IJMMS 28:1 (2001) 51–55
PII. S0161171201006202
http://ijmms.hindawi.com

ON THE STRONGLY STARLIKENESS OF MULTIVALENTLY CONVEX FUNCTIONS OF ORDER α

MAMORU NUNOKAWA, SHIGEYOSHI OWA, and AKIRA IKEDA

(Received 16 November 2000)

Abstract. The object of the present paper is to derive some sufficient conditions for strongly starlikeness of multivalently convex functions of order α in the open unit disc.

2000 Mathematics Subject Classification. 30C45.

1. Introduction. Let $\mathcal{A}(p)$ denote the class of the functions $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ which are analytic in the open unit disc $\mathcal{E} = \{z : |z| < 1\}$. A function $f(z) \in \mathcal{A}(p)$ is called p-valently starlike if and only if the inequality

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \tag{1.1}$$

holds for $z \in \mathcal{E}$. A function $f(z) \in \mathcal{A}(p)$ is called p-valently convex of order α $(0 \le \alpha < p)$ if and only if the inequality

$$1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > \alpha \tag{1.2}$$

holds for $z \in \mathcal{E}$. We denote by $\mathcal{C}(p,\alpha)$ the family of such functions. A function $f(z) \in \mathcal{A}(p)$ is said to be strongly starlike of order α $(0 < \alpha \le 1)$ if and only if the inequality

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\alpha \tag{1.3}$$

holds for $z \in \mathcal{E}$. We also denote by $\mathrm{STS}(p,\alpha)$ the family of functions which satisfy the above inequality for the argument. From the definition, it follows that if $f(z) \in \mathrm{STS}(p,\alpha)$, then we have

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad \text{in } \mathcal{E} \tag{1.4}$$

or $f(z)$ is p-valently starlike in $\mathcal{E}$ and therefore $f(z)$ is p-valent in $\mathcal{E}$ (see [1, Lemma 7]).

Nunokawa [2, 3] proved the following theorems.

Theorem 1.1 (see [2]). *If $f(z) \in \mathcal{A}(p)$ satisfies*

$$1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} < p + \frac{\alpha}{2}, \tag{1.5}$$

where $0 < \alpha \le 1$, then $f(z) \in \mathrm{STS}(p,\alpha)$.

Theorem 1.2 (see [3]). *If $f(z) \in \mathcal{A}(1)$ satisfies*

$$\left|\arg\left\{1+\frac{zf''(z)}{f'(z)}\right\}\right| < \frac{\pi}{2}\alpha(\beta) \quad \text{in } \mathcal{E}, \tag{1.6}$$

then

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\beta \quad \text{in } \mathcal{E}, \tag{1.7}$$

where

$$\begin{aligned} \alpha(\beta) &= \beta+\frac{2}{\pi}\tan^{-1}\left\{\frac{\beta q(\beta)\sin(\pi/2)(1-\beta)}{p(\beta)+\beta q(\beta)\cos(\pi/2)(1-\beta)}\right\}, \\ p(\beta) &= (1+\beta)^{(1+\beta)/2}, \qquad q(\beta) = (1-\beta)^{(\beta-1)/2}. \end{aligned} \tag{1.8}$$

It is the purpose of the present paper to prove that if $f(z) \in \mathcal{C}(1, 1-(\alpha/2))$, then $f(z) \in \mathrm{STS}(1,\alpha)$.

In this paper, we need the following lemma.

Lemma 1.3. *Let $f(z) \in \mathcal{A}(1)$ be starlike with respect to the origin in $\mathcal{E}$. Let $C(r,\theta) = \{f(te^{i\theta}) : 0 \le t \le r < 1\}$ and $T(r,\theta)$ be the total variation of $\arg f(te^{i\theta})$ on $C(r,\theta)$, so that*

$$T(r,\theta) = \int_0^r \left|\frac{\partial}{\partial t}\arg\{f(te^{i\theta})\}\right| dt. \tag{1.9}$$

Then

$$T(r,\theta) < \pi. \tag{1.10}$$

We owe this lemma to Sheil-Small [6, Theorem 1].

2. Main theorem.

Our main theorem for the starlikeness of multivalently convex functions of order α is the following.

Theorem 2.1. *Let $f(z) \in \mathcal{A}(1)$ and*

$$1+\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > 1-\frac{\alpha}{2} \quad \text{in } \mathcal{E}, \tag{2.1}$$

where $0 < \alpha \le 1$. Then

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\alpha \quad \text{in } \mathcal{E}, \tag{2.2}$$

or $f(z)$ is strongly starlike of order α in $\mathcal{E}$.

Proof. We put

$$\frac{2}{\alpha}\left\{1+\frac{zf''(z)}{f'(z)}-1+\frac{\alpha}{2}\right\} = \frac{zg'(z)}{g(z)}, \tag{2.3}$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. From assumption (2.1), we have

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > 0 \quad \text{in } \mathscr{E}. \tag{2.4}$$

This shows that $g(z)$ is starlike and univalent in $\mathscr{E}$. With an easy calculation (cf. [4]), (2.3) gives us that

$$f'(z) = \left\{\frac{g(z)}{z}\right\}^{\alpha/2}. \tag{2.5}$$

Since

$$f'(z) \neq 0, \quad 0 < |z| < 1, \tag{2.6}$$

we easily have

$$\frac{f(z)}{zf'(z)} = \int_0^1 \frac{f'(tz)}{f'(z)}\,dt = \int_0^1 t^{-\alpha/2}\left\{\frac{g(tre^{i\theta})}{g(re^{i\theta})}\right\}^{\alpha/2} dt, \tag{2.7}$$

where $z = re^{i\theta}$ and $0 < r < 1$. Since $g(z)$ is starlike in $\mathscr{E}$, from Lemma 1.3, we have

$$-\pi < \arg\left\{g(tre^{i\theta})\right\} - \arg\left\{g(re^{i\theta})\right\} < \pi \tag{2.8}$$

for $0 < t \leq 1$. Putting

$$\xi = \left\{\frac{g(tre^{i\theta})}{g(re^{i\theta})}\right\}^{\alpha/2}, \tag{2.9}$$

we have

$$\arg s = \frac{\alpha}{2}\arg\left\{\frac{g(tre^{i\theta})}{g(re^{i\theta})}\right\}. \tag{2.10}$$

From (2.8) and (2.10), s lies in the convex sector

$$\left\{s : |\arg s| \leq \frac{\pi}{2}\alpha\right\} \tag{2.11}$$

and the same is true of its integral mean of (2.7), (cf. [5, Lemma 1]). Therefore, we have

$$\left|\arg\left\{\frac{f(z)}{zf'(z)}\right\}\right| < \frac{\pi}{2}\alpha \quad \text{in } \mathscr{E} \tag{2.12}$$

or

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\alpha \quad \text{in } \mathscr{E}. \tag{2.13}$$

This shows that

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad \text{in } \mathscr{E}, \tag{2.14}$$

which completes the proof of our main theorem. □

Remark 2.2. This result is sharp for the case $\alpha \to 0$ and $\alpha = 1$.

(a) For the case $\alpha \to 0$, put $f(z) = z$, then $f(z)$ is a convex function of order $1 - (\alpha/2) \to 1$ and $f(z)$ then $f(z)$ is a strongly starlike function of order $\alpha \to 0$.

(b) For the case $\alpha = 1$, put

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1}{1-z}. \tag{2.15}$$

Then we have

$$1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2} \quad \text{in } \mathscr{E}, \tag{2.16}$$

and therefore $f(z)$ is a convex function of order $1/2$. From (2.10), we easily have

$$f'(z) = \frac{1}{1-z}, \qquad f(z) = \log\left\{\frac{1}{1-z}\right\}. \tag{2.17}$$

Putting $|z| = 1$, $z = e^{i\theta}$, $0 \le \theta < 2\pi$, then it follows that

$$\frac{z}{1-z} = -\frac{1}{2} + i\frac{\cos(\theta/2)}{2\sin(\theta/2)},$$

$$\log\left\{\frac{1}{1-z}\right\} = \log\left|\frac{1}{2} + i\frac{\cos(\theta/2)}{2\sin(\theta/2)}\right| + i\arg\left\{\frac{1}{2} + i\frac{\cos(\theta/2)}{2\sin(\theta/2)}\right\}.$$

$$\begin{aligned}
\lim_{\theta\to+0}\arg\left\{\frac{zf'(z)}{f(z)}\right\} &= \lim_{\theta\to+0}\arg\left\{\frac{z/(1-z)}{\log(1/(1-z))}\right\}\\
&= \lim_{\theta\to+0}\arg\left\{-\frac{1}{2} + i\frac{\cos(\theta/2)}{2\sin(\theta/2)}\right\}\\
&\quad - \lim_{\theta\to+0}\arg\left\{\log\left|\frac{1}{2} + i\frac{\cos(\theta/2)}{2\sin(\theta/2)}\right| + i\arg\left(\frac{1}{2} + i\frac{\cos(\theta/2)}{2\sin(\theta/2)}\right)\right\}\\
&= \frac{\pi}{2}.
\end{aligned} \tag{2.18}$$

The above shows that the main theorem is sharp for the case $\alpha \to 0$ and $\alpha = 1$.

Applying the same method as above and [2], we can obtain the following result.

Theorem 2.3. *If $f(z) \in A(p)$ and satisfies*

$$p - \frac{\alpha}{2} < 1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} \quad \text{in } \mathscr{E}, \tag{2.19}$$

where $0 < \alpha \le 1$, then $f(z) \in \mathrm{STS}(p,\alpha)$.

References

[1] M. Nunokawa, *On the theory of multivalent functions*, Tsukuba J. Math. **11** (1987), no. 2, 273–286. MR 89d:30013. Zbl 639.30014.

[2] ______, *On certain multivalently starlike functions*, Tsukuba J. Math. **14** (1990), no. 2, 275–277. MR 92b:30016. Zbl 728.30014.

[3] ______, *On the order of strongly starlikeness of strongly convex functions*, Proc. Japan Acad. Ser. A Math. Sci. **69** (1993), no. 7, 234–237. MR 95f:30019. Zbl 793.30007.

[4] M. Nunokawa and S. Owa, *On certain subclass of analytic functions*, Indian J. Pure Appl. Math. **19** (1988), no. 1, 51–54. MR 89c:30029. Zbl 646.30020.

[5] C. Pommerenke, *On close-to-convex analytic functions*, Trans. Amer. Math. Soc. **114** (1965), 176–186. MR 30#4920. Zbl 132.30204.

[6] T. Sheil-Small, *Some conformal mapping inequalities for starlike and convex functions*, J. London Math. Soc. (2) **1** (1969), 577–587. MR 40#2842. Zbl 201.40803.

Mamoru Nunokawa: Department of Mathematics, University of Gunma, Aramaki Maebashi Gunma, 371-8510, Japan
E-mail address: nunokawa@edu.gunma-u.ac.jp

Shigeyoshi Owa: Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan
E-mail address: owa@math.kindai.ac.jp

Akira Ikeda: Department of Applied Mathematics, Fukuoka University, Nanakuma Jonan-ku Fukuoka, 814-0180, Japan
E-mail address: aikeda@sf.sm.fukuoka-u.ac.jp