be a unit vector tangent to the quaternion Kaehler manifold $\bar{M}$. Then the vectors X, IX, JX, KX form an orthonormal frame. Let $Q(X)$ be the quaternion section determined by X. Any plane in a quaternion section is called a quaternion plane and the sectional curvature of a quaternion plane is called a quaternion sectional curvature. A quaternion Kaehler manifold is called a quaternion space form, which is denoted by $\bar{M}(c)$, if its quaternion sectional curvature is equal to a constant c at any point of the manifold. The curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given by, [7],

$$\bar{R}(X,Y)Z = \frac{c}{4}\Big[g(Y,Z)X - g(X,Z)Y + \sum_{r=1}^{3} g(\psi_r Y,Z)\psi_r X - g(\psi_r X,Z)\psi_r Y + 2g(X,\psi_r Y)\psi_r Z\Big], \tag{2.1}$$

where $\psi_1 = I$, $\psi_2 = J$, $\psi_3 = K$.

Let M be a Riemannian manifold isometrically immersed in a quaternion Kaehler manifold $\bar{M}$. We also denote by g the metric tensor induced on M. If ∇ is the covariant differentiation induced on M, the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \qquad \bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.2}$$

respectively, for any X, Y tangent to M and N normal to M. Here h and $\nabla^{\perp}$ are the second fundamental form associated with M, and the connection of the normal bundle, respectively. The second fundamental tensor A_N is related to h by

$$g(A_N X, Y) = g(h(X,Y), N). \tag{2.3}$$

A differentiable distribution D_x on M such that $\psi_r(D_x) \subseteq D_x$ for all $r = 1,2,3$ is called a quaternion distribution. In other words, D_x is a quaternion distribution if D_x is contained into itself by its quaternion structure.

It is known [1] that a submanifold M of a quaternion Kaehler manifold $\bar{M}$ is called a quaternion CR-submanifold if it admits a quaternion distribution D_x such that its orthogonal complementary distribution $D_x^{\perp}$, is totally real, that is, $\psi_r(D_x^{\perp}) \subseteq T_x^{\perp}M$ for all $x \in M$ and $r = 1,2,3$, where $T_x^{\perp}M$ denotes the normal space of M at x.

A submanifold M of a quaternion Kaehler manifold $\bar{M}$ is called a quaternion (resp., totally real) submanifold if $\dim D_x^{\perp}=0$ (resp., $\dim D_x=0$). A quaternion CR-submanifold is said to be proper if it is neither quaternion nor totally real.

We denote by μ the subbundle of the normal bundle $T^{\perp}M$ which is the orthogonal complement of $\psi_1 D^{\perp} \oplus \psi_2 D^{\perp} \oplus \psi_3 D^{\perp}$, that is,

$$T^{\perp}M = \psi_1 D^{\perp} \oplus \psi_2 D^{\perp} \oplus \psi_3 D^{\perp} \oplus \mu; \quad g(\mu, \psi_r D^{\perp}) = 0. \tag{2.4}$$

The mean curvature vector H of M in $\bar{M}$ is defined by $H = (1/n)\operatorname{trace} h$, where n denotes the dimension of M. If we have

$$h(X,Y) = g(X,Y)H \tag{2.5}$$

for any $X, Y \in TM$, then M is called a totally umbilical submanifold. In particular, if $h(X,Y) = 0$ identically for all $X, Y \in TM$, M is called a totally geodesic submanifold. Finally M is called mixed totally geodesic if $h(X,Y) = 0$ for $X \in D$, $Y \in D^{\perp}$. For totally umbilical CR-submanifolds, equations (2.2) take the forms

$$\bar{\nabla}_X Y = \nabla_X Y + g(X,Y)H, \qquad \bar{\nabla}_X N = -g(H,N)X + \nabla^{\perp}_X N. \tag{2.6}$$

The Codazzi equation for a totally umbilical CR-submanifold M, is given by

$$\bar{R}(X,Y;Z,N) = g(Y,Z)g(\nabla^{\perp}_X H, N) - g(X,Z)g(\nabla^{\perp}_Y H, N). \tag{2.7}$$

Definition 2.1 (see [1]). Let M be a quaternion CR-submanifold of a quaternion Kaehler manifold $\bar{M}$. Then M is called a QR-product, if M is locally the Riemannian product of a quaternion submanifold and a totally real submanifold of $\bar{M}$.

For any $X \in TM$ and $N \in T^{\perp}M$, we put

$$\psi_r X = P_r X + Q_r X, \tag{2.8}$$
$$\psi_r N = t_r N + f_r N, \tag{2.9}$$

where $P_r X$, $t_r N$ (resp., $Q_r X$, $f_r N$) are the tangential (resp., the normal) components of $\psi_r X$ and $\psi_r N$ for $r = 1, 2, 3$.

For the second fundamental form h, the covariant differentiation is defined by

$$(\bar{\nabla}_X h)(Y, Z) = \nabla_X^{\perp} h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{2.10}$$

and the Gauss-Codazzi equations are given by

$$R(X, Y, Z, W) = \bar{R}(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \tag{2.11}$$
$$[R(X, Y)Z]^{\perp} = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z), \quad \forall X, Y, Z, W \text{ tangent to } \bar{M}, \tag{2.12}$$

where R is the curvature tensor associated with ∇ and $\perp$ in (2.12) denotes the normal component.

We collect from Barros et al. [1] the following results which we shall need in the sequel.

Lemma 2.2. *Every quaternion submanifold of a quaternion Kaehler manifold is totally geodesic.*

Lemma 2.3. *The quaternion distribution D of a quaternion CR-submanifold M in a quaternion Kaehler manifold $\bar{M}$ is integrable if and only if $h(D, D) = 0$.*

Lemma 2.4. *Let M be a quaternion CR-submanifold of a quaternion Kaehler manifold $\bar{M}$. Then the leaf $M^{\perp}$ of $D^{\perp}$ is totally geodesic in M if and only if $g(h(D, D^{\perp}), \psi_r D^{\perp}) = 0$, $r = 1, 2, 3$.*

Lemma 2.5. *Let M be a quaternion CR-submanifold of a quaternion Kaehler manifold $\bar{M}$. Then*

$$A_{\psi_r W} Z = A_{\psi_r Z} W \quad \text{for any } W, Z \in D^{\perp}. \tag{2.13}$$

3. Canonical parallel structures and QR-product.

Let P_r, f_r, Q_r, and t_r be the endomorphisms and the vector-bundle-valued 1-forms defined in (2.8), respectively. We define the covariant differentiation of P_r, Q_r, t_r, and f_r as follows:

$$\begin{aligned} (\bar{\nabla}_X P_r)(Y) &= \nabla_X (P_r Y) - P_r \nabla_X Y, \qquad (\bar{\nabla}_X Q_r)(Y) = \nabla_X^{\perp} (Q_r Y) - Q_r \nabla_X Y, \\ (\bar{\nabla}_X t_r)(N) &= \nabla_X (t_r N) - t_r \nabla_X^{\perp} N, \qquad (\bar{\nabla}_X f_r)(N) = \nabla_X^{\perp} (f_r N) - f_r \nabla_X^{\perp} N, \end{aligned} \tag{3.1}$$

for any vector fields $X, Y \in TM$ and $N \in T^{\perp}M$.

The endomorphisms P_r (resp., the endomorphisms f_r, the 1-forms Q_r and t_r) are parallel if $\bar{\nabla} P_r = 0$ (resp., $\bar{\nabla} f_r = 0$, $\bar{\nabla} Q_r = 0$, and $\bar{\nabla} t_r = 0$).

Now using the definition of a quaternion Kaehler manifold and taking account of (2.2), (2.8), we can easily obtain the following:

$$(\bar{\nabla}_X P_r)(Y) = A_{Q_r Y} X + t_r h(X,Y), \tag{3.2}$$

$$(\bar{\nabla}_X Q_r)(Y) = f_r h(X,Y) - h(X, P_r Y), \tag{3.3}$$

$$(\bar{\nabla}_X t_r)(N) = A_{f_r N} X - P_r A_N X, \tag{3.4}$$

$$(\bar{\nabla}_X f_r)(N) = -h(X, t_r N), \tag{3.5}$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$.

Remark 3.1. Since the second fundamental form is symmetric, it follows from (3.2) that P_r is parallel if and only if

$$A_{\psi_r U} V = A_{\psi_r V} U, \quad \forall U, V \in TM. \tag{3.6}$$

Now if we set $V = X \in D$ in (3.6), we find that $A_{\psi_r U} X = 0$ for all $U \in TM$, which is equivalent to $g(h(X,Y), \psi_r U) = 0$ for any $X \in D$, and $Y, U \in TM$. In particular $g(h(X,Y), \psi_r Z) = 0$ for any $X \in D$ and $Y, Z \in D^{\perp}$.

Thus, using Lemma 2.4 we obtain the following lemma.

Lemma 3.2. *Let M be a quaternion CR-submanifold of a quaternion Kaehler manifold $\bar{M}$. If P_r is parallel then the leaf $M^{\perp}$ of $D^{\perp}$ is totally geodesic in M.*

Now we state and prove the following proposition.

Proposition 3.3. *Let M be a quaternion CR-submanifold of a quaternion Kaehler manifold $\bar{M}$. Then Q_r is parallel if and only if t_r is parallel.*

Proof. Suppose t_r is parallel. Then from (3.4) we have

$$A_{f_r N} U = P_r A_N U, \quad \text{for any } U \in TM. \tag{3.7}$$

Thus for any vector fields $U, V \in TM$ and $N \in T^{\perp}M$, we get

$$g(A_{f_r N} U, V) = g(P_r A_N U, V), \tag{3.8}$$

or equivalently

$$f_r h(U,V) - h(U, P_r V) = 0, \tag{3.9}$$

that is, $\bar{\nabla} Q_r = 0$.

The proof of the converse statement is similar. □

Lemma 3.4. *Let M be a QR-product of a quaternion Kaehler manifold $\bar{M}$. Then*

(a) $\nabla_Z X \in D$,

(b) $\nabla_X Z \in D^{\perp}$,

for all $X \in D$ and $Z \in D^{\perp}$.

Proof. By using (2.2) and the definition of a quaternion Kaehler manifold, we find

$$\psi_r \nabla_Z X = \nabla_Z \psi_r X + h(Z, \psi_r X) - \psi_r(X, Z) \quad \text{for } X \in D,\ Z \in D^{\perp}. \tag{3.10}$$

The above equation yields

$$\begin{aligned} g(\psi_r \nabla_Z X, \psi_r W) &= g(\nabla_Z \psi_r X, \psi_r W) + g(h(Z, \psi_r X), \psi_r W), \\ g(\nabla_Z X, W) &= g(h(Z, \psi_r X), \psi_r W) \quad \text{for } X \in D,\ W, Z \in D^{\perp}. \end{aligned} \tag{3.11}$$

Since M is a QR-product the leaf $M^{\perp}$ of $D^{\perp}$ is totally geodesic. Thus using Lemma 2.4 we get (a).

Next for $X \in D$, $Z \in D^{\perp}$ we have

$$\bar{\nabla}_X \psi_r Z = \psi_r \bar{\nabla}_X Z \tag{3.12}$$

which, by virtue of (2.2), gives

$$\psi_r \nabla_X Z = -A_{\psi_r Z} X + \nabla^{\perp}_X \psi_r Z - \psi_r h(X, Z). \tag{3.13}$$

Taking inner products with $Y \in D$ and using the fact that the leaf $M^{\perp}$ of $D^{\perp}$ is totally geodesic, we find

$$g(\psi_r \nabla_X Z, Y) = -g(A_{\psi_r Z} X, Y) = -g(h(X, Y), \psi_r Z) \quad \text{for } X, Y \in D,\ Z \in D^{\perp}. \tag{3.14}$$

On the other hand, for $X \in D$ and $W, Z \in D^{\perp}$ and the use of Lemma 2.5, (3.13) gives

$$\begin{aligned} g(\psi_r \nabla_X Z, W) &= -g(\psi_r h(X, Z), W) - g(h(X, W), \psi_r Z) \\ &= g(A_{\psi_r W} Z, X) - g(A_{\psi_r Z} W, X) \\ &= g(A_{\psi_r W} Z - A_{\psi_r Z} W, X) \\ &= 0. \end{aligned} \tag{3.15}$$

Thus from (3.14) and (3.15) we see that $\psi_r \nabla_X Z$ is normal to M. So $\nabla_X Z \in D^{\perp}$ for all $X \in D$ and $Z \in D^{\perp}$. □

Theorem 3.5. *Let M be a quaternion CR-submanifold of a quaternion Kaehler manifold $\bar{M}$. Then M is a QR-product if and only if P_r is parallel.*

Proof. Suppose P_r is parallel, then from (3.2), we have

$$A_{Q_r Y} X + t_r h(X, Y) = 0 \quad \forall X, Y \in TM. \tag{3.16}$$

If $Y \in D$, then $Q_r Y = 0$. Hence (3.16) is reduced to $t_r h(X, Y) = 0$ for all $X \in TM$, $Y \in D$. Therefore by virtue of [1, Lemma 5.1, page 403], we get $h(D, D^{\perp}) = 0$ or $h(D, D) = 0$. So the quaternion distribution D is integrable by virtue of Lemma 2.3. Thus it follows that each leaf $M^{\perp}$ is totally geodesic in $\bar{M}$ and in particular $M^{\perp}$ is totally geodesic in M by virtue of Lemma 2.2.

Again from (3.2), we have

$$A_{\psi_r W} Z + t_r h(W,Z) = 0 \quad \forall W, Z \in D^{\perp}. \tag{3.17}$$

So for $X \in D$, we have

$$g(A_{\psi_r W} Z, X) + g(t_r h(W,Z), X) = 0 \tag{3.18}$$

which means

$$g(h(X,Z), Q_r W) - g(h(W,Z), Q_r X) = 0, \tag{3.19}$$

that is,

$$g(h(X,Z), Q_r W) = 0 \tag{3.20}$$

or

$$g(h(D, D^{\perp}), Q_r D^{\perp}) = 0. \tag{3.21}$$

Thus using Lemma 2.4, it follows that the leaf $M^{\perp}$ of $D^{\perp}$ is totally geodesic. Hence M is a QR-product.

Conversely, let M be a QR-product. First we show that $\nabla_U X \in D$ for any $X \in D$ and U tangent to M. Since M is a QR-product, that is, locally a Riemannian product of a quaternion submanifold and a totally real submanifold, it is sufficient to show that $\nabla_Z X \in D$ for any $X \in D$, $Z \in D^{\perp}$ but this was proved in Lemma 3.4(a). Using this fact, we have

$$\nabla_U \psi_r X + h(U, \psi_r X) = \psi_r \nabla_U X + \psi_r h(X,U) \quad \text{for any } X \in D,\ U \text{ tangent to } M, \tag{3.22}$$

which yields

$$\psi_r h(U,X) = h(U, \psi_r X), \qquad \nabla_U \psi_r X = \psi_r \nabla_U X. \tag{3.23}$$

Thus $(\bar{\nabla}_U P_r)(X) = \nabla_U P_r X - P_r \nabla_U X = 0$, for any $X \in D$, and U tangent to M.

Similarly, by using Lemma 3.4(b), it follows that $\nabla_U Z \in D^{\perp}$ for any $Z \in D^{\perp}$, and U tangent to M. But since M is a QR-product, it follows that $\nabla_X Z \in D^{\perp}$ for $U = X \in D$ and $Z \in D^{\perp}$.

Thus, we have $(\bar{\nabla}_U P_r)(Z) = 0$ for any $Z \in D^{\perp}$, U tangent to M. Therefore $\bar{\nabla} P_r = 0$, which completes the proof. □

Corollary 3.6. *Let M be a QR-product of a quaternion Kaehler manifold $\bar{M}$. Then M is mixed totally geodesic, that is, $h(D, D^{\perp}) = 0$.*

Remark 3.7. If M is a proper QR-product of a quaternion space form $\bar{M}(c)$, then the ambient manifold $\bar{M}$ is necessarily a space of zero curvature. Hence there does not exist a proper QR-product of a quaternion space form $\bar{M}(c)$ with $c \neq 0$.

4. Canonical foliations and index of a quaternion CR-submanifold

Definition 4.1 (see [8])**.** Let D be a distribution on the Riemannian manifold M, $D^{\perp}$ the orthogonal distribution, $\Pi^{\perp} : TM \to D^{\perp}$ the projection and ∇ the Levi-Civita connection. Then the second fundamental form of the plane field D, is defined by

$$S_\nabla(X,Y) = \frac{1}{2}\Pi^\perp(\nabla_X Y + \nabla_Y X). \tag{4.1}$$

The distribution D is called a totally geodesic plane field, if the geodesics tangent to it at one point remain tangent for all their length.

Thus we say that the distribution D is a totally geodesic plane field if

$$S_\nabla(X,Y) = \Pi^\perp(\nabla_X Y + \nabla_Y X) = 0 \quad \forall X, Y \in D. \tag{4.2}$$

A geometric definition of this notion is given in [9].

A foliation f on a Riemannian manifold M is called a Riemannian foliation, if the Bott connection $\mathring{\nabla}_X Y = \Pi[X,Y]$ in the normal bundle of f preserves the Riemannian metric. Also f is a Riemannian foliation if and only if the second fundamental form S_∇ of the plane field D vanishes (see [9, page 157]).

Theorem 4.2. *Let M be a quaternion CR-submanifold of a quaternion Kaehler manifold $\bar{M}$ such that $D_M^\perp$ is a totally real foliation of M. Then the Bott connection of $D_M^\perp$ preserves the volume form ψ of D_M, that is, $\mathring{\nabla}_Z\psi = 0$, for all $Z \in D_M^\perp$.*

Proof. For any $X, Y \in D$ and $Z \in D^\perp$, we have

$$\begin{aligned}
g\Big(\big(\mathring{\nabla}_Z\psi_r\big)(X),Y\Big) &= g\Big(\mathring{\nabla}_Z\psi_r X,Y\Big) - g\Big(\psi_r\mathring{\nabla}_Z X,Y\Big)\\
&= g([Z,\psi_r X],Y) + g([Z,X],\psi_r Y)\\
&= g(\bar{\nabla}_Z\psi_r X,Y) - g(\bar{\nabla}_{\psi_r X}Z,Y)\\
&\quad + g(\bar{\nabla}_Z X,\psi_r Y) - g(\bar{\nabla}_X Z,\psi_r Y)\\
&= g(X,\psi_r\bar{\nabla}_Z Y) + g(\bar{\nabla}_{\psi_r X}Y,Z)\\
&\quad - g(X,\bar{\nabla}_Z\psi_r Y) + g(\bar{\nabla}_X\psi_r Y,Z)\\
&= g(\bar{\nabla}_{\psi_r X}Y,Z) + g(\bar{\nabla}_X\psi_r Y,Z)\\
&= g(\bar{\nabla}_{\psi_r X}Y,Z) + g(\bar{\nabla}_X\psi_r Z,Y)\\
&= g(\bar{\nabla}_{\psi_r X}Y,Z) - g(A_{\psi_r Z}X,Y).
\end{aligned} \tag{4.3}$$

Also,

$$\begin{aligned}
g(\nabla_X X,Z) &= g(\bar{\nabla}_X X,Z)\\
&= g(\psi_r\bar{\nabla}_X X,\psi_r Z)\\
&= g(\bar{\nabla}_X\psi_r X,\psi_r Z)\\
&= -g(\bar{\nabla}_X\psi_r Z,\psi_r X)\\
&= g(A_{\psi_r Z}X,\psi_r X).
\end{aligned} \tag{4.4}$$

If $D_M^\perp$ is Riemannian then D_M is a totally geodesic plane field and so (4.4) gives $g(A_{\psi_r Z}X,\psi_r X) = 0$.

Therefore $g(A_{\psi_r Z}(X+Y), \psi_r(X+Y)) = 0$, and hence we obtain

$$g(A_{\psi_r Z}X, \psi_r Y) + g(A_{\psi_r Z}Y, \psi_r X) = 0. \tag{4.5}$$

Thus using (4.3) and (4.5), we have

$$\begin{aligned} g((\mathring{\nabla}_Z \psi_r)(X), \psi_r Y) &= g(\bar{\nabla}_{\psi_r X}\psi_r Y, Z) - g(A_{\psi_r Z}X, \psi_r Y) \\ &= g(\bar{\nabla}_{\psi_r X}\psi_r Y, Z) + g(A_{\psi_r Z}Y, \psi_r X) \\ &= 0. \end{aligned} \tag{4.6}$$

Moreover, it is known that D_M is a minimal distribution [2], which implies that

$$(d\psi)(Z, X_1, \dots, X_{4n}) = 0 \quad \text{for } Z \in D^\perp,\ X_1, \dots, X_{4n} \in D. \tag{4.7}$$

Hence

$$\begin{aligned} (\mathring{\nabla}_Z \psi)(X_1, \dots, X_{4n}) &= Z\psi(X_1, \dots, X_{4n}) - \sum_{a=1}^{4n} \psi(X_1, \dots, \Pi[Z, X_a], \dots, X_{4n}) \\ &= (d\psi)(Z, X_1, \dots, X_{4n}) = 0, \end{aligned} \tag{4.8}$$

which completes the proof. □

Now, let M be a compact totally geodesic quaternion CR-submanifold of a quaternion Kaehler manifold $\bar{M}$. Let N be a normal vector field and denote by $\nu''(N)$ the second normal variation of M induced by N. Then we have (see [6, Chapter 1]),

$$\nu''(N) = \int_M \left\{ \|\nabla^\perp N\|^2 - \sum_{i=1}^{n} \bar{R}(X_i, N, N, X_i) - \|A_N\|^2 \right\} dV, \tag{4.9}$$

where $N \in T^\perp M$, dV denotes the volume element of M and $\{X_i\}$ is an orthonormal frame in TM. Applying the Stokes theorem to the integral of the first term of (4.9), we have

$$I(N,N) =: \nu''(N) = \int_M g(LN, N) * 1, \tag{4.10}$$

where L is a selfadjoint, strongly elliptic linear differential operator of the second order. The differential operator L is called the Jacobi operator of M in $\bar{M}$ and has discrete eigenvalues $\lambda_1 < \lambda_2 < \cdots$. We put $E_\lambda = \{N \in T^\perp M : L(N) = \lambda N\}$. The dimension of the space E_λ, $\dim(E_\lambda)$, is called the index of M in $\bar{M}$. For two normal vector fields N_1, N_2 to a minimal submanifold M in $\bar{M}$, their index form is defined by

$$I(N_1, N_2) = \int_M g(LN_1, N_2) * 1. \tag{4.11}$$

It is easy to see that the index form I is a symmetric bilinear form; $I : T^\perp M \times T^\perp M \to R$.

Now we prove the following theorem.

Theorem 4.3. *Let M be a compact n-dimensional minimal quaternion CR-submanifold of a quaternion Kaehler manifold $\bar{M}$. If M has nonpositive holomorphic bisectional curvature, then the index form satisfies*

$$I(N,N) + I(\psi_r N, \psi_r N) \geq 0 \quad \textit{for any } N \in \mu. \tag{4.12}$$

Proof. By using the Weingarten equation we have that for all $X, Y \in D^{\perp}$,

$$\begin{aligned} g(\nabla_X^{\perp} N, \psi_r Y) &= g(\bar{\nabla}_X N, \psi_r Y) \\ &= -g(\psi_r \bar{\nabla}_X N, Y) \\ &= -g(\bar{\nabla}_X \psi_r N, Y) \\ &= g(A_{\psi_r N} X, Y) \end{aligned} \tag{4.13}$$

which implies that

$$\|\nabla^{\perp} N\|^2 \geq \|A_{\psi_r N}\|^2, \qquad \|\nabla^{\perp} \psi_r N\|^2 \geq \|A_N\|^2 \quad \text{for any } N \in \mu, \tag{4.14}$$

where μ is defined in (2.4). Thus by using (4.9), (4.10), (4.13), and (4.14) we get

$$I(N,N) + I(\psi_r N, \psi_r N) \geq -\int_M \sum_{i=1}^{n} \{\bar{R}(N, e_i, e_i, N) + \bar{R}(\psi_r N, e_i, e_i, \psi_r N)\} * 1 \tag{4.15}$$

from which the proof follows, since M has nonpositive holomorphic bisectional curvature. □

Finally, we prove a classification theorem for the totally umbilical quaternion CR-submanifolds of a quaternion Kaehler manifold.

Theorem 4.4. *Let M be a compact totally umbilical quaternion CR-submanifold of a quaternion Kaehler manifold $\bar{M}$. Then*

(a) *M is a totally geodesic submanifold, or,*

(b) *M is locally the Riemannian product of a quaternion submanifold and a totally real submanifold, or,*

(c) *M is a totally real submanifold, or,*

(d) *the totally real distribution is one dimensional, that is, $\dim D^{\perp} = 1$,*

(e) *$\nabla_X^{\perp} H \in \mu$, for $X \in D$.*

Proof. We take $X, W \in D^{\perp}$ and using (2.6) with the fact that $\bar{M}$ is a quaternion Kaehler manifold, we have

$$\psi_r \nabla_X W + g(X, W)\psi_r H = -A_{\psi_r W} X + \nabla_X^{\perp} \psi_r W. \tag{4.16}$$

Taking inner product with X we get

$$g(H, \psi_r W)\|X\|^2 = g(X, W) g(H, \psi_r X). \tag{4.17}$$

Exchanging X and W in (4.17) we have

$$g(H, \psi_r X)\|W\|^2 = g(X, W) g(H, \psi_r W). \tag{4.18}$$

This together with (4.17) gives

$$g(H, \psi_r W) = \frac{g(X, W)^2}{\|X\|^2 \|W\|^2} g(H, \psi_r W). \tag{4.19}$$

The possible solutions of (4.19) are

(i) $H=0$,

(ii) $H\perp\psi_r W$,

(iii) $X\|W$.

Suppose that condition (i) holds, that is, $H=0$. This implies that M is totally geodesic which proves (a). Combining (ii) with a result in [1, page 407] we get (b) of the theorem.

Now from (2.7) we have

$$\begin{aligned} O &= \bar{R}(IX,JX,KX,N) \\ &= \bar{R}(KX,N,IX,JX) \\ &= -\bar{R}(KX,N,X,KX) \\ &= -\bar{R}(X,KX,KX,N) \\ &= -g(\nabla^{\perp}_X H,N)\|X\|^2 \end{aligned} \tag{4.20}$$

which implies that

$$\nabla^{\perp}_X H \in \mu \quad \forall X \in D \tag{4.21}$$

proving (e). Next we have

$$\bar{\nabla}_X \psi_r H = \psi_r \bar{\nabla}_X H \quad \text{for } X \in D \tag{4.22}$$

which, by (2.6) gives

$$\nabla^{\perp}_X \psi_r H = -g(H,H)\psi_r X + \psi_r \nabla^{\perp}_X H. \tag{4.23}$$

Since $\nabla^{\perp}_X H \in \mu$, from (4.23) we have $\psi_r X = 0$ for all $X \in D$. Hence $D = \{0\}$ which proves (c). Finally if (iii) is valid then $\dim D^{\perp} = 1$, which completes the proof. □

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