IJMMS 27:4 (2001) 215–220
PII. S0161171201006615
http://ijmms.hindawi.com
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HIGHER-ORDER KdV-TYPE EQUATIONS AND THEIR STABILITY

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(Received 29 January 2001)

ABSTRACT. We have derived solitary wave solutions of generalized KdV-type equations of fifth order in terms of certain hyperbolic functions and investigated their stability. It has been found that the introduction of more dispersive effects increases the stability range.

2000 Mathematics Subject Classification. 35Q53.

1. Introduction. Since the inverse scattering technique was established [5], several methods of obtaining solitary wave solutions have been developed. Due to the availability of symbolic manipulation packages such as Maple, Mathematica, and so forth [6], the search for exact solutions of nonlinear evolution equations became more and more interesting as well as attractive.

In Section 2, we consider a generalized Korteweg-de Vries equation of third order [7] which has the maximum nonlinearity term $u^{2p}u_x$. This equation is known to have stable soliton solutions for $p < 4$. In Section 3, we have a fifth-order KdV-type equation [4] with the maximum nonlinearity term $u^p u_x$. This equation has been found to have stable soliton solutions for $p \geq 4$ from which it is clear that the fifth-order dispersive term has increased the stability range. In Section 4, we investigate a generalized fifth-order KdV-type equation [2] with a maximum nonlinearity term $u^{2p}u_x$. It has been shown for the case $p = 1/2$, that when the parameter in the highest-order dispersive term and the traveling wave velocity have both positive signs, the soliton solution is stable, when they both have negative signs, the soliton solution is unstable and when both have opposite signs, the solution is stable with a constraint on parameters.

2. Generalized Korteweg-de Vries equations. We consider a generalized Korteweg-de Vries (gKdV) equation of the form

$$u_t + (\alpha + \beta u^p) u^p u_x + \gamma u_{xxx} = 0, \tag{2.1}$$

where α, β, γ, and p are real constants. For $p = 1$, (2.1) is a combined KdV-mKdV equation and it further simplifies to KdV equation when $\beta = 0$. For $p = 2$, this equation has been solved by the method of direct integration as well as the series method [1, 3]. Equation (2.1) with p being any positive integer is often referred to as the gKdV equation. This equation describes an anharmonic lattice with a nearest-neighbour interaction force $F \sim \Delta^{p+1}$, where Δ is the extension or compression of the spring between two neighbouring masses. Here, we will discuss the case when p is a positive real number.

Traveling wave solutions of (2.1) are of the form $u = u(z)$, where $z = x - vt$. Integrating (2.1) with respect to z and using the solitary wave boundary condition that $u \to 0$ as $z \to \pm\infty$, we get

$$-vu + \frac{\alpha u^{p+1}}{p+1} + \frac{\beta u^{2p+1}}{2p+1} + \gamma u_{zz} = 0. \tag{2.2}$$

We look for a solution of the form

$$u = [A + B\cosh(mz)]^{-1/p}. \tag{2.3}$$

Substituting (2.3) in (2.2) it can be easily shown that

$$\begin{aligned} A &= \frac{\alpha}{(p+1)(p+2)v}, \\ B &= \frac{\sqrt{(2p+1)\alpha^2 + (p+1)(p+2)^2\beta v}}{(p+1)(p+2)v}, \\ m &= p\sqrt{\frac{v}{\gamma}}. \end{aligned} \tag{2.4}$$

Assuming v to be positive, we should take $\gamma > 0$ for the solution in the form (2.3) to exist. If $\beta > 0$, the solution (2.3) is valid for all v. For $\beta < 0$, the solution is valid for all v but

$$v = -\frac{(2p+1)\alpha^2}{(p+1)(p+2)^2\beta}. \tag{2.5}$$

A special case of this equation is when $\alpha = 1$, $\beta = 0$, $p = 1/2$, given by

$$u_t + u^{1/2}u_x + \gamma u_{xxx} = 0, \tag{2.6}$$

which describes ion-acoustic waves in a cold-ion plasma where the electrons do not behave isothermally during their passage of the wave. In this case, the solution (2.3) reduces to the solitary wave solution [7]

$$u(x,t) = \frac{225v^2}{64}\operatorname{sech}^4\left[\frac{1}{4}\sqrt{\frac{v}{\gamma}}(x - vt)\right]. \tag{2.7}$$

Another special case is the equation when $p = 1/2$ with nonzero α and β, given by

$$u_t + (\alpha + \beta u^{1/2})u^{1/2}u_x + \gamma u_{xxx} = 0, \tag{2.8}$$

where u refers to the perturbed ion density in a plasma with non-isothermal electrons. The solitary wave solution (2.3) then becomes

$$u(x,t) = \left[\frac{4\alpha}{15v} + \frac{\sqrt{75\beta v + 16\alpha^2}}{15v}\cosh\left(\frac{1}{2}\sqrt{\frac{v}{\gamma}}(x - vt)\right)\right]^{-2}. \tag{2.9}$$

As noted earlier, this solution is valid for all positive v when $\beta > 0$ and is valid for all $v \neq -16\alpha^2/75\beta$ when $\beta < 0$.

To see whether there exists a solitary wave solution for this critical wave velocity, we consider a solution for (2.2) in the form

$$u = A[1 - \tanh(mz)]^{1/p}. \tag{2.10}$$

By substituting (2.10) in (2.2) one can show that

$$\begin{aligned} A &= \left(-\frac{\alpha(2p+1)}{\beta(2p+4)}\right)^{1/p}, \\ m^2 &= -\frac{p^2(2p+1)\alpha^2}{(p+1)(2p+4)^2\beta\gamma}, \\ v &= -\frac{4(2p+1)\alpha^2}{(p+1)(2p+4)^2\beta}. \end{aligned} \tag{2.11}$$

For the special case, $p = 1/2$, we get the solution

$$u(x,t) = \frac{4\alpha^2}{25\beta^2}\left[1 - \tanh\left(\frac{\alpha}{15}\sqrt{-\frac{3}{\beta\gamma}}(x - vt)\right)\right]^2, \tag{2.12}$$

which is valid for all v with $v = -16\alpha^2/75\beta$.

3. Fifth-order KdV-type equations. We consider a fifth-order KdV-type equation in the form

$$\frac{\partial u}{\partial t} + \alpha u^p \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} + \gamma \frac{\partial^5 u}{\partial x^5} = 0, \tag{3.1}$$

with $p > 0$. The last term in the equation describes higher-order dispersive effects which influences the properties of the solitons. The equation with $p = 1$ is the fifth-order KdV equation and the equation with $p = 2$ is a fifth-order modified KdV equation both of which have applications in fluid mechanics, plasma physics, and so forth. The equation with $p \geq 4$ has great theoretical interest in the general problem of collapse of nonlinear waves.

Without loss of generality, we assume $\alpha = \beta = 1$. Traveling wave solutions of (3.1) in the form $u = u(z)$, where $z = x - vt$ with the solitary wave boundary conditions give rise to the equation

$$-vu + \frac{u^{p+1}}{p+1} + \frac{d^2u}{dz^2} + \gamma \frac{d^4u}{dz^4} = 0. \tag{3.2}$$

We look for a solution of the form

$$u(z) = A\,\mathrm{sech}^{4/p}(mz). \tag{3.3}$$

By substituting (3.3) in (3.2), we can easily show that,

$$u(z) = \left[\frac{v}{8}\,\frac{(p+1)(p+4)(3p+4)}{p+2}\right]^{1/p} \operatorname{sech}^{4/p}\left(\frac{pz\sqrt{v(p^2+4p+8)}}{4(p+2)}\right), \tag{3.4}$$

$$A = \left[\frac{v(p+1)(p+4)(3p+4)}{8(p+2)}\right]^{1/p}, \tag{3.5}$$

$$m = \frac{p\sqrt{v(p^2+4p+8)}}{4(p+2)}, \tag{3.6}$$

with $v > 0$ and $\gamma < 0$.

Equation (3.1) is a Hamiltonian system for which the momentum is given by

$$M = \frac{1}{2}\int_{-\infty}^{\infty} u^2\,dx, \tag{3.7}$$

For the solution (3.4), we can show that

$$M = \frac{A^2 2^{8/p}\Gamma^2(4/p)}{4m\Gamma(8/p)}, \tag{3.8}$$

where A and m are given by (3.5) and (3.6).

The sufficient condition for soliton stability is,

$$\frac{\partial M}{\partial v} > 0. \tag{3.9}$$

Here, we have

$$\frac{\partial M}{\partial v} = \frac{1}{v}\left(\frac{2}{p} - \frac{1}{2}\right)M, \tag{3.10}$$

so that

$$\frac{\partial M}{\partial v} > 0 \quad \text{iff } p < 4. \tag{3.11}$$

The $\gamma = 0$ soliton was only stable for $p \geq 4$ and so it is evident that the fifth-order term has increased the stability range.

4. Generalized fifth-order KdV-type equations. We introduce higher-order dispersive effects into (2.1) and write the equation in the form

$$u_t + (\alpha + \beta u^p)u^p u_x + \gamma u_{xxx} + \delta u_{xxxxx} = 0. \tag{4.1}$$

Traveling wave solutions of (4.1) in the form $u = u(z)$, where $z = x - vt$, with the solitary wave boundary conditions give rise to the equation

$$-vu + \frac{\alpha u^{p+1}}{p+1} + \frac{\beta u^{2p+1}}{2p+1} + \gamma u_{zz} + \delta u_{zzzz} = 0. \tag{4.2}$$

We look for a solution of (4.2) in the form

$$u = A[1 + \cosh(mz)]^{-1/p}. \tag{4.3}$$

Substituting (4.3) in (4.2), we get

$$m = \left[-\frac{\gamma p^2 \pm p^2\sqrt{\gamma^2+4v\delta}}{2\delta} \right]^{1/2},$$
$$A = \left[-\frac{m^4\delta(3p+2)(2p+1)(p+2)(p+1)}{p^4\beta} \right]^{1/2p}. \tag{4.4}$$

Now, we will consider the problem of stability of the soliton solution of (4.3) for different values of p. Equation (4.1) is a Hamiltonian system for which the momentum is given by

$$M = \frac{1}{2}\int_{-\infty}^{\infty} u^2\,dx, \tag{4.5}$$

For $p = 1/2$, we have,

$$m = \left[\frac{-\gamma \pm \sqrt{\gamma^2+4v\delta}}{2\delta} \right]^{1/2}, \tag{4.6}$$

$$A = \sqrt{-\frac{90\delta}{\beta}}\,m^2. \tag{4.7}$$

Thus the momentum M is given by

$$M = \frac{2}{3}\left(-\frac{90\delta}{\beta}\right)m^3. \tag{4.8}$$

The sufficient condition for soliton stability is,

$$\frac{\partial M}{\partial v} > 0. \tag{4.9}$$

We assume δ to be positive so that β should be negative from (4.7). Now, we can easily see that

$$\frac{\partial M}{\partial v} = -90\frac{m\delta}{\beta\sqrt{\gamma^2+4v\delta}} > 0 \tag{4.10}$$

for positive v and any real value of γ and so is stable.

When δ and v have opposite signs, the solution is stable with the condition that $|\gamma| \geq \sqrt{-4v\delta}$.

When both δ and v have negative signs, the solution is unstable for all real values of γ.

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