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AN EXTENSION OF A THEOREM OF SAHAB, KHAN, AND SESSA

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ABSTRACT. A fixed point theorem of Fisher and Sessa is generalized to locally convex spaces and the new result is applied to extend a recent theorem on invariant approximation of Sahab, Khan, and Sessa.

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1. Introduction and terminology. In 1988, Mukherjee and Verma [10] obtained the following generalization of a theorem of Fisher and Sessa [4].

Theorem 1.1. *Let T and I be two weakly commuting mappings of a closed convex subset C of a Banach space X into itself satisfying the inequality*

$$\|Tx-Ty\| \le a\|Ix-Iy\| + (1-a)\max\{\|Tx-Ix\|, \|Ty-Iy\|\}, \tag{1.1}$$

for all $x,y \in C$, where $a \in (0,1)$. If I is affine and nonexpansive in C and if $T(C) \subseteq I(C)$, then T and I have a unique common fixed point in C.

In this note, we first prove that Theorem 1.1 can appreciably be extended to the setup of a Hausdorff locally convex space. An application of new result is presented to best approximation theory; our work extends earlier results of Brosowski [3], Sahab et al. [12], Singh [14] and many others.

In the sequel, (E,τ) will be a Hausdorff locally convex topological vector space. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E is said to be an associated family of seminorms for τ if the family $\{\gamma U : \gamma > 0\}$, where $U = \bigcap_{i=1}^{n} U_{\alpha_i}$ and $U_{\alpha_i} = \{x : p_{\alpha_i}(x) < 1\}$, forms a base of neighbourhoods of zero for τ. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E is called an augmented associated family for τ if $\{p_\alpha : \alpha \in I\}$ is an associated family with the property that the seminorm $\max\{p_\alpha, p_\beta\} \in \{p_\alpha : \alpha \in I\}$ for any $\alpha, \beta \in I$. The associated and augmented associated families of seminorms will be denoted by $A(\tau)$ and $A^*(\tau)$, respectively. It is well known that given a locally convex space (E,τ), there always exists a family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E such that $\{p_\alpha : \alpha \in I\} = A^*(\tau)$ (see [9, page 203]).

The following construction will be crucial. Suppose that M is a τ-bounded subset of E. For this set M we can select a number $\lambda_\alpha > 0$ for each $\alpha \in I$ such that $M \subset \lambda_\alpha U_\alpha$ where $U_\alpha = \{x : p_\alpha(x) \le 1\}$. Clearly $B = \bigcap_\alpha \lambda_\alpha U_\alpha$ is τ-bounded, τ-closed, absolutely convex and contains M. The linear span E_B of B in E is $\bigcup_{n=1}^{\infty} nB$. The Minkowski functional of B is a norm $\|\cdot\|_B$ on E_B. Thus $(E_B, \|\cdot\|_B)$ is a normed space with B as its closed unit ball and $\sup_\alpha p_\alpha(x/\lambda_\alpha) = \|x\|_B$ for each $x \in E_B$.

Following Sessa [13], we say, two selfmaps I and T of a locally convex space (E,τ) are weakly commuting if and only if

$$p_\alpha(ITx - TIx) \le p_\alpha(Ix - Tx), \tag{1.2}$$

for each $x \in E$ and $p_\alpha \in A^*(\tau)$. Clearly, commuting maps are weakly commuting but not conversely in general (see [10, 13]). A mapping $T : E \to E$ is said to be nonexpansive on E if $p_\alpha(Tx - Ty) \le p_\alpha(x - y)$ for all x, y in E and $p_\alpha \in A^*(\tau)$. The set of fixed points of T on E is denoted by $F(T)$. If $u \in E$, $M \subseteq E$, then for $0 < a \le 1$, we define the set D_a of best (M,a)-approximants to u as follows:

$$D_a = \{y \in M : a p_\alpha(y - u) = d_{p_\alpha}(u,M), \forall p_\alpha \in A^*(\tau)\}, \tag{1.3}$$

where

$$d_{p_\alpha}(u,M) = \inf\{p_\alpha(x - u) : x \in M\}. \tag{1.4}$$

Let D denote the set of best approximations to u. For $a = 1$, our definition reduces to the set D of best M-approximants to u. A mapping $T : M \to E$ is called demiclosed at 0 if whenever $\{x_n\}$ converges weakly to x and $\{Tx_n\}$ converges to 0, we have $Tx = 0$.

2. Results

Lemma 2.1. *Let T and I be weakly commuting selfmaps of a τ-bounded subset M of a Hausdorff locally convex space (E,τ). Then T and I are weakly commuting on M with respect to $\|\cdot\|_B$.*

Proof. By hypothesis for any $x \in M$,

$$p_\alpha(ITx - TIx) \le p_\alpha(Ix - Tx), \quad \text{for each } p_\alpha \in A^*(\tau). \tag{2.1}$$

Taking supremum on both sides, we get

$$\begin{gathered} \sup_\alpha p_\alpha\left(\frac{ITx - TIx}{\lambda_\alpha}\right) \le \sup_\alpha p_\alpha\left(\frac{Ix - Tx}{\lambda_\alpha}\right), \\ \|ITx - TIx\|_B \le \|Ix - Tx\|_B \quad \text{as desired.} \end{gathered} \tag{2.2}$$

□

Note that if I is nonexpansive on a τ-bounded subset M of E, then I is also nonexpansive with respect to $\|\cdot\|_B$ (cf. [8, 15]).

We use a technique of Tarafdar [15] to obtain the following common fixed point theorem which generalizes Theorem 1.1 and the main result of Fisher and Sessa [4].

Theorem 2.2. *Let M be a nonempty τ-bounded, τ-complete, and convex subset of a Hausdorff locally convex space (E,τ) and T, I two weakly commuting selfmaps of M satisfying the inequality*

$$p_\alpha(Tx - Ty) \le a p_\alpha(Ix - Iy) + (1 - a)\max\{p_\alpha(Tx - Ix),\ p_\alpha(Ty - Iy)\}, \tag{2.3}$$

for all $x, y \in M$ and for all $p_\alpha \in A^*(\tau)$ and for some $a \in (0,1)$. If I is affine and nonexpansive on M and $T(M) \subseteq I(M)$, then T and I have a unique common fixed point.

Proof. Since M is τ-complete, it follows that $(E_B, \|\cdot\|_B)$, is a Banach space and M is complete in it. By Lemma 2.1, T and I are $\|\cdot\|_B$-weakly commuting maps of M. From (2.3) we obtain for $x, y \in M$,

$$\begin{aligned} \sup_\alpha p_\alpha\left(\frac{Tx-Ty}{\lambda_\alpha}\right) \le a \sup_\alpha p_\alpha\left(\frac{Ix-Iy}{\lambda_\alpha}\right) \\ + (I-a)\max\left\{\sup_\alpha p_\alpha\left(\frac{Tx-Ix}{\lambda_\alpha}\right), \sup_\alpha p_\alpha\left(\frac{Ty-Iy}{\lambda_\alpha}\right)\right\}. \end{aligned} \tag{2.4}$$

Thus

$$\|Tx-Ty\|_B \le a\|Ix-Iy\|_B + (1-a)\max\{\|Tx-Ix\|_B, \|Ty-Iy\|_B\}. \tag{2.5}$$

It can be shown easily that I is $\|\cdot\|_B$-nonexpansive on M. A comparison of our hypothesis with that of Theorem 1.1 tells that we can apply Theorem 1.1 to M as a subset of $(E_B, \|\cdot\|_B)$ to conclude that there exists a unique $a \in M$ such that $a = Ia = Ta$. □

An application of Theorem 2.2 establishes the following result in best approximation theory.

Theorem 2.3. *Let T and I be selfmaps of a Hausdorff locally convex space (E, τ) and M a subset of E such that $T(\partial M) \subseteq M$, where ∂M denotes boundary of M and $u \in F(T) \cap F(I)$. Suppose that T and I satisfy (2.3) for all x, y in $D'_a = D_a \cup \{u\}$ and I is nonexpansive and affine on D_a. For each $p_\alpha \in A^*(\tau)$,*

$$p_\alpha(TIx - ITx) \le \frac{1}{k} p_\alpha((kTx + (1-k)q) - Ix), \tag{2.6}$$

for all $k \in (0,1)$, $x \in D_a$ and for some $q \in D_a$. If D_a is nonempty convex, $q \in F(I)$ and $I(D_a) = D_a$, then I and T have a common fixed point in D_a provided one of the following conditions holds:

(i) *D_a is τ-compact.*

(ii) *D_a is weakly compact in (E, τ), I is weakly continuous and $I - T$ is demiclosed at* 0.

Proof. Let $y \in D_a$. Then $Iy \in D_a$, since $I(D_a) = D_a$. Further, if $y \in \partial M$, then $Iy \in M$ for $T(\partial M) \subseteq M$. From (2.3), it follows that for each $p_\alpha \in A^*(\tau)$,

$$\begin{aligned} p_\alpha(Ty-u) &= p_\alpha(Ty-Tu) \\ &\le a p_\alpha(Iy-Iu) + (1-a)\max\{p_\alpha(Ty-Iy), p_\alpha(Tu-Iu)\} \\ &\le a p_\alpha(Iy-u) + (1-a)\big(p_\alpha(Ty-u) + p_\alpha(Iy-u)\big). \end{aligned} \tag{2.7}$$

So we have, $a p_\alpha(Ty-u) \le p_\alpha(Iy-u)$ for all $p_\alpha \in A^*(\tau)$. Hence $Ty \in D_a$ which implies that T maps D_a into itself.

Let $\{k_n\}$ be a monotonically nondecreasing sequence of real numbers such that $0 < k_n < 1$ and $\limsup k_n = 1$. Define for each $n \in \mathbb{N}$, a mapping $T_n : D_a \to D_a$ by

$$T_n(x) = k_n Tx + (1-k_n)q. \tag{2.8}$$

It is possible to define such a mapping T_n for each $n \in \mathbb{N}$, since D_a is convex and

$q \in D_a$. The map I is affine so we have

$$T_nIx = k_nTIx + (1-k_n)q, \qquad IT_nx = k_nITx + (1-k_n)q. \tag{2.9}$$

From (2.6), it follows that

$$\begin{aligned} p_\alpha(T_nIx - IT_nx) &= k_np_\alpha(TIx - ITx) \\ &\le k_n\left(\left(\frac{1}{k_n}\right)p_\alpha(k_nTx + (1-k_n)q) - Ix\right) \\ &= p_\alpha(T_nx - Ix), \quad \forall x \in D_a,\ p_\alpha \in A^*(\tau). \end{aligned} \tag{2.10}$$

Thus I and T_n are weakly commuting on D_a for each n and $T_n(D_a) \subseteq D_a = I(D_a)$. For all $x, y \in D_a$, $p_\alpha \in A^*(\tau)$ and for all $j \ge n$, (n fixed), we obtain from (2.3),

$$\begin{aligned} p_\alpha(T_nx - T_ny) &= k_np_\alpha(Tx - Ty) \le k_jp_\alpha(Tx - Ty) \\ &\le p_\alpha(Tx - Ty) \\ &\le ap_\alpha(Ix - Iy) + (1-a)\max\{p_\alpha(Tx - Ix), p_\alpha(Ty - Iy)\} \\ &\le ap_\alpha(Ix - Iy) + (1-a)\max\{p_\alpha(Tx - T_nx) + p_\alpha(T_nx - Ix), \\ &\qquad p_\alpha(Ty - T_ny) + p_\alpha(T_ny - Iy)\} \\ &\le ap_\alpha(Ix - Iy) + (1-a)\max\{(1-k_n)p_\alpha(Tx - q) + p_\alpha(T_nx - Ix), \\ &\qquad (1-k_n)p_\alpha(Ty - q) + p_\alpha(T_ny - Iy)\}. \end{aligned} \tag{2.11}$$

Hence for all $j \ge n$, we have

$$\begin{aligned} p_\alpha(T_nx - T_ny) \le{}& ap_\alpha(Ix - Iy) \\ &+ (1-a)\max\{(1-k_j)p_\alpha(Tx - q) + p_\alpha(T_nx - Ix), \\ &\qquad (1-k_j)p_\alpha(Ty - q) + p_\alpha(T_ny - Iy)\}. \end{aligned} \tag{2.12}$$

As $\lim k_j = 1$, from (2.12), for every $n \in \mathbb{N}$, we have

$$\begin{aligned} p_\alpha(T_nx - T_ny) &= \lim_j p_\alpha(T_nx - T_ny) \\ &\le \lim_j \{ap_\alpha(Ix - Iy) + (1-a) \\ &\quad \times \max\{(1-k_j)p_\alpha(Tx - q) + p_\alpha(T_nx - Ix), \\ &\qquad (1-k_j)p_\alpha(Ty - q) + p_\alpha(T_ny - Iy)\}\}. \end{aligned} \tag{2.13}$$

This implies that for every $n \in \mathbb{N}$,

$$p_\alpha(T_nx - T_ny) \le ap_\alpha(Ix - Iy) + (1-a)\max\{p_\alpha(T_nx - Ix), p_\alpha(T_ny - Iy)\}, \tag{2.14}$$

for all $x, y \in D_a$ and for all $p_\alpha \in A^*(\tau)$.

(i) D_a being τ-compact is τ-bounded and τ-complete. Thus by Theorem 2.2, for every $n \in \mathbb{N}$, T_n and I have unique common fixed point x_n in D_a. Now the τ-compactness of D_a ensures that $\{x_n\}$ has a convergent subsequence $\{x_{n_j}\}$ which converges to a point $x_o \in D_a$. Since

$$x_{n_j} = T_{n_j} x_{n_j} = k_{n_j} T x_{n_j} + (1 - k_{n_j}) q \tag{2.15}$$

and T is continuous, so we have, as $j \to \infty, T x_0 = x_0$. The continuity of I implies that

$$I x_o = I\left(\lim_j x_{n_j}\right) = \lim_j I\left(x_{n_j}\right) = \lim_j x_{n_j} = x_0. \tag{2.16}$$

(ii) Weakly compact sets in (E, τ) are τ-bounded and τ-complete so again by Theorem 2.2, T_n and I have a common fixed point x_n in D_a for each n. The set D_a is weakly compact so there is a subsequence $\{x_j\}$ of $\{x_n\}$ converging weakly to some $y \in D_a$. The map I being weakly continuous gives that $I y = y$. Now

$$x_j = I(x_j) = T_j(x_j) = k_j T x_j + (1 - k_j) q \tag{2.17}$$

implies that $I x_j - T x_j = (1 - k_j)[q - T x_j] \to 0$ as $j \to \infty$. The demiclosedness of $I - T$ at 0 implies that $(I - T)(y) = 0$. Hence $I y = T y = y$. □

Example 2.4 (cf. MR.89h:54030)**.** Let $M = [1, \infty)$ and d be the absolute value metric on M. Define f and g on M by $fx = 1 + x$, $g(x) = 1 + 2x$. As $d(fgx, gfx) = 1 \le x = d(fx, gx)$ for all x in M so f and g are weakly commuting but evidently there exists no sequence $\{x_n\}$ in M for which the condition of compatibility is satisfied (f and g are compatible (see [6]) if $d(fgx_n, gfx_n) \to 0$, as $n \to \infty$, for any sequence $\{x_n\}$ in M satisfying $\lim_n fx_n = \lim_n gx_n = t \in M$).

Remarks 2.5. (i) In the light of Example 2.4, the classes of weakly commuting and compatible maps are different and so the statement "weakly commuting maps are compatible" on page 977 in [6] is not valid. Hence Theorem 2.3 cannot be implied by Theorem 5 of Pathak et al. [11] even in Banach space setting.

(ii) Commuting maps satisfy (2.6) so Theorem 2.3(i) is a proper generalization of the main results of Sahab et al. [12] and Singh [14].

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